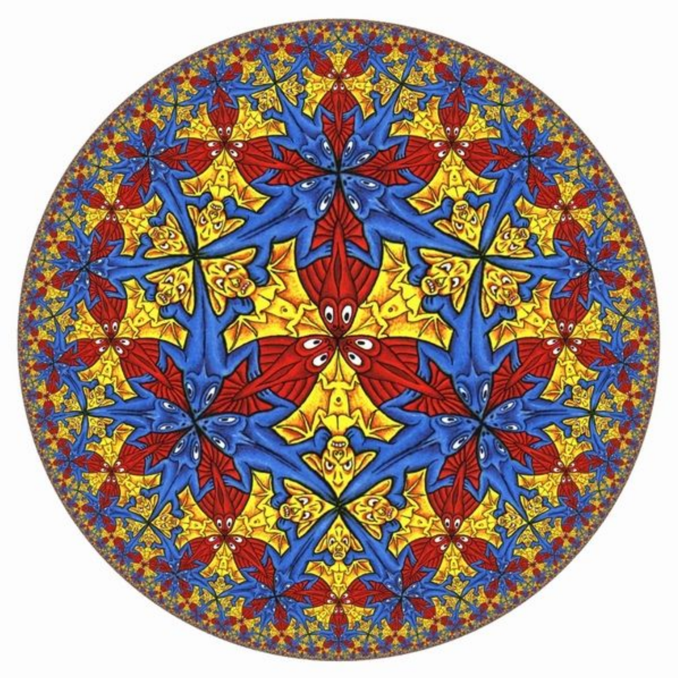
Positivity, higher Teichmüller spaces and (non-commutative) cluster algebras



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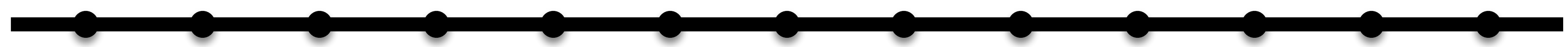
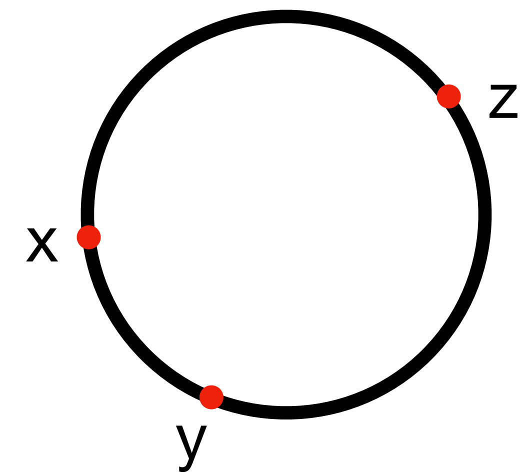
The circle

The real line $(\mathbb{R}, <)$ 

covers the circle and induces a cyclic order



(x,y,z) is a positive triple

Recast this ordering on $\mathbb{RP}^1$

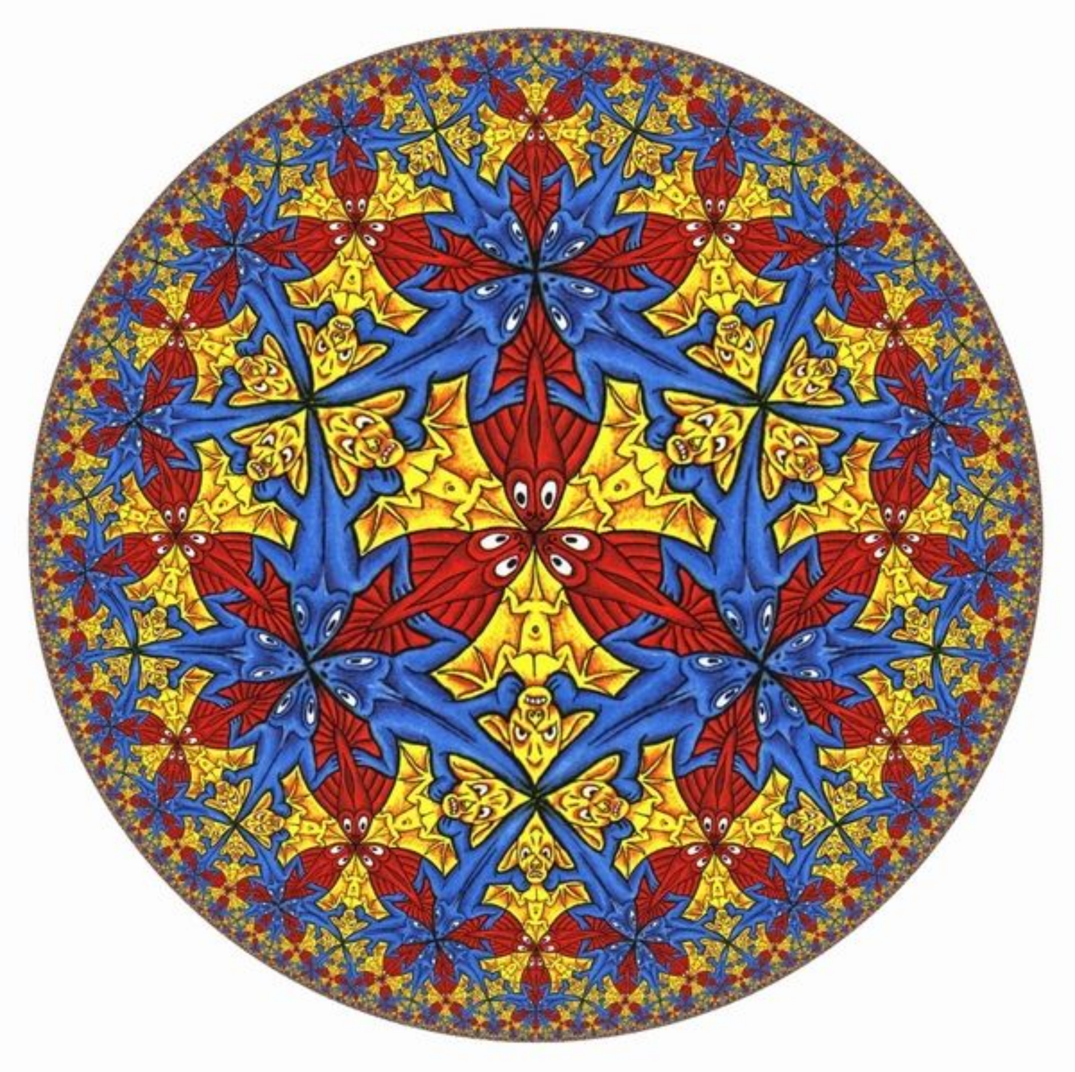
Assume $x = \mathbb{R}e_2, z = \mathbb{R}e_1$

every $y \neq z$ can be written as $y = \begin{pmatrix} 1 & t_y \\ 0 & 1 \end{pmatrix} \cdot e_2$ with $t_y \in \mathbb{R}$.

The triple (x,y,z) is positive if and only if $t_y \in \mathbb{R}_{>0}$.

Inside: the hyperbolic plane

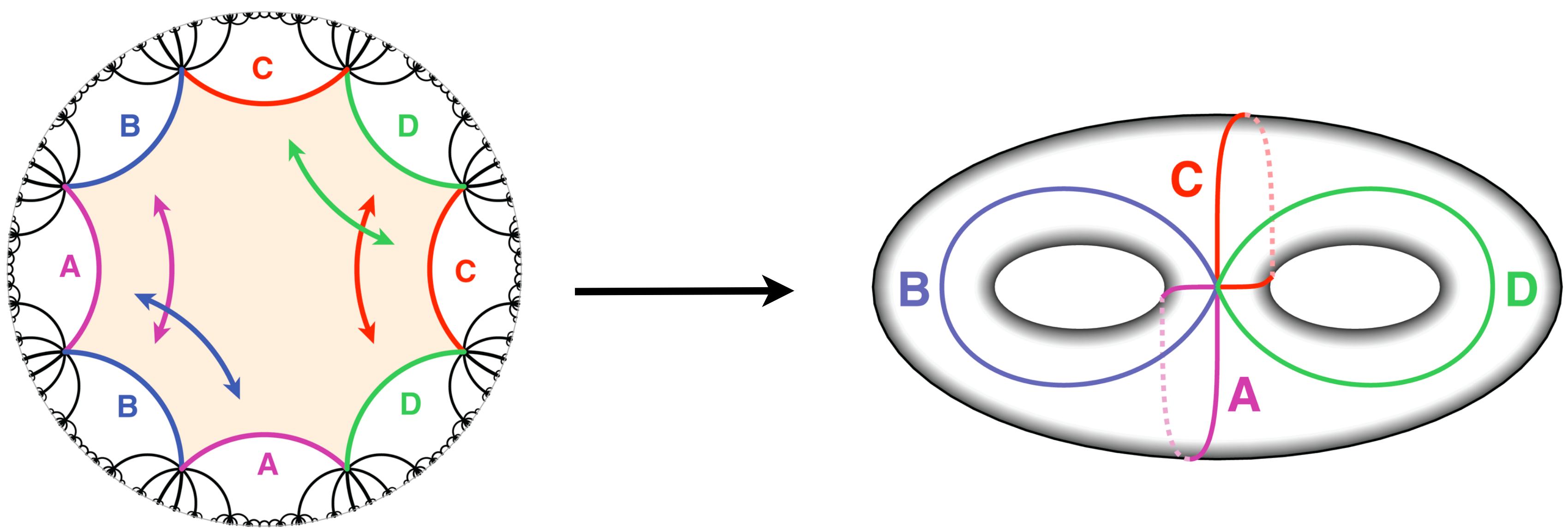
Poincare disk model $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\} = \{z \in \mathbb{C} \mid 1 - \bar{z}z > 0\}$



There is a rich interplay between the interior and the boundary circle

Hyperbolic structures on surfaces

Surfaces of genus >1 naturally carry a hyperbolic structure.



$$\mathrm{Hyp}(S) = \{(X, f) \mid X \text{ hyperbolic surface}, f : S \to X \text{ homeo}\}/\sim$$

holonomy ↓

quotient by mapping class group is moduli space $\mathscr{M}(S)$

$$\mathrm{Hom}(\pi_1(S), \mathrm{PSp}(2,\mathbb{R}))/\mathrm{PSp}(2,\mathbb{R})$$

Hyperbolic structures

The space of hyperbolic structures $\mathrm{Hyp}(S) \subset \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ is a connected component consisting entirely of discrete and faithful representations.

$\mathrm{Hyp}(S)$ can be identified with the Teichmüller space of S.

Representations in $\mathrm{Hyp}(S)$ are positive representations

The action of $\rho(\pi_1(S))$ on the circle is by orientation preserving homeomorphism.

For every representation $\rho \in \mathrm{Hyp}(S)$ there is an equivariant map $\xi : \mathbb{RP}^1 \cong \partial\pi_1(S) \to \mathbb{RP}^1$ sending positive triples to positive triples.

Representations in $\mathrm{Hyp}(S)$ are those of maximal Euler number [Goldman]

The Euler number is the obstruction to lifting to universal covering $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$

$$\rho : \pi_1(S) = \{a_1, b_1, \cdots a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1\} \to \mathrm{PSL}(2,\mathbb{R})$$

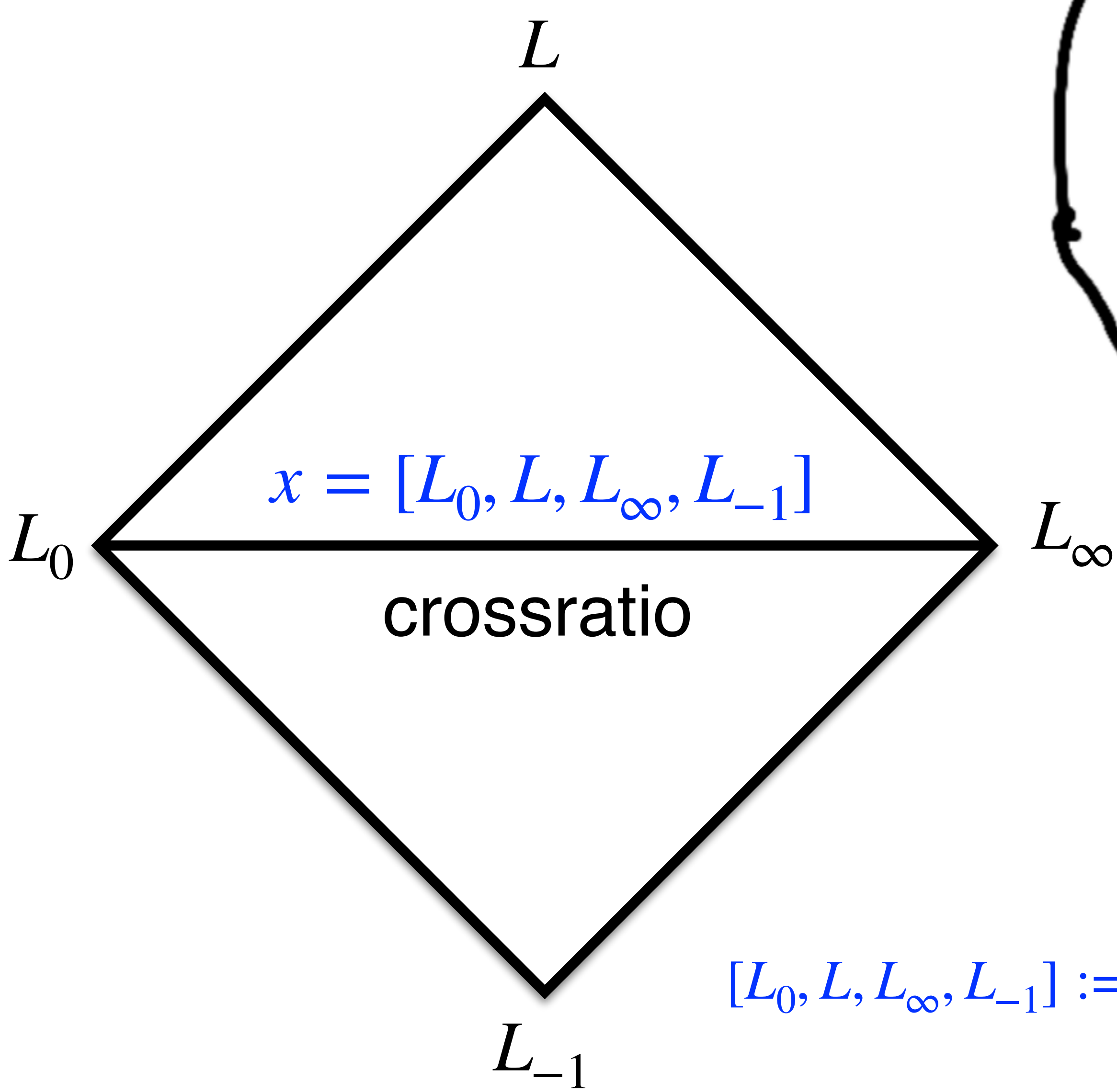
$$eu(\rho) = \widetilde{A_1}\widetilde{B_1}\widetilde{A_1^{-1}}\widetilde{B_1^{-1}} \cdots \widetilde{A_g}\widetilde{B_g}\widetilde{A_g^{-1}}\widetilde{B_g^{-1}} \in \mathbb{Z} \cap [2-2g, 2g-2]$$

Coordinates

S = S(g,n) surface with punctures — pick an ideal triangulation

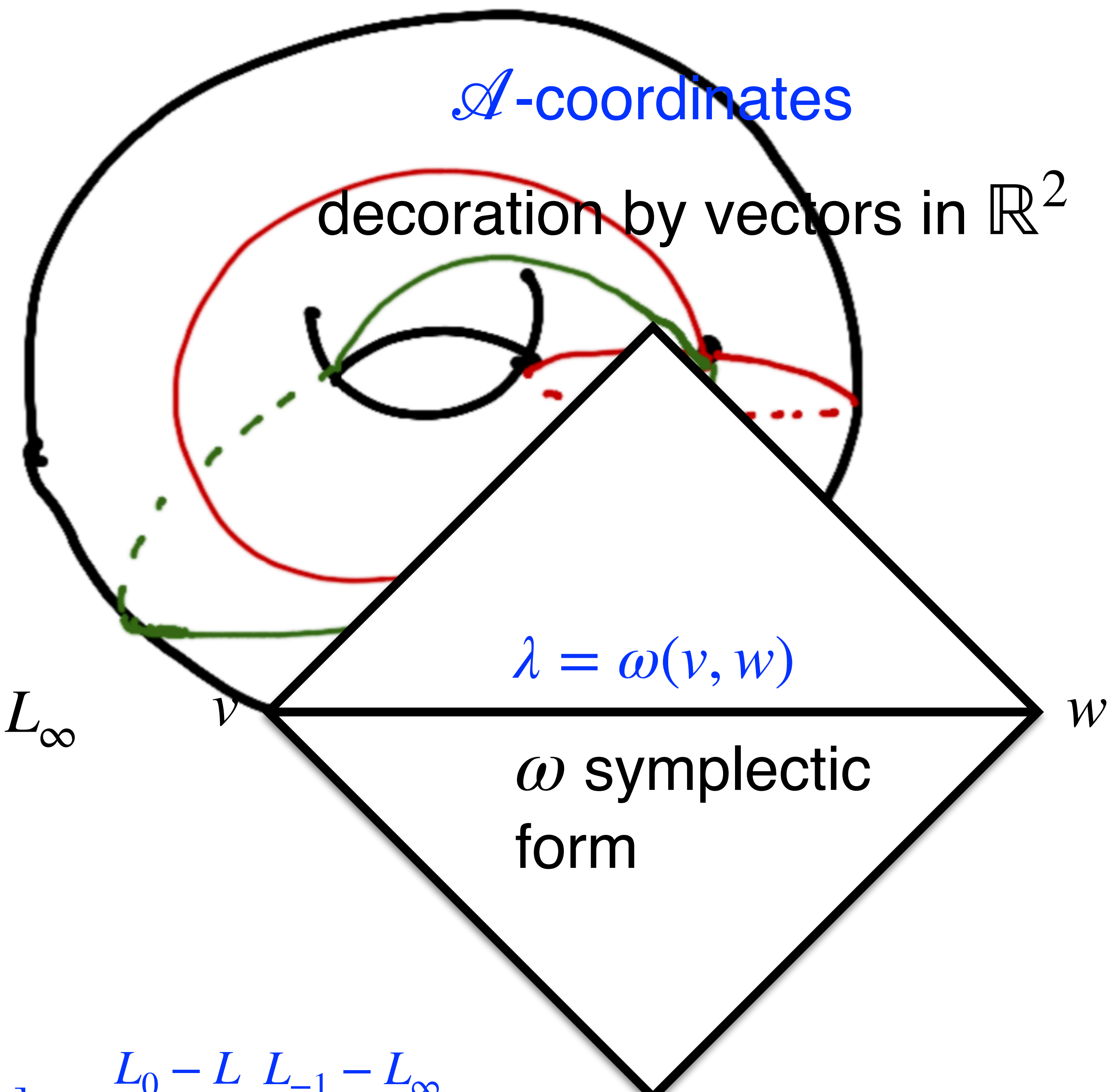
$\mathscr{X}$-coordinates

framing by $\mathbb{RP}^1$



$\mathscr{A}$-coordinates

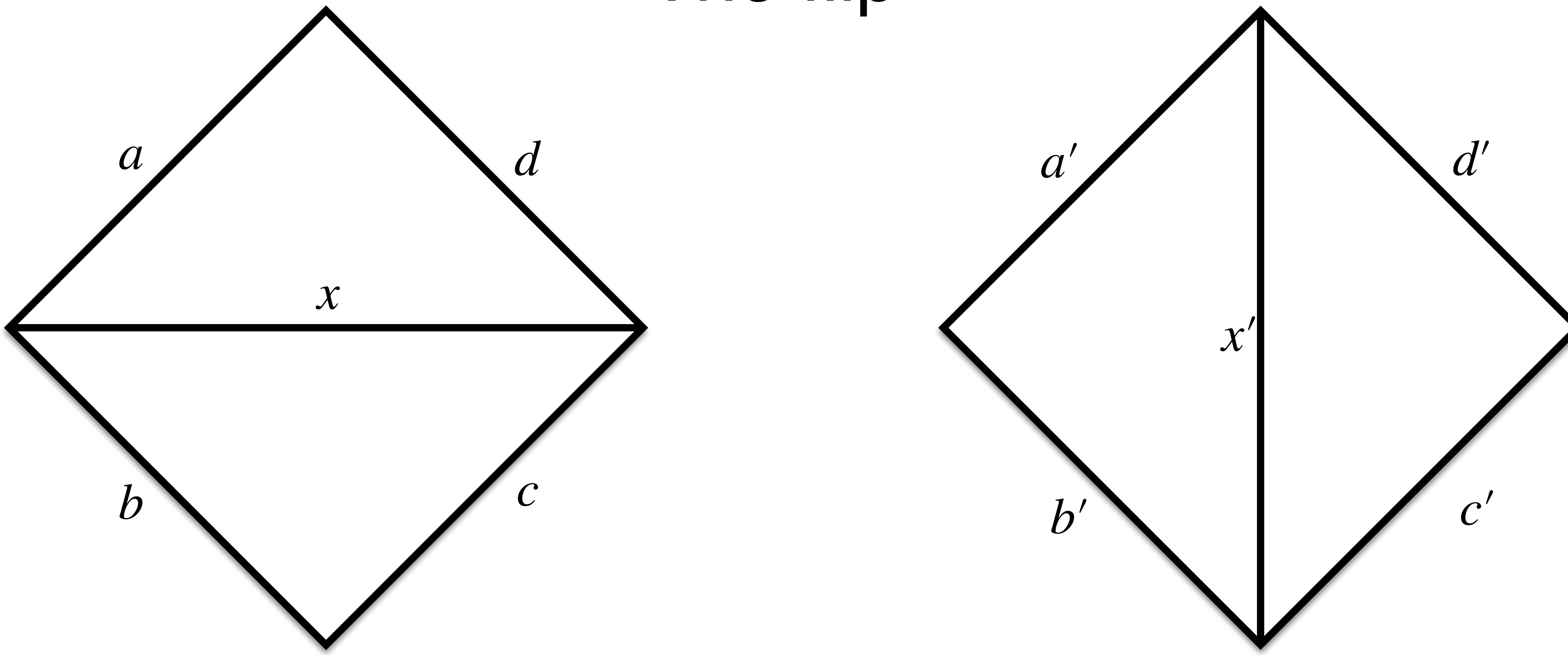
decoration by vectors in $\mathbb{R}^2$



$$[L_0, L, L_\infty, L_{-1}] := \frac{L_0 - L}{L_\infty - L}\,\frac{L_{-1} - L_\infty}{L_{-1} - L_0}$$

[Thurston, Bonahon] [Fock-Goncharov] [Penner]

The flip



$\mathscr{X}$-coordinates: $x' = x^{-1}$ $\quad a' = a(1+x)^{-1}$ $\quad d' = d(1+x)$

$b' = b(1+x^{-1})$ $\quad c' = c(1+x^{-1})^{-1}$

$\mathscr{A}$-coordinates: $$x' = \frac{ac+bd}{x}$$ Ptolemy equation

These are examples of cluster transformations

Representation varieties

Representation variety
$\mathrm{Hom}(\pi_1(S), G)/G$

$\rho : \pi_1(S) \to G$

$E_\rho = (\widetilde{S} \times G)/\pi_1(S)$

G-local systems on S
$\mathscr{M}(S, G)$

[Donaldson, Corlette]
[Hitchin, Simpson]

fix $X \cong S$

G-Higgs bundles on X
$\mathscr{M}_{Higgs}(X, G)$

Many different tools to study the same object !

(E, ϕ)

E holomorphic vector bundle

ϕ holomorphic one form with values in $\mathrm{End}(E)$

Hitchin fibration

$$\mathscr{M}_{Higgs}(X, G) \longrightarrow \bigoplus_{i=1}^{r_G} \mathrm{H}^0(X, K^{m_i})$$

Higher Teichmüller spaces

(Unions of) connected components $\mathrm{Hom}(\pi_1(S), G)/G$
consisting entirely of discrete and faithful representations

Hitchin component

G split real group

$\mathrm{SL}(n,\mathbb{R}), \mathrm{Sp}(2n,\mathbb{R}), \mathrm{SO}(n,n+1), \mathrm{SO}(n,n)$

deformations of
$\pi_{princ} \circ \iota : \pi_1(S) \to \mathrm{SL}(2,\mathbb{R}) \to G$

section of Hitchin fibration

$\mathrm{Hit}(S,G) \cong \mathbb{R}^{(2g-2)\dim G}$

[Hitchin, Goldman, Choi-Goldman]
[Labourie, Fock-Goncharov]

Maximal representations

G Hermitian type

$\mathrm{Sp}(2n,\mathbb{R}), \mathrm{SU}(n,m), \mathrm{SO}(2,n), \mathrm{SO}^*(2n)$

$\mathrm{Max}(S,G) = \tau^{-1}((2g-2)\mathrm{rk}_G)$

several components
nontrivial topology

[Goldman, Toledo, Hernandez]
[Burger-Iozzi-W] [Bradlow-GarciaPrada-Gothen]

connection to total positivity,
cluster algebras, physics …
[Gaiotto-Moore-Neitzke]

??? $\longrightarrow$!!?

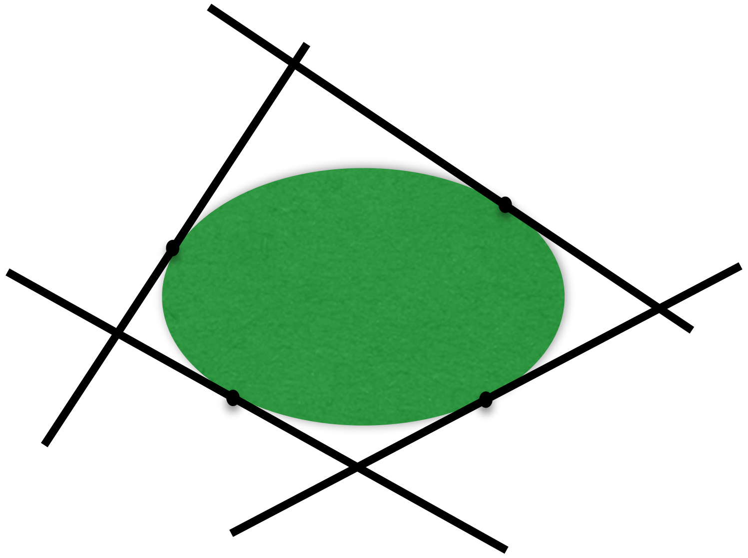
Total positivity and Hitchin components

An invertible matrix is totally positive if every minor is positive.
A totally positive lower triangular unipotent matrix is one, where every possibly nonzero minor is positive.

Triple of flags $F_0, F, F_\infty \in \mathscr{F}(\mathbb{R}^n)$ is positive if

$$F = u_F \cdot F_0$$

for a totally positive unipotent matrix u_F



A representation $\rho : \pi_1(S) \to \mathrm{SL}(n, \mathbb{R})$ is Hitchin iff there exists a positive ρ-equivariant boundary map $\xi : \mathbb{RP}^1 \to \mathscr{F}(\mathbb{R}^n)$ [Fock-Goncharov, Labourie, Guichard]

General characterization in terms of Lusztig total positivity. [Fock-Goncharov]
Fock-Goncharov cluster coordinates associated to ideal triangulation.

Maximal representations

Associated to $\rho : \pi_1(S) \to \mathrm{Sp}(2n, \mathbb{R})$ there is an invariant

$$\begin{array}{ccc} & & \mathbb{Z} \\ & & \downarrow \\ & & \widetilde{\mathrm{Sp}(2n,\mathbb{R})} \\ & \overset{\tilde{\rho}}{\nearrow} & \downarrow \\ \rho : \pi_1(S) & \to & \mathrm{Sp}(2n,\mathbb{R}) \\ & & \downarrow \\ & & 1 \end{array}$$

$$\tau(\rho) = \Pi_{i=1}^{g}[\tilde{\rho}(a_i), \tilde{\rho}(b_i)] \in \mathbb{Z} \cap [(2-2g)n, (2g-2)n]$$

$$\pi_1(S) = < a_1, b_1, \cdots a_g, b_g \mid \Pi_{i=1}^{g}[a_i, b_i] = 1 >$$

Maximal representations $\mathrm{Max}(S, \mathrm{Sp}(2n, \mathbb{R})) := \tau^{-1}((2g-2)n)$

1) $\mathrm{Max}(S, \mathrm{Sp}(2,\mathbb{R})) = \mathrm{Hyp}(S)$ [Goldman]

2) Maximal representations are discrete and faithful [Burger-Iozzi-W]

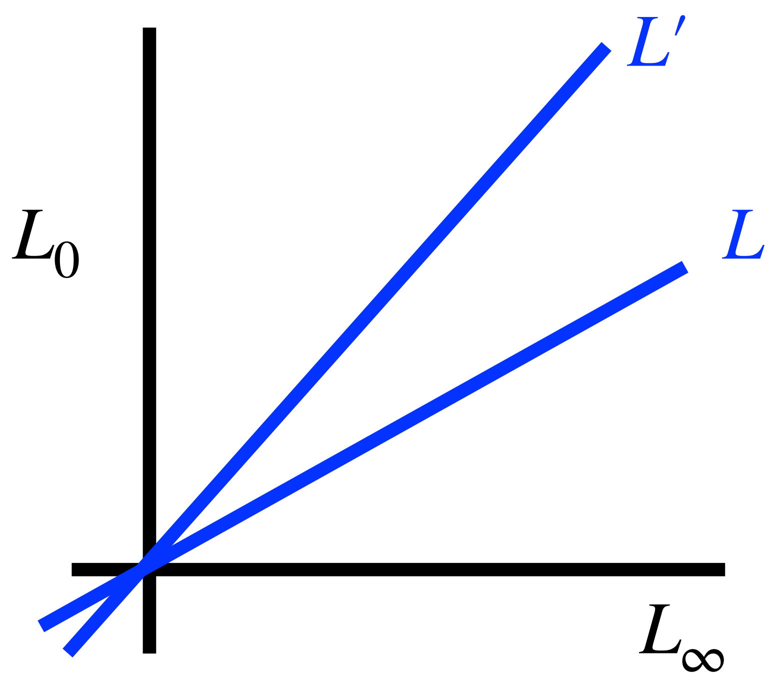
3) For $\mathrm{Sp}(4,\mathbb{R})$ there are components where every representation is Zariski-dense. [Gothen,Guichard-W,Bradlow-GarciaPrada-Gothen]

Maslov index

Space of Lagrangians: $\mathrm{L}(\mathbb{R}^{2n}) = \{L < \mathbb{R}^{2n} \mid \dim L = n, \omega|_{L\times L} = 0\}$

$L_0, L_\infty \in \mathrm{L}(\mathbb{R}^{2n}),\ L_0 \oplus L_\infty = \mathbb{R}^{2n}$

Any L is the graph of a map $M_L : L_0 \to L_\infty$, set $\alpha_L(v) := \omega(v, M_l(v))$



Maslov index $\mu(L_0, L, L_\infty) = \mathrm{sign}(\alpha_L)$

Triple L_0, L, L_∞ is positive if $\mu(L_0, L, L_\infty) = n$

Crossratio $[L_0, L, L_\infty, L'] = \mathrm{ev}(M_{L'}^{-1} \circ M)$

A representation $\rho : \pi_1(S) \to \mathrm{Sp}(2n, \mathbb{R})$ is maximal iff there is a positive ρ-equivariant boundary map $\xi : \mathbb{RP}^1 \to \mathrm{L}(\mathbb{R}^{2n})$, i.e. for all (x, y, z) in $\mathbb{RP}^1$ positively oriented, $\mu(\xi(x), \xi(y), \xi(z)) = n$

[Burger-Iozzi-W, Burger-Iozzi-Labourie-W]

Unifying picture

G simple non-compact real Lie group, Δ set of simple roots, $\Theta \subset \Delta$

P_Θ parabolic subgroup, L_Θ its Levi subgroup, U_Θ its unipotent radical

Decompose $\mathfrak{u}_\Theta$ as a L_Θ° representation, simple pieces $\mathfrak{u}_\alpha, \alpha \in \Theta$

G has Θ-positive structure if $\exists\, L_\Theta^\circ$-invariant convex cone $c_\alpha \subset \mathfrak{u}_\alpha, \forall \alpha \in \Theta$

There are four families of G admitting a Θ-positive structure:

1) G split real Lie group, $\Theta = \Delta$, P_Δ minimal parabolic
2) G Hermitian Lie group of tube type, $\Theta = \{\alpha_\Theta\}$, P_{α_Θ} maximal parabolic
3) $G = \mathrm{SO}(p,q), p < q$, $\Theta = \Delta - \{\alpha_p\}$, P_Θ stabilizer of $(F_1, \cdots, F_{p-1})$ partial isotropic flag
4) $G = F_4, E_6, E_7, E_8$, $\Theta = \{\alpha_1, \alpha_2\}$

[Guichard-W]

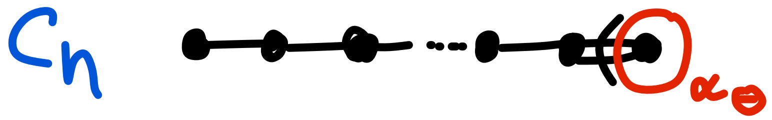
The non-negative semigroup $U_\Theta^{\geq 0}$ is generated by $\exp(c_\alpha), \alpha \in \Theta$

The Θ-Weyl group

1) G split real Lie group, P_Δ minimal parabolic $\color{blue}{W(\Theta) = W}$

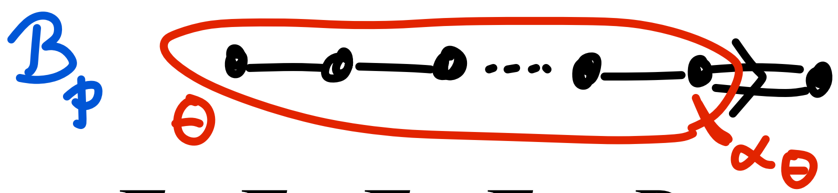
non-commutative Sp_2-theory

2) G Hermitian of tube type, P_{α_Θ} maximal parabolic $\color{blue}{W(\Theta) = W_{A_1}}$

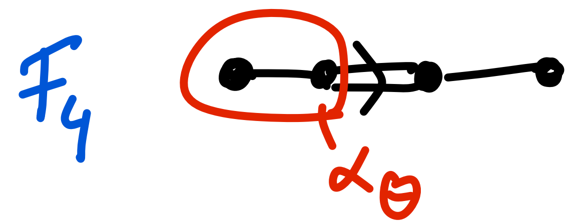


3) $G = \mathrm{SO}(p,q), p < q$, $P_{\Delta - \{\alpha_p\}}$ stabilizer of $(F_1, \cdots, F_{p-1})$ $\color{blue}{W(\Theta) = W_{B_{p-1}}}$

non-commutative B_n-theory



4) $G = F_4, E_6, E_7, E_8$, $P_{\{\alpha_1, \alpha_2\}}$ $\color{blue}{W(\Theta) = W_{G_2}}$



non-commutative G_2-theory

We have $\mathfrak{u}_\alpha = \mathfrak{g}_\alpha$ if $\alpha \neq \alpha_\Theta$, and $\dim \mathfrak{u}_{\alpha_\Theta} > 1$.

There is a subgroup $W(\Theta) < W$ of the Weyl group that governs the combinatorics and gives a parametrization of the positive semigroup $U_\Theta^{>0}$

[Guichard-W]

"Almost everything that works for total positivity, works for Θ-positivity with W replaced by $W(\Theta)$"

Higher Teichmüller spaces

A representation $\rho : \pi_1(S) \to G$ is Θ-positive if there is a ρ-equivariant positive boundary map $\xi : \mathbb{RP}^1 \to G/P_\Theta$

A Θ-positive representation $\rho : \pi_1(S) \to G$ is discrete and faithful.
(In fact it is Θ-Anosov)
The set of Θ-positive representations is open in $\mathrm{Hom}(\pi_1(S), G)/G$.

[Guichard-Labourie-W]

Conjecture:

1) Higher Teichmüller spaces exist iff G admits a Θ-positive structure.
2) Additional connected components in $\mathrm{Hom}(\pi_1(S), G)/G$ exist iff G admits a Θ-positive structure.

[Guichard-Labourie-W]

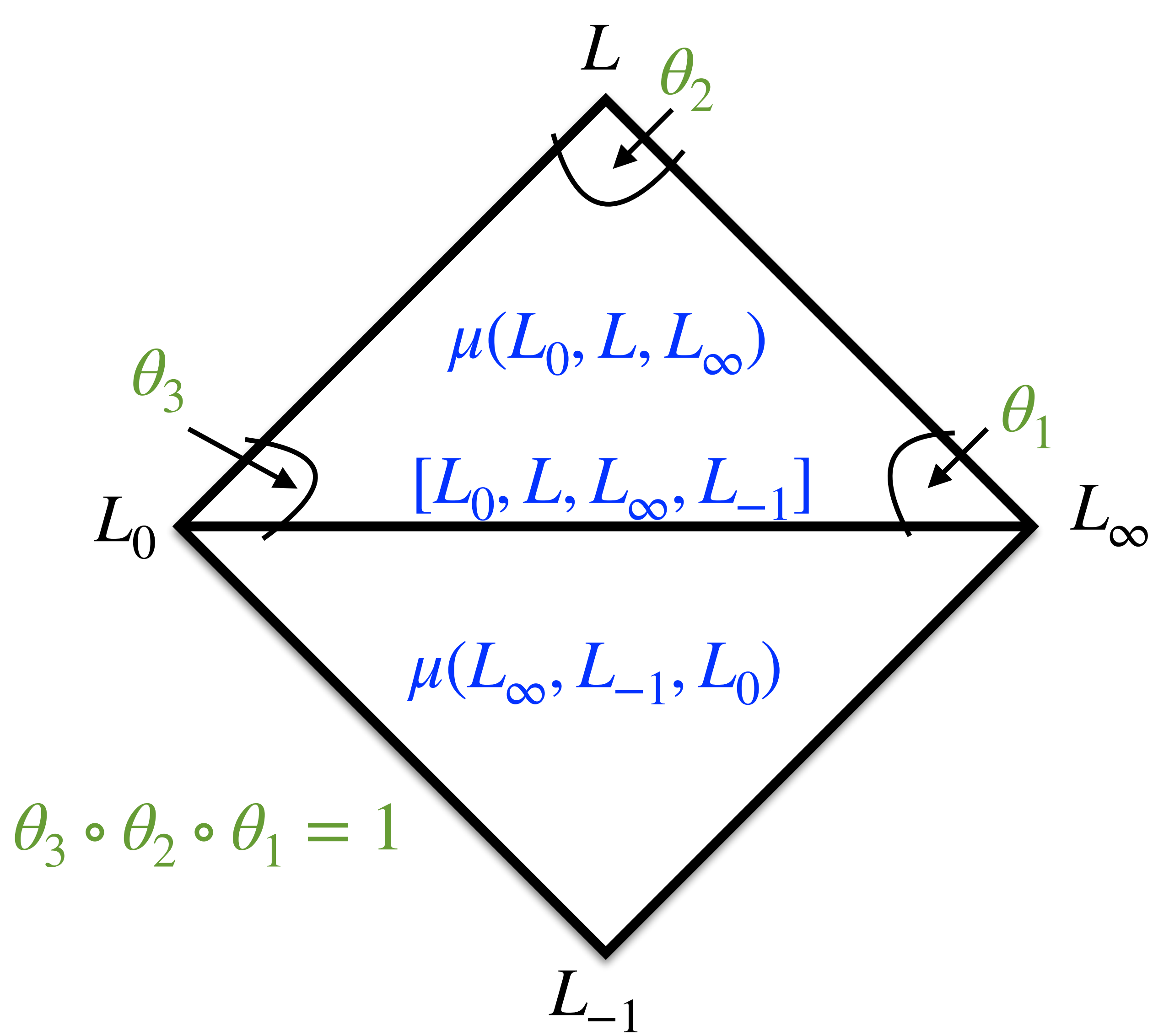
Additional components confirmed for G admitting a Θ-positive structure by Higgs bundle methods
[Aparicio-Arroyo-Bradlow-Collier-García-Prada-Gothen-Oliveira]

Non-commutative coordinates

S = S(g,n) surface with punctures — pick ideal triangulation

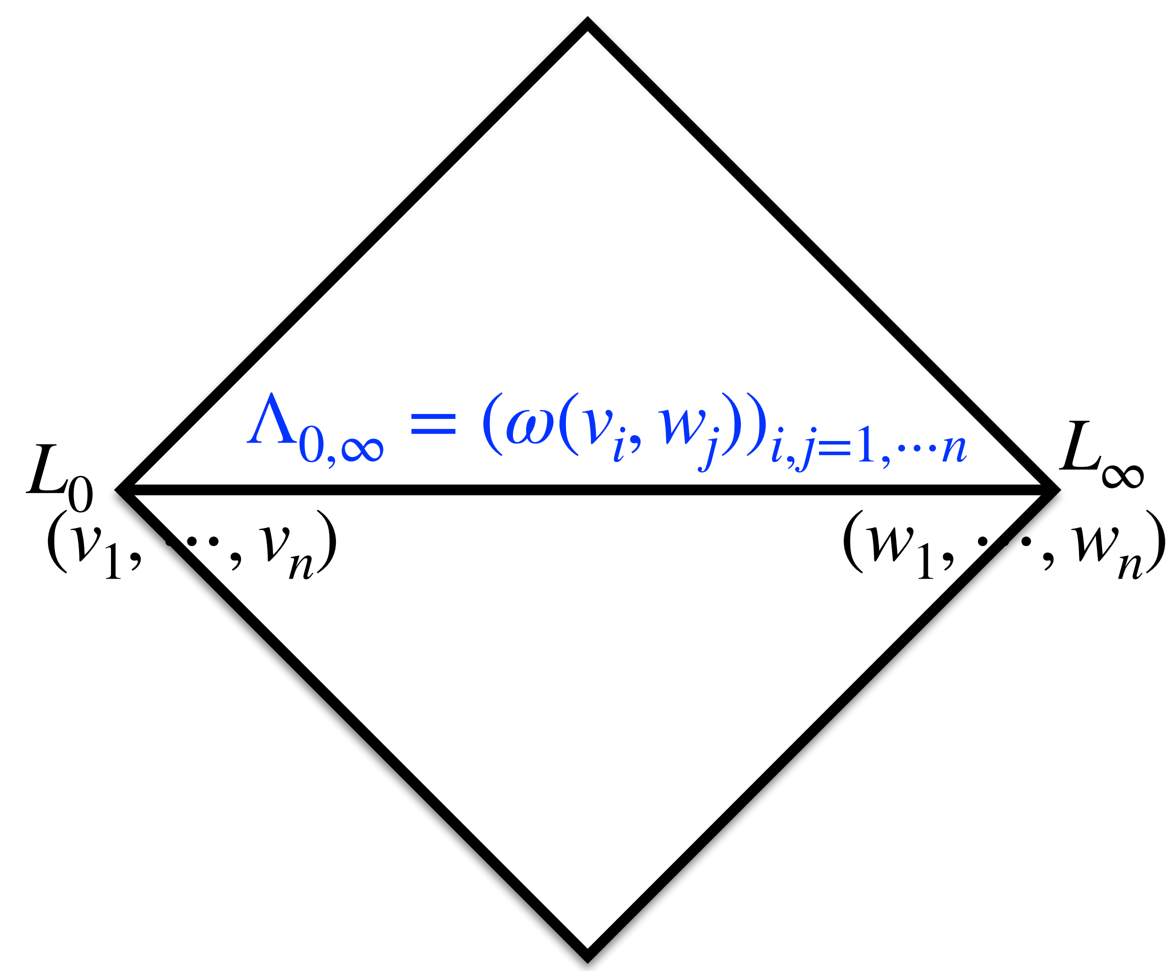
$\mathscr{X}$-coordinates

framing by $\mathrm{Lag}(\mathbb{R}^{2n})$



$\mathscr{A}$-coordinates

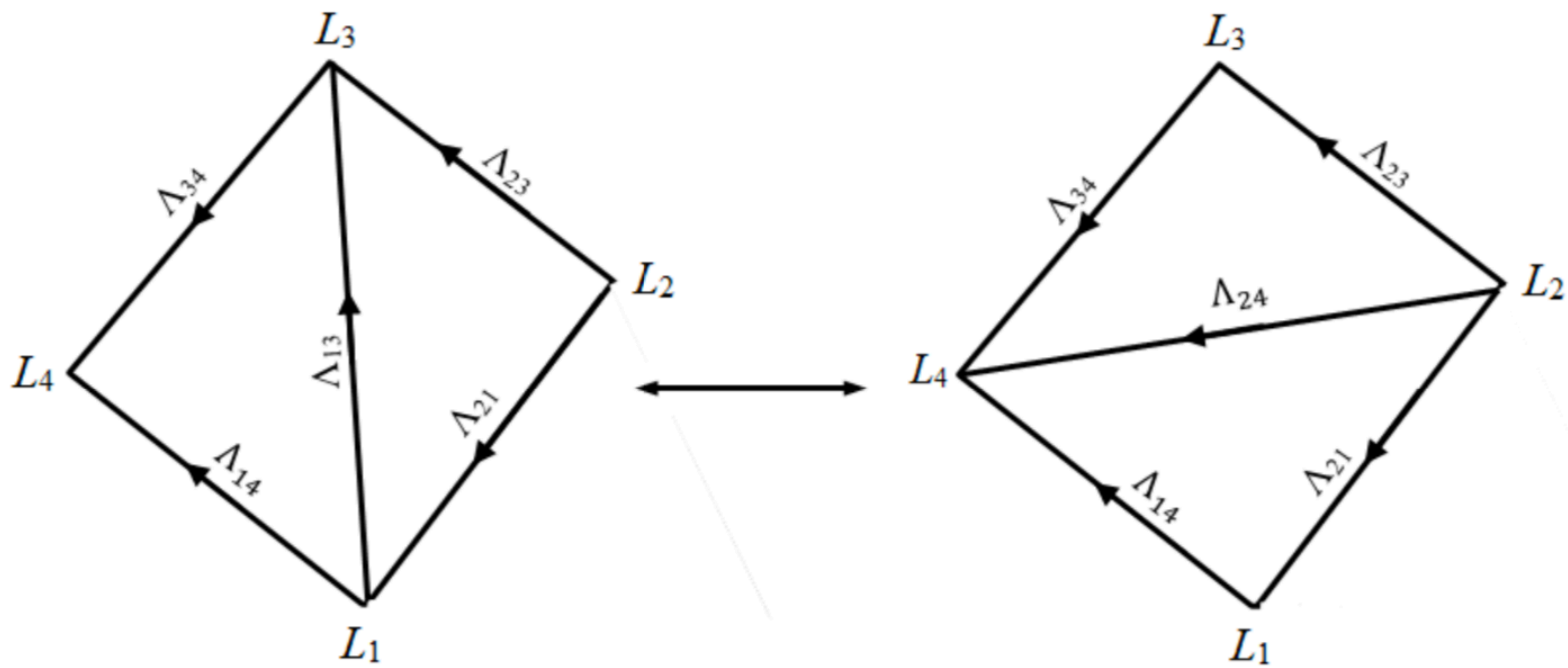
framing by isotropic bases $(v_1, \cdots, v_n)$



[Alessandrini-Guichard-Rogozinnikov-W]

The $\mathscr{A}$-flip

$\mathscr{A}$-coordinates: $\Lambda_{ij} \in \mathrm{GL}(n, \mathbb{R}),\ \Lambda_{ji} = -\Lambda_{ij}^T$



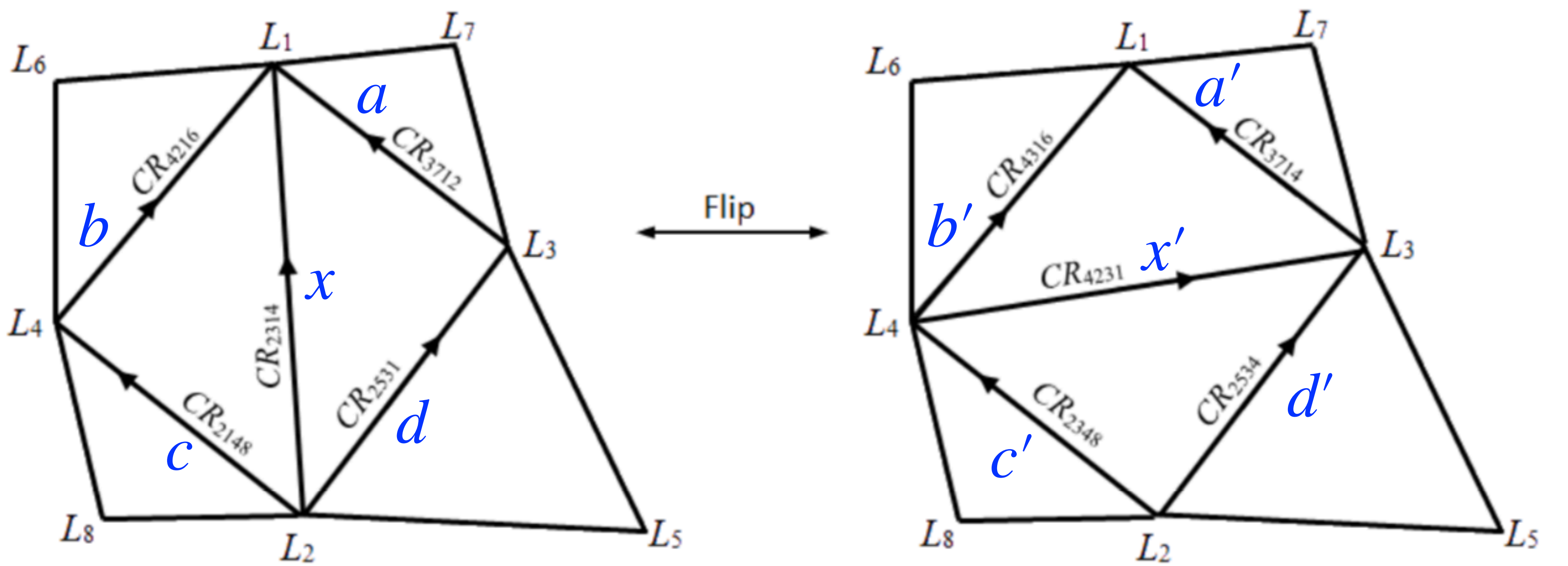
Ptolemy equation $\Lambda_{24} = \Lambda_{23}\Lambda_{13}^{-1}\Lambda_{14} + \Lambda_{21}\Lambda_{31}^{-1}\Lambda_{34}$

Triangle equation

$\Lambda_{32}^{-1}\Lambda_{31}\Lambda_{21}^{-1}\Lambda_{23}\Lambda_{13}^{-1}\Lambda_{12} = -1 \qquad \Lambda_{23}\Lambda_{13}^{-1}\Lambda_{12} + \Lambda_{21}\Lambda_{31}^{-1}\Lambda_{32} = 0$

Realization of Berenstein-Retakh non-commutative cluster algebra

The $\mathscr{X}$-flip



$$[L_1, L_2, L_3, L_4] = \mathrm{ev}(-CR_{1234}),\ CR_{1234} = -\Lambda_{41}^{-1}\Lambda_{43}\Lambda_{23}^{-1}\Lambda_{21}$$

$$CR_{4231} = \Lambda_{24}^{-1} \cdot CR_{2314}^{-T} \cdot \Lambda_{24} \qquad x' = x^{-1}$$

$$CR_{2534} = (\mathrm{Id} + CR_{2314})CR_{2531} \qquad d' = d(1+x)$$

$$CR_{4316} = CR_{4216}(\mathrm{Id} + CR_{4231}^{-1}) \qquad b' = b(1+x^{-1})$$

$$CR_{2348} = CR_{2148}(\mathrm{Id} + CR_{2314}^{-1})^{-1} \qquad c' = c(1+x^{-1})^{-1}$$

$$CR_{3714} = (\mathrm{Id} + CR_{3142})^{-1}CR_{3712} \qquad a' = a(1+x)^{-1}$$

Upshot: This is a noncommutative ~~SL(2)~~ Sp(2)-Theory

Symplectic groups over non-commutative rings

A non-commutative associative algebra, $\sigma : A \to A$ involution

$$\omega : A^2 \times A^2 \to A\ ,\ \ \omega(v,w) := \sigma(v)^T \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} w$$

$$\mathrm{Sp}_2(A,\sigma,\omega) := \{g \in \mathrm{GL}_2(A) \mid \omega(g\cdot v, g\cdot w) = \omega(v,w)\ \forall v,w \in A^2\}$$

Example: $\mathrm{Sp}(2n,\mathbb{R}) = \mathrm{Sp}_2(A,\sigma,\omega)$ with $A = \mathrm{Mat}(n,\mathbb{R}), \sigma(X) = X^T$

Symplectic group is a Sp_2 over a non-commutative ring

[Berentstein-Retakh-Rogozinnikov-W]

$A_{sym} = \{X \in A \mid \sigma(X) = X\}$, $A^+_{sym} < A_{sym}$ open cone

$\mathscr{H} := \{Z \in A^{\mathbb{C}}_{sym} \mid \mathrm{Im(Z)} \in A^+_{sym}\}$ hyperbolic plane over A.

$\mathrm{Sp}_2(A,\sigma,\omega)$ acts on $\mathscr{H}_n$ by fractional linear transformations

New model for the symmetric space associated to $\mathrm{Sp}(2n,\mathbb{C})$

$$\mathscr{H}_{\mathbb{C}} := \{\mathrm{Z}_1 + \mathrm{Z}_2 J \mid \mathrm{Z}_1 \in \mathrm{Sym}_n(\mathbb{C}), \mathrm{Z}_2 \in \mathrm{Herm}^+_n(\mathbb{C})\} \subset \mathrm{Mat}_n(\mathbb{H})$$

Conjectures and Questions

1) There are (partially) non-commutative cluster algebras for B_n and G_2 (arising geometrically $\mathscr{A}$-coordinates for Θ-positive representations).

2) Are there non-commutative Hitchin fibrations ?

3) Are there quantum groups and canonical bases adapted to Θ-positivity ?

4) There is a theory of generalized opers. [Collier-Sanders]

5) What is the connections to physics ? [Gaiotto-Moore-Neitzke]