Embeddings of manifolds in Euclidean space and Feynman diagrams

Thomas Willwacher

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Goal

Let M, N be smooth manifolds of dimension m, n. Long standing problems: Understand

- Embedding spaces (knot spaces)

$$\mathrm{Emb}(M,N) = \{f : M \to N \mid f \text{ smooth embedding}\} \subset C^\infty(M,N)$$

- Diffeomorphism groups $\mathrm{Diff}(M)$

with the C^∞ topology.

Goal

Concrete questions:

- $\pi_0(\mathrm{Emb}(M, N)) =?$, i.e., classify embeddings modulo isotopy. (knot theory)
- Higher $\pi_k(-) =?$
- Simpler question: Rational homotopy groups $\pi_k(-) \otimes \mathbb{Q} =?$ for $k \geq 2$, or rational homotopy type.
  Hope: Possible for wide class of manifolds in a few years.

Rational homotopy theory I

- One has a (Quillen) equivalence

  $$|-| : \mathrm{sSet} \rightleftarrows \mathrm{Top} : S$$

  and a (Quillen) adjunction

  $$\Omega : \mathrm{sSet} \rightleftarrows \mathrm{dgca}^{op} : G$$

  between the categories of topological spaces, simplicial sets and differential graded commutative algebras $/\mathbb{Q}$.
- In particular, for X a topological space the differential graded commutative algebra $\Omega(X)$ are the (PL) differential forms on X.

Rational homotopy theory II

- Let X, Y be (simply connected) spaces. A map $f : X \to Y$ is
  - a weak homotopy equivalence if f induces bijections $\pi_k(X) \to \pi_k(Y)$.
  - a rational (homotopy) equivalence if f induces bijections $\pi_k^{\mathbb{Q}}(X) \to \pi_k^{\mathbb{Q}}(Y)$, with

$$\pi_k^{\mathbb{Q}}(X) := \pi_k(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

- A *model* for X is a dg comm. alg. A that is connected to $\Omega(X)$ via a chain of quasi-isomorphisms.

$$A \xrightarrow{\sim} \cdots \xleftarrow{\sim} \Omega(X).$$

  (Quasi-isomorphism:=dg comm. alg. morphism inducing isom. on cohomology)
- For good X (e.g. simply connected) one can recover $\pi_k^{\mathbb{Q}}(X)$ from a model for X, and X, Y are rationally equivalent iff they have quasi-isomorphic models.

Mapping spaces

- For X, Y topological spaces (simplicial sets) we may consider the mapping space $\mathrm{Map}(X, Y) = \{f : X \to Y \mid f \text{ continuos}\}$.
- Since dgca is a model category, we may also define the (derived) mapping space $\mathrm{Map}(A, B)$ for $A, B \in \mathrm{dgca}$.
- By functoriality we have a map

$$\mathrm{Map}(X, Y) \to \mathrm{Map}(\Omega(Y), \Omega(X)).$$

- (Haefliger, Sullivan '80) For good X, Y the above map induces componentwise rational homotopy equivalences.

$$\mathrm{Map}(X, Y)_f \xrightarrow{\sim_{\mathbb{Q}}} \mathrm{Map}(\Omega(Y), \Omega(X))_{\Omega(f)},$$

and a finite-to-one map on π_0.

Mapping spaces – Example

- $\mathrm{Map}(S^1, S^1) \simeq \mathbb{Z} \times S^1$.
- Model for S^1:
$$A = \mathbb{Q}[\omega] \xrightarrow{\sim} \Omega(S^1),$$
with ω a variable of degree 1.
- $$\mathrm{Map}(A, A) \simeq |\mathrm{Hom}(A, A \otimes \Omega(\Delta^\bullet))|$$
- Any dgca morphism $A \to B$ is determined by image of ω, hence one can show
$$\pi_k(\mathrm{Map}(A, A)) = \begin{cases} \mathbb{Q} & k = 0 \\ \mathbb{Q} & k = 1 \\ * & k \geq 2. \end{cases}$$

Towards $\mathrm{Emb}(M, N)$

- We "can understand" mapping spaces.
- We would like to see $\mathrm{Emb}(M, N)$ as an upgraded version of $\mathrm{Map}(M, N)$.

Goodwillie-Weiss embedding calculus

- Let

$$\mathrm{conf}_M^{m-fr}(r) = \{(x_1, F_1, \dots, x_r, F_r) \mid x_j \in M, x_i \neq x_j \text{ for } i \neq j\}$$

  the space of configurations of r points on M, with an m-frame F_j in the tangent space at x_j.
- Any embedding $f : M \to N$ induces maps $\mathrm{conf}_M^{m-fr}(r) \to \mathrm{conf}_N^{m-fr}(r)$.
- Main idea: Study $\mathrm{Emb}(M, N)$ via

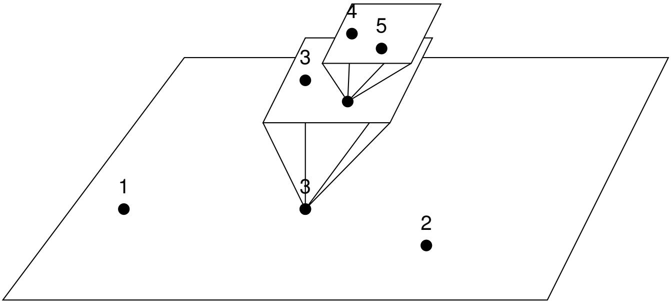
$$\mathrm{Emb}(M, N) \to \{\mathrm{Map}(\mathrm{conf}_M^{m-fr}(r), \mathrm{conf}_N^{m-fr}(r))\}_{r \geq 1}.$$

- Problem: ...still need coherences between the various r and the points.

Fulton-MacPherson operad and action

- The framed Fulton-MacPherson–Axelrod-Singer operad FM_m^{fr} is a compactification

$$\mathrm{conf}_{\mathbb{R}^m}^{m-fr} \to \mathsf{FM}_m^{fr}(r).$$



- Gluing yields operations

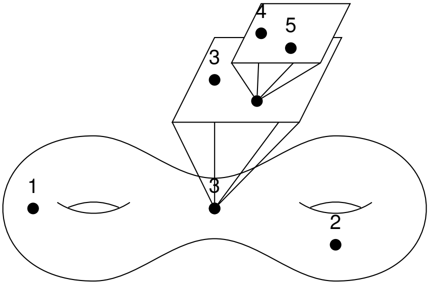
$$\mathsf{FM}_m^{fr}(r) \times \mathsf{FM}_m^{fr}(s) \to \mathsf{FM}_m^{fr}(r+s-1)$$

that assemble into an operad structure. (FM_m^{fr} is equivalent to the framed little m-disks operad.)

Fulton-MacPherson operad and action

- Similarly one has a compactification

$$\mathrm{conf}_{\mathbb{R}^m}^{m-fr} \to \mathsf{FM}_M^{fr}(r) = \mathsf{FM}_M^{m-fr}(r).$$



- Gluing produces right actions

$$\mathsf{FM}_M^{fr}(r) \times \mathsf{FM}_m^{fr}(s) \to \mathsf{FM}_M^{fr}(r+s-1)$$

that assemble into a right operadic FM_m^{fr}-module structure on FM_M^{fr}.

Goodwillie-Weiss embedding calculus III

For an embedding $f : M \to N$ the induced map $\mathsf{FM}_M^{fr} \to \mathsf{FM}_N^{m-fr}$ is compatible with the right FM_m^{fr}-actions.

Theorem (Goodwillie, Weiss, Klein, Boavida)

The map

$$\mathrm{Emb}(M, N) \to \mathrm{Map}_{\mathsf{FM}_m^{fr}-mod}(\mathsf{FM}_M^{fr}, \mathsf{FM}_N^{m-fr})$$

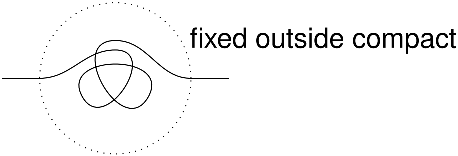
is a weak homotopy equivalence if $dim N - dim M \geq 3$.

Here $\mathrm{Map}_{\mathsf{FM}_m^{fr}-mod}(-)$ is the derived mapping space in the model category of right operadic FM_m^{fr}-modules.

Our results

To state our results, we will slightly adjust the setting as follows:

- Specialize to $N = \mathbb{R}^n$.
- $M \subset \mathbb{R}^m$ is the complement of a compact submanifold (with boundary), i.e., open, extending to ∞. $\Rightarrow$ incorporate mixed dimensions by taking tubular neighborhood, and ”long” objects.
- Consider $\mathrm{Emb}_\partial(M, \mathbb{R}^n) \subset \mathrm{Emb}(M, \mathbb{R}^n)$ be the embeddings that agree with the given embedding outside a compact.



Our results

- Let $\mathrm{Imm}_\partial(M, \mathbb{R}^n)$ be the immersions (supported on a compact) and consider the homotopy fiber:

$$\overline{\mathrm{Emb}}_\partial(M, \mathbb{R}^n) = \mathrm{hofiber}(\mathrm{Emb}_\partial(M, \mathbb{R}^n) \to \mathrm{Imm}_\partial(M, \mathbb{R}^n))$$

- Advantage 1: For $M = \mathrm{Tube}(M')$ (tubular neighborhood), $\overline{\mathrm{Emb}}(M, \mathbb{R}^n) = \overline{\mathrm{Emb}}(M', \mathbb{R}^n)$.
- Advantage 2: Since M, $\mathbb{R}^n$ are framed, we can consider the non-framed analogs $\mathrm{FM}_M \subset \mathrm{FM}_M^{fr}, \mathrm{FM}_n \subset \mathrm{FM}_n^{fr}$.

Our results II

Theorem (Fresse-Turchin-W., arxiv:2008.08146)

For $M \subset \mathbb{R}^m$ as above, $n \geq m+3$:

$$\overline{\mathrm{Emb}}_\partial(M, \mathbb{R}^n) \xrightarrow[(1)]{\simeq} \mathrm{Map}_{\mathsf{FM}_m-bmod}(\mathsf{FM}_M, \mathsf{FM}_n)$$

$$\xrightarrow[(2)]{\simeq_{\mathbb{Q}}} \mathrm{Map}_{\Omega(\mathsf{FM}_m)-bmod}(\Omega(\mathsf{FM}_n), \Omega(\mathsf{FM}_M)) \xrightarrow[(3)]{\simeq} |\mathsf{MC}_\bullet(\mathsf{HGC}_{\bar{A},n})|,$$

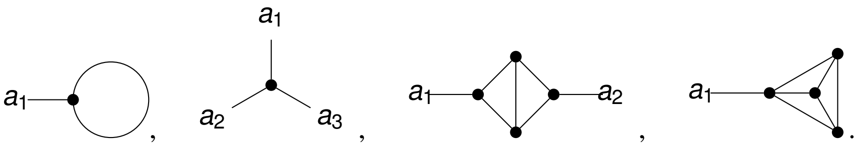
with A a model for $M \cup \{\infty\}$ and $\bar{A} \subset A$ the augmentation ideal.

- (1) is due to Goodwillie, Weiss, Klein, Arone, Turchin
- (2) is an operadic extension of Haefliger's result (Fresse, Mienné), and a rational weak equivalence componentwise, finite-to-one on π_0.
- (3) is our main result and holds if $n - m \geq 2$ ... with the right-hand side to be defined...

Hairy graph L_∞-algebra $\mathrm{HGC}_{A,n}$

For A a dgca, n an integer:

$\mathrm{HGC}_{A,n} = \mathit{span}\{$isom. classes of admissible A-decorated hairy graphs$\}$

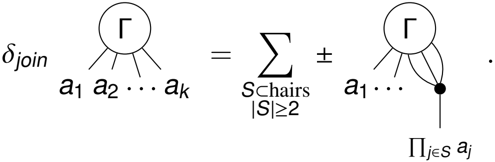


- Vertices have degree $-n$, edges $n-1$, $a_j \in A$ carry their (homological, non-positive) degree.
- *Admissible:* (i) Valence of vertices ≥ 3. (ii) No odd symmetries.
- Carries natural homotopy Lie- (L_∞-)algebra structure.

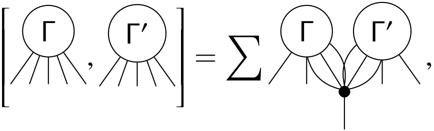
L_∞-algebra structure

- Differential $\delta = d_A + \delta_{split} + \delta_{join}$,

$$\delta_{split}\Gamma = \sum_{v \text{ vertex}} \pm\Gamma \text{ split } v$$



- Lie bracket:



- Higher L_∞-operations $[-, \dots, -]$ are defined similarly.

Interlude: dg Lie algebras and Maurer-Cartan spaces

- Maurer-Cartan elements of L_∞-algebra $\mathfrak{g}$

$$\mathrm{MC}(\mathfrak{g}) = \left\{ x \in \mathfrak{g}_0 \mid dx + \frac{1}{2}[x,x] + \frac{1}{3!}[x,x,x] + \cdots = 0 \right\}$$

- Maurer-Cartan space (simplicial set)

$$\mathrm{MC}_\bullet(\mathfrak{g}) = \mathrm{MC}(\mathfrak{g} \hat{\otimes} \Omega_{poly}(\Delta^\bullet))$$

- For $x \in \mathrm{MC}(\mathfrak{g})$ we can consider the twisted L_∞-algebra with operations

$$[a_1, \dots, a_r]^x = \sum_{k \geq 0} \frac{1}{k!} [\underbrace{x, \dots, x}_{k \times}, a_1, \dots, a_r]$$

Interlude: Two important results about MC spaces

Theorem (Berglund)

$\mathfrak{g}$: *pro-nilpotent L_∞-algebra, $x \in \mathrm{MC}(\mathfrak{g})$. Then for $k \geq 1$*

$$\pi_k(\mathrm{MC}_\bullet(\mathfrak{g}), x) \cong H_k(\mathfrak{g}^x),$$

where for $k = 1$ the rhs. is equipped with the PBW group structure.

Remark: Furthermore, in good cases that cohomology $H(\mathrm{MC}_\bullet(\mathfrak{g})_x)$ is computed by the Chevalley complex of the truncation

$$H(\mathrm{MC}_\bullet(\mathfrak{g})_x) \cong H(C(\mathfrak{g}^x_{trunc})).$$

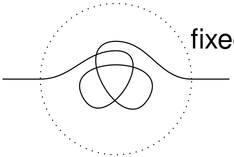
Consequences

Recall: $\overline{\mathrm{Emb}}_\partial(M, \mathbb{R}^n) \simeq_\mathbb{Q} |\mathsf{MC}_\bullet(\mathsf{HGC}_{\bar{A},n})|$ for A a model for $M \cup \{*\}$.
Consequences:

- For $n \geq m + 3$, $k \geq 1$ one can compute $\pi_k^\mathbb{Q}\overline{\mathrm{Emb}}(M, \mathbb{R}^n)$ in terms of diagrams.
- One has characterization of $\pi_0\overline{\mathrm{Emb}}(M, \mathbb{R}^n)$ (invariants) extending classical Vassiliev invariants, that are complete up to finite ambiguity if $n \geq m + 3$.

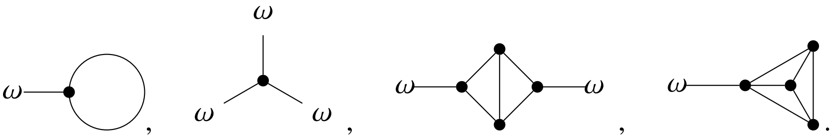
Example I - long knots

$M = \mathbb{R}^m$, then $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)$ is the space of m-dimensional long knots in $\mathbb{R}^n$.



- Then $\mathbb{R}^m \cup \{\infty\} \cong S^m$.
- We can take $A = \mathbb{Q}[\omega]$ with ω of degree m, $\omega^2 = 0$.
- $\bar{A}$ is one-dimensional, spanned by ω, hence all hairs carry same decoration.

Example I - long knots, ctd.

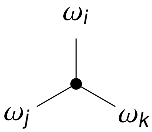


- All L_∞-operations joining hairs are 0 $\Rightarrow$ abelian L_∞-algebra.
- $\Rightarrow \pi_0(\mathrm{MC}_\bullet(\mathrm{HGC}_{\bar{A},n})) \cong H_0(\mathrm{HGC}_{\bar{A},n})$.
- In the case $n = 3$, $m = 1$ one recovers the diagrams enumerating Vassiliev knot inveriants, i.e., uni-trivalent diagrams modulo IHX.

Example II - string links

$M = \mathrm{Tube}(\mathbb{R}^{m_1} \sqcup \mathbb{R}^{m_2} \sqcup \cdots \sqcup \mathbb{R}^{m_r}) \subset \mathbb{R}^m$ (tubular neighborhod), then $\overline{\mathrm{Emb}_\partial}(M, \mathbb{R}^n)$ is (essentially) the space of string links with k components of dimensions $m_1, \ldots, m_r$. (see ”board”)

- $M \cup \{\infty\} \cong S^{m_1} \vee \cdots \vee S^{m_r}$ wedge product of spheres.
- $A = \mathbb{Q}[\omega_1, \ldots, \omega_r]/\langle \omega_i \omega_j = 0 \rangle$, and $\bar{A} = \mathrm{span}(\omega_1, \ldots, \omega_r)$.
- $\mathrm{HGC}_{\bar{A},n}$ is given by hairy graphs with hairs of r ”colors”, and is still abelian.



- Recovers results of Turchin-Tsopméné

General remark

- Let $M' \subset \mathbb{R}^m$ be an m'-dimensional submanifold.
- Then we can consider a tubular neighborhood $M = \mathrm{Tube}(M') \subset \mathbb{R}^m$, an open submanifold.
- The embedding space $\mathrm{Emb}(M, \mathbb{R}^n)$ is identified with the space of framed embeddings of M' in $\mathbb{R}^m$.
- However, $\overline{\mathrm{Emb}}(M, \mathbb{R}^n) \cong \overline{\mathrm{Emb}}(M', \mathbb{R}^n)$, that is, the homotopy fiber "eats" the framing.
- $\Rightarrow$ our result applies to embeddings of general compact submanifolds $M' \subset \mathbb{R}^m$, with the choice $M = \mathrm{Tube}(M' \sqcup \{\infty\})$, then $\overline{\mathrm{Emb}}_\partial(M, \mathbb{R}^n) \cong \overline{\mathrm{Emb}}(M', \mathbb{R}^n)$

MC elements and trees

- Let m' the cohomological dimension of M, i.e., the smallest number such that $H^k(M) = 0$ for all $k \leq m'$, and suppose $m' \leq n - 3$.
- Degree counting $\Rightarrow$ all elements of $\mathrm{HGC}_{\bar{H}(M),n}$ of non-positive degree are trees.
- Hence $\pi_0 \mathrm{MC}_\bullet(\mathrm{HGC}_{\bar{H}(M),n})$ is determined by tree diagrams.

Example - Unknotting

Classical unknotting theorem:

Theorem (Whitney-Wu)

M' compact k-connected of dimension m', $n \geq 2m' + 1$, then $\pi_0(\mathrm{Emb}(M', \mathbb{R}^n)) = *$.

We can "see" this on graphs: In case $n > 2m' + 1$ there are no graphs of degree ≤ 0, hence we can conclude from our result that $\pi_0(\overline{\mathrm{Emb}}(M', \mathbb{R}^n))$ is finite.

Example - Nonlinear MC equation

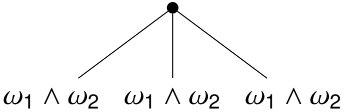
$M' = S^2 \times S^2 \subset \mathbb{R}^5$, $n = 7$.

- $A = \mathbb{Q}[\omega_1, \omega_2]/\langle \omega_1^2 = \omega_2^2 = 0 \rangle$, with $|\omega_1| = |\omega_2| = 2$.
- List of (relevant) degree 0 graphs:

$$L_1 = \omega_1 \text{ ------ } \omega_1 \wedge \omega_2 \; ; \qquad L_2 = \omega_2 \text{ ------ } \omega_1 \wedge \omega_2 \; .$$

- MC equation for $x = \lambda_1 L_1 + \lambda_2 L_2$:

$$0 = \frac{1}{2}[x, x] = \lambda_1 \lambda_2$$



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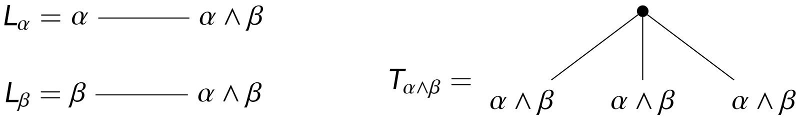
- Thus

$$\pi_o(\mathrm{MC}_\bullet(\mathrm{HGC}_{\bar{A},n})) = \{\lambda_1 L_1 + \lambda_2 L_2 \,|\, \lambda_1 = 0 \text{ or } \lambda_2 = 0\}.$$

Example - with gauge transformations

We consider $M' = S^1 \times S^2$, $n = 6$.

- $A = H^*(S^1 \times S^2) = \mathbb{Q}[\alpha, \beta] / \langle \beta^2 = 0 \rangle$, with $|\alpha| = 1$, $|\beta| = 2$.
- Relevant graphs:



  with L_α of degree 1, L_β, $T_{\alpha\wedge\beta}$ of degree -1.
- Any $\lambda L_\beta + \mu T_{\alpha\wedge\beta}$ is an MC element.
- Gauge equivalence $\lambda L_\beta + \mu T_{\alpha\wedge\beta} \sim \lambda L_\beta$ if $\lambda \neq 0$. Hence

$$\pi_0(\mathrm{MC}_\bullet(\mathrm{HGC}_{\bar{A},n})) \cong \{\lambda L_\beta + \mu T_{\alpha\wedge\beta} \,|\, \mu = 0 \text{ or } \lambda = 0\}.$$

Future directions

- Generalize $\mathbb{R}^n$ to N. Probably possible with similar techniques. Expected result: Need to decorate vertices of graphs by $H(N)$, with more complicated L_∞-structure.
- Attack codimension restriction $n - m \geq 3$, and target in particular $\mathrm{Diff}(M)$ ($n = m$): Can likely be done using tricks akin Weiss fiber sequence.

The End

Thanks for listening!