# Lecture 1 — Matrix models, correlators, maps

① Random Hermitian matrix $M$ size $N$ $\mathcal{H}_N$

$U \in U_N$ $M \underset{\mathbb{P}}{\sim} UMU^\dagger$

$$\mathbb{C}[\mathcal{H}_N]^{U_N} = \mathbb{C}[p_1, p_2, \dots] / \text{char poly} = 0 \qquad p_k = \mathrm{Tr}(M^k)$$

$$N \longrightarrow +\infty \qquad = \mathbb{C}[p_1, p_2, p_3, \dots]$$

$$d\mu(M) = \frac{dM}{Z} \boxed{\exp\Big(-\boxed{N}\,\mathrm{Tr}\frac{M^2}{2}} + \boxed{N}\,\mathrm{Tr}[T_{0,1}(M)]\Big)$$

$Z$ partition function $\int_{\mathcal{H}_N} d\mu(M) = 1$

$$T_{0,1}(x) = \sum_{\ell \geq 3} \frac{t_\ell}{\ell} x^\ell$$

Solving the matrix model means being able to compute all moments of any observables $U_N$-invariant

$$\mathbb{E}[p_{L_1}(M), \dots, p_{L_n}(M)] \qquad L_1, \dots, L_n > 0 \qquad \text{moments.}$$

Rather use n-point correlators

$$W_n(x_1, \dots, x_n) = \mathbb{E}_c\Big[\mathrm{Tr}\frac{1}{x_1 - M}, \dots, \mathrm{Tr}\frac{1}{x_n - M}\Big] \qquad n \geq 1$$

cumulant

$$\mathbb{E}_c[A_1, \dots, A_n] = \partial_{s_1=0} \cdots \partial_{s_n=0} \ln \mathbb{E}\Big[\exp\Big(\sum_{i=1}^n s_i A_i\Big)\Big]$$

$n=1$ $\mathbb{E}_c[A] = \mathbb{E}[A]$

$n=2$ $\mathbb{E}_c[A_1, A_2] = \mathbb{E}[A_1 A_2] - \mathbb{E}[A_1]\mathbb{E}[A_2]$

etc ....

Two situations • $x_1, \dots, x_n \in \mathbb{C} - \mathbb{R}$

• $x_i \longrightarrow +\infty$ $\quad \dfrac{1}{x - M} = \displaystyle\sum_{k \geq 0} \frac{M^k}{x^{k+1}}$

$$W_n(x_1, \dots, x_n) = \sum_{\ell_1, \dots, \ell_n > 0} \frac{\mathbb{E}_c[p_{\ell_1}(M), \dots, p_{\ell_n}(M)]}{x_1^{\ell_1+1} \cdots x_n^{\ell_n+1}} + \frac{N\delta_{n,1}}{x_1}$$

Two differents settings

- formal matrix models (combinatorics) $\quad t_\ell$ formal variables

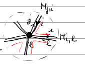
$\neq$
$N \to +\infty$

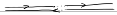
- convergent matrix models (probability). $\quad d\mu$ probability measure $(t_\ell \in \mathbb{R})$
  + assumptions making integral converge.
  asymptotics large $N$. $\quad V(x)$ $x \to \pm\infty$ goes to $-\infty$
  "faster" than $-2\ln|x|$.

## Formal matrix models

Feynman diagram expansion + Wick theorem.

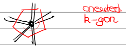
$$\mathrm{Tr}(M^k) = \sum_{i_1, \dots, i_k = 1}^{N}$$



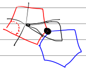


pairing edges
preserving indices.

Maps    building blocks



glue pairs of ~~edges~~
with opposite orientations

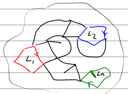


⇝ discretised surface
made out of polygons.

$$W_n(x_1,\dots,x_n) = \sum_{g\geq 0} N^{\boxed{2-2g-n}} \underbrace{W_{g,n}}_{\text{independent of } N}(x_1,\dots,x_n)$$

identity of formal
power series in t's

$$W_{g,n}(x_1,\dots,x_n) = \sum_{\substack{\text{connected maps}\\ \text{of genus } g\\ n \text{ labeled, rooted faces}\\ L_1,\dots,L_n \text{ lengths}}}' \frac{1}{\#\mathrm{Aut}} \cdot \frac{\prod_{k\geq 3} t_k^{\#k\text{-gons}}}{\prod_{i=1}^n x_i^{L_i+1}}$$



closed surface
(all edges paired)

[BIPZ 78]
t'Hooft 74.

Counting powers of $N$ : # faces (not labelled) = $2-2g-n$
− # edges
+ # vertices

"Topological expansion"

## ② Multi-trace matrix models.

Most general $U_N$-invariant measure which has a topological expansion.

$$d\mu(M) = \frac{1}{Z} \exp\Big(-N\,\mathrm{Tr}\,\frac{M^2}{2} + \sum_{h,k\geq 0} \frac{N^{2-\textcircled{2h}-k}}{k!} \mathrm{Tr}\big[T_{h,k}(M^{(1)},\dots,M^{(k)})\big]$$

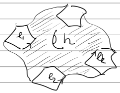
$$T_{h,k}(x_1,\dots,x_k) = \sum_{\ell_1,\dots,\ell_k>0}' \frac{t_{h;\ell_1,\dots,\ell_k}}{\ell_1\cdots\ell_k} x_1^{\ell_1} \cdots x_k^{\ell_k}$$

$$M^{(i)} = \overbrace{1_N \otimes \cdots \otimes \underset{\substack{\uparrow\\ i\text{-th position.}}}{M} \otimes \cdots \otimes 1_N}^{k \text{ tensor factors}}$$

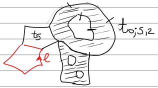
$$\mathrm{Tr}\big[T_{h,k}(M^{(1)},\dots,M^{(k)})\big] = \sum_{\ell_1\dots\ell_k>0}' \frac{t_{h;\ell_1,\dots,\ell_k}}{\ell_1\cdots\ell_k} \mathrm{Tr}(M^{\ell_1})\cdots\mathrm{Tr}(M^{\ell_k})$$

$$\mathrm{Det}(M+c) = \exp(\mathrm{Tr}\ln(M+c)) = \exp\Big(N\ln c + \sum_{\ell>0} \frac{(-1)^{\ell+1}}{\ell} \frac{\mathrm{Tr}\,M^\ell}{c^\ell}\Big)$$

Stuffed maps    building blocks



glue along pairs of
edges with opposite
orientation



$W_1(x)$

Goal    compute $W_{g,n}(x_1,\dots,x_n)$ ?? for all $g,n$

## ③ Fully simple maps

In free probability, one is interested in some non-$U_N$-invariant observables but for $U_N$-invariant measure

$\gamma$ cycle of $\{1,\dots,N\}$ $\qquad \mathcal{P}_\gamma(M) = \prod_{i \in \text{cycle } \gamma} M_{i,\gamma(i)}$

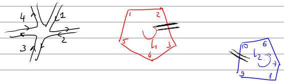
$L_1,\dots,L_n > 0 \qquad N > \sum L_i$

$\gamma_1,\dots,\gamma_n$ pairwise disjoint cycles of lengths $L_1,\dots,L_n$

$\mathbb{E}_c\left[\underline{\mathcal{P}_{\gamma_1}(M)},\dots,\underline{\mathcal{P}_{\gamma_n}(M)}\right]$

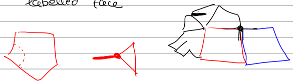
$= \sum_{\text{fully simple maps}}$

$U_N$-invariance is independent of the choice of cycles. (dpd on $L$'s only).



Fully simple means that a vertex on a labelled face is incident to exactly 2 edges along this labelled face

Forbidden is



Ordinary $\neq$ fully simple

**Theorem** $\hat{Z}(A) = \mathbb{E}\left[\exp(N\,\mathrm{Tr}(MA))\right] = \exp\left(\sum'_{\substack{n\geq 1\\ \ell_1,\dots,\ell_n>0}} \frac{N^{\sum \ell_i}}{n!} \frac{\mathbb{E}_c\left[\mathcal{P}_{\gamma_1}(M),\dots,\mathcal{P}_{\gamma_n}(M)\right]}{\ell_1\cdots\ell_n} p_{\ell_1}(A)\cdots p_{\ell_n}(A)\right)$

$A \in H_N$ fixed

$\frac{1}{Z}\int d\mu(M)\, e^{N\mathrm{Tr}(MA)}$ matrix model with external field $A$

Idea :
- Project on $U_N$-invariant observables

  $\mathbb{E}\left[f(M)\right] = \int_{U_N} \underline{\underline{dU}}\; \mathbb{E}\left[f(UMU^\dagger)\right]$ Weingarten calculus.

  $\mathbb{E}\left[\mathcal{P}_{\gamma_1}(M),\dots,\mathcal{P}_{\gamma_n}(M)\right]$ in terms of $\mathbb{E}\left[p_{k_1}(M)\cdots p_{k_n}(M)\right]$

  Monotone Hurwitz numbers.
- $\int_{U_N} dU \exp\left(N\,\mathrm{Tr}(AUBU^\dagger)\right)$ $A, B$. HCIZ integral.

  Guay-Paquet, Goulden, Novak. 2011

Assume (convergent matrix models) $\quad W_n(x_1,\dots,x_n) \underset{N\to\infty}{\sim} N^{2-n}\, \underline{W_{0,n}(x_1,\dots,x_n)}$

n-th order cumulants.

$\Rightarrow \quad \mathbb{E}_c\left[\mathcal{P}_{\gamma_1}(M),\dots,\mathcal{P}_{\gamma_n}(M)\right] \underset{N\to\infty}{\sim} N^{2-n-\sum \ell_i} K_{\ell_1,\dots,\ell_n}$

Collins, Mingo, Śniady, Speicher : $K_{\ell_1,\dots,\ell_n}$ are n-th order free cumulants.

[Side question : compute $K_{\ell_1,\dots,\ell_n}$ ?
and beyond leading order ?]

# Examples of matrix models.

- $T_{h,k} = 0$ unless $(h,k) \neq (0,1),(0,2)$

  $N\mathrm{Tr}\,M^k$ $\quad \mathrm{Tr}\,M^{k_1}\cdot\mathrm{Tr}\,M^{k_2}$

polygons + cylinders.

$W_{g,n}$ (when exists) are given Topological recursion.

- In general: Blobbed topological recursion.

Source of examples

* Two matrix model

$$\int_{M_2\in H_N} d\mu_1(M_1)\boxed{d\mu_2(M_2)\,\exp(N\mathrm{Tr}(M,M_2))} = d\mu(M_1)\boxed{\hat{Z}(M_1)}$$

even with $d\mu(M_1) = \exp\left(-N\mathrm{Tr}\frac{M^2}{2} + N\mathrm{Tr}\,T_{0,1}(M)\right)$

stuffed maps – polygons of type 1

– fully simple maps of type 2.

→ * Colored tensor models Bonzom, Dartois 2016 ...

→ * random spectral triples Azarfar, Khalkhali 2019