Lecture 2 — Schwinger-Dyson equations and loop equations

①
$$d\mu(M) = \frac{1}{Z}\, dM \exp\left[N \operatorname{Tr} T_{0,1}(M) + \frac{1}{2} \operatorname{Tr} T_{0,2}(M\otimes 1, 1\otimes M) + \cdots \right]$$

Diagonalisation $M = \underline{U} \operatorname{diag}(\lambda_1, \dots, \lambda_n) \underline{U}^+$ $\quad U \in U_N$ $\quad \frac{1}{N}\sum_i \delta_{\lambda_i} \longrightarrow \mu_{eq}$

random.

$$dM = dU\, c_N \prod_{i<j} (\lambda_i - \lambda_j)^{②} \prod d\lambda_i \quad \text{hermitian}$$

$$d\mu(M) = \frac{1}{Z}\, dU\, c_N \prod d\lambda_i \underbrace{\prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left[N \sum_i T_{0,1}(\lambda_i) + \frac{1}{2} \sum_{i,j} \cancel{T_{0,2}(\lambda_i, \lambda_j)} + \cdots \right]}_{\tilde{\mu}(\lambda_i)}$$

f analytic function near $\mathbb{R}$

$$\to \mathbb{E}\Big[\sum_i f(\lambda_i) \Big] = \oint f(\xi) W_1(\xi) \frac{d\xi}{2i\pi} \qquad W_1(x) = \mathbb{E}\left[\operatorname{Tr} \frac{1}{x - M} \right]$$
$$= \mathbb{E}\left[\sum_i \frac{1}{x - \lambda_i} \right]$$

Schwinger-Dyson equations $\quad Z = \tilde{c}_N \int_{\mathbb{R}^N} \prod d\lambda_i\, \tilde{\mu}(\lambda_1, \dots, \lambda_N)$.

$$\lambda_i \longrightarrow \lambda_i + \frac{\varepsilon}{x - \lambda_i} \qquad x \in \mathbb{C} - \mathbb{R} \qquad \lambda_i \longrightarrow \lambda_i + \varepsilon \lambda_i^k \quad k \geqslant 1.$$

invariant at first order in ε.

$$\prod d\lambda_i \longrightarrow \prod d\lambda_i \left(1 + \varepsilon \sum_i \frac{1}{(x - \lambda_i)^2} + O(\varepsilon^2) \right)$$

$$\prod_{i<j} (\lambda_i - \lambda_j)^2 \rightarrow \prod_{i<j} (\lambda_i - \lambda_j)^2 \cdot \left[1 + \varepsilon\, 2 \sum_{i<j} \left(\frac{1}{x - \lambda_i} - \frac{1}{x - \lambda_j} \right) \frac{1}{\lambda_i - \lambda_j} + O(\varepsilon^2) \right]$$

$$\exp\Big(N \sum_i T_{0,1}(\lambda_i) + \frac{1}{2} \sum_{i,j} T_{0,2}(\lambda_i, \lambda_j)\Big) \longrightarrow \exp\Big(N \sum_i T_{0,1}(\lambda_i) + \frac{1}{2} \sum_{i,j} T_{0,2}(\lambda_i, \lambda_j)\Big)$$
$$\cdot \left(1 + \varepsilon \left(N \sum_i \frac{T_{0,1}'(\lambda_i)}{x - \lambda_i} + \sum_{i,j} \frac{\partial_{\lambda_i} T_{0,2}(\lambda_i, \lambda_j)}{x - \lambda_i} \right) + O(\varepsilon^2) \right]$$

$$\rightsquigarrow \mathbb{E}\left[\sum_i \frac{1}{(x - \lambda_i)^2} + \sum_{i<j} \frac{2}{(x - \lambda_i)(x - \lambda_j)} + N \sum_i \frac{T_{0,1}'(\lambda_i)}{x - \lambda_i} + \sum_{i,j} \frac{\partial_{\lambda_i} T_{0,2}(\lambda_i, \lambda_j)}{x - \lambda_i} \right] = 0$$

$$\mathbb{E}\Big[\sum_{i,j} \frac{1}{(x - \lambda_i)(x - \lambda_j)} + N \sum_i \frac{T_{0,1}'(\lambda_i)}{x - \lambda_i} + \sum_{i,j} \frac{\partial_{\lambda_i} T_{0,2}(\lambda_i, \lambda_j)}{x - \lambda_j} \Big] = 0.$$

$$\mathbb{E}_c\left[\sum_i \frac{1}{x - \lambda_i} \sum_j \frac{1}{x - \lambda_j} \right] + \mathbb{E}\left[\sum_i \frac{1}{x - \lambda_i} \right] \mathbb{E}\left[\sum_j \frac{1}{x - \lambda_j} \right] + \quad = 0.$$

$$\boxed{\begin{aligned} & W_2(x, x) + (W_1(x))^2 + N \oint \frac{d\xi}{2i\pi} \frac{T_{0,1}'(\xi) W_1(\xi)}{x - \xi} + \oint\!\!\oint \frac{d\xi_1 d\xi_2}{(2i\pi)^2} \frac{\partial_{\xi_1} T_{0,2}(\xi_1, \xi_2)}{x - \xi_1} \\ & \qquad \big(W_2(\xi_1, \xi_2) + W_1(\xi_1) W_1(\xi_2) \big) = 0 \end{aligned}}$$

First SD equation

- δ_{x_i} $\quad \frac{d}{ds}\Big|_{s=0}$ $\quad T_{0,1}(\lambda) \to T_{0,1}(\lambda_i) + \frac{s}{N} \frac{1}{x_i - \lambda}$ $\quad x_i \in \mathbb{C} - \mathbb{R}$.

$$\delta_y W_n(x_1, \dots, x_n) = W_{n+1}(x_1, \dots, x_n, y)$$

Let's apply $\delta_{x_1} \cdots \delta_{x_n}$ on 1st SD equation :

$$\boxed{\begin{aligned}&W_{n+2}(x, x, x_1, \dots, x_n) + \sum'_{J \cup J' = \{x_1, \dots, x_n\}} W_{1+|J|}(x, J)\, W_{1+|J'|}(x, J') \\
&+ \sum_{i=1}^{n} \oint \frac{d\xi}{2\pi} \frac{1}{(x_i - \xi)^2} \frac{W_n(\xi, x_1, \dots \hat{x}_i \dots x_n)}{x - \xi} + N \oint \frac{T'_{0,1}(\xi)\, W_{n+1}(\xi, x_1, \dots x_n)}{x - \xi} \frac{d\xi}{2\pi} \\
&+ \oiint \frac{d\xi_1 d\xi_2}{(2\pi)^2} \frac{\partial_{\xi_1} T_{0,2}(\xi_1, \xi_2)}{x - \xi_1} \left(W_{n+2}(\xi_1, \xi_2, x_1, \dots x_n) + \sum'_{J \cup J' = \{x_1, \dots, x_n\}} W_{1+|J|}(\xi_1, J)\, W_{1+|J'|}(\xi_2, J') \right) = 0\end{aligned}}$$

(n+1)-th SD equation involves $W_{n+2}, W_{n+1}, W_n, \dots, W_1$

② Large N limit

- under weak assumptions . $W_1(x) \underset{N \to \infty}{\sim} N W_{0,1}(x)$ (limit exists).

concentration of measure . $W_2(x_1, x_2) = O(N \ln N)$

leading order (N^2) of 1st SD equation:

$$W_{0,1}^2(x) + \oint \frac{d\xi}{2\pi} \frac{T'_{0,1}(\xi)\, W_{0,1}(\xi)}{x - \xi} + \oiint \frac{d\xi_1 d\xi_2}{(2\pi)^2} \frac{\partial_{\xi_1} T_{0,2}(\xi_1, \xi_2)}{x - \xi_1} W_{0,1}(\xi_1)\, W_{0,1}(\xi_2) = 0$$

non-linear, integral equation for $\underline{W_{0,1}}$

$$W_{0,1}^2(x) + T'_{0,1}(x)\, W_{0,1}(x) + \underbrace{\oint \frac{d\xi}{2\pi} \boxed{\frac{T'_{0,1}(\xi) - T'_{0,1}(x)}{x - \xi}}\, W_{0,1}(\xi)}_{P_{01}(x)}.$$

analytic
x approaches $\mathbb{R}$

$$+ \underbrace{\left(\oint \frac{d\xi_2}{2\pi} \partial_x T_{0,2}(x, \xi_2)\, W_{0,1}(\xi_2) \right)} W_{0,1}(x) + \underbrace{P_{02}(x)} = 0$$

analytic
when x approaches $\mathbb{R}$.

imagine that $T_{0,1}$ polynomial

otherwise, $T_{0,1}$ analytic near $\mathbb{R}$.

$$W_{0,1}^2(x) + \tilde{T}'_{0,1}(x)\, W_{0,1}(x) + P(x) = 0.$$

$$\tilde{T}_{0,1}(x) = T_{0,1}(x) + \oint \frac{d\xi_2}{2\pi} T_{0,2}(x, \xi_2)\, W_{0,1}(\xi_2) \qquad \text{analytic near } \mathbb{R}.$$

$$W_{0,1}(x) = -\frac{\tilde{T}'_{0,1}(x)}{2} \pm \frac{1}{2} \sqrt{\underbrace{\tilde{T}'_{0,1}(x)^2 - 4P(x)}}$$

$$W_{0,1}(x) = -\frac{\tilde{T}'_{0,1}(x)}{2} \pm M(x) \sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4) \cdots (x-a_{2h})}$$

$$a_1 < \cdots < a_{2h}.$$

$\Rightarrow$ eigenvalue condensate on $S = \bigcup_{i=1}^{h} [a_{2i+1}, a_{2i}]$ h-cut regime.

If $M > 0$ on S, off-critical regime

Theorem • 1-cut regime + off critical (+ assumptions)

$$W_n(x_1, \dots, x_n) \underset{N \to +\infty}{=} \sum_{g=0}^{G} N^{2-2g-n} W_{g,n}(x_1, \dots, x_n) + o(N^{2-2G-n})$$

to all orders ε

in this case $W_{g,n}$ have power series expansion in t's
and the corresponding formal power series are the
generating series of (stuffed) maps (correlators of formal matrix model)

- multi-cut regime + off-critical (+ ...).

wrong : there are oscillatory terms

1-cut regime — Albeverio, Pastur, Shcherbina 2001.
B., Guionnet, Kozlowski ($T_{0,2}$, $T_{h,k}$). 2011 – 2015.
Multi-cut regime — B., Guionnet 13 + Kozlowski ←

Then, the coefficients are computed by blobbed topological recursion using the spectral curve

$$y = W_{0,1}(x) \qquad \text{(Riemann surface on which } W_{0,1} \text{ admits analytic continuation)}$$

$$\text{“}\omega_{0,2}\text{”} = B = \left[W_{0,2}(x_1, x_2) + \frac{1}{(x_1-x_2)^2} \right] dx_1 dx_2 \qquad \text{(analytic continuation on the same sp curve).}$$

Eynard 04 (no $T_{0,2}$)
B., Eynard, Orantin 13 ($T_{0,2}$ as well).
B. 15 (all $T_{h,k}$ allowed).

Multicut : non-perturbative TR
David, Eynard 06 predicted.
Eynard, Mariño

Proved by heavy analysis of SD equations.

③ All-order expansion.

1st SD $\quad W_2 + W_1^2 + N W_1 \qquad [N^2] \quad W_{0,1}^2 + W_{0,1} \to$ nonlinear equation for $W_{0,1}$.

$[N^0] \quad \boxed{W_{0,2}} + \boxed{W_{1,1}}\,\underline{W_{0,1}} + W_{1,1} \leftarrow$ linear for $W_{1,1}$

2nd SD $\quad W_3 + \underbrace{W_1}W_2 + \underline{N}\boxed{W_2} + \boxed{W_1} \qquad [N] \quad \boxed{W_{0,2}}\boxed{W_{0,1}} + W_{0,2} + \boxed{W_{0,1}} \to$ linear for $W_{0,2}$

General feature of SD — all the other equations form a triangular system of the form $K[W_{g,n}] = \underset{(\mathcal{P}(x))}{\text{analytic}} + \mathcal{F}[W_{g',n'} \quad 2g'-2+n' < 2g-2+n]$

The work is to invert K.

K invertible ⟺ off-critical + 1-cut.

Tomorrow ⇝ loop equations for $W_{g,n}$