$$\boxed{\begin{aligned} & W_{n+2}(x,x,x_1,\dots,x_n) + \sum_{J\cup J'=\{x_1,\dots,x_n\}} W_{1+|J|}(x,J)\,W_{1+|J'|}(x,J') \longleftarrow \\ & + \sum_{i=1}^{n} \oint \frac{d\xi}{2\pi}\, \frac{1}{(x_i-\xi)^2}\, \frac{W_n(\xi,x_1,\dots\hat{x}_i\dots x_n)}{x-\xi} + N\oint \frac{T_{0,1}'(\xi)\,W_{n+1}(\xi,x_1,\dots,x_n)}{x-\xi}\,\frac{d\xi}{2\pi} \\ & + \oint\!\!\oint \frac{d\xi_1 d\xi_2}{(2\pi)^2}\, \frac{\partial_{\xi_1} T_{0,2}(\xi_1,\xi_2)}{x-\xi_1} \Big(W_{n+2}(\xi_1,\xi_2,x_1,\dots,x_n) + \sum_{J\cup J'=\{x_1,\dots,x_n\}} W_{1+|J|}(\xi_1,J)\,W_{1+|J'|}(\xi_2,J')\Big) = 0 \end{aligned}}$$

$(n+1)$-th SD equation involves $W_{n+2}, W_{n+1}, W_n, \dots, W_1$

$$W_n = \sum_{g\geq 0} N^{2-2g-n}\, W_{g,n}$$

coeff of $N^{2-2g-(n+1)}$

$[N^k]$

Lecture 3 – loop equations and topological recursion

① Reminder : 1st Schwinger-Dyson equation at leading order

$$* \quad \underbrace{W_{0,1}^2(x)} + \underbrace{\tilde{T}_{0,1}'(x)\,W_{0,1}(x)} + \overset{\text{analytic near } \mathbb{R}}{P(x)} = 0.$$

$$\tilde{T}_{0,1}(x) = T_{0,1}(x) + \oint \frac{d\xi_2}{2\pi}\, T_{0,2}(x,\xi_2)\,W_{0,1}(\xi_2) \qquad \text{analytic near } \mathbb{R}$$

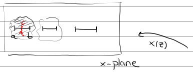
$$W_{0,1}(x^\pm) = \lim_{\varepsilon\to 0^\pm} W_{0,1}(x\pm i\varepsilon) \qquad \text{boundary values exist}$$

Discontinuity of $*$ $\quad (x^+) - (x^-)$

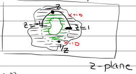
$$\big(\underbrace{W_{0,1}(x^+) - W_{0,1}(x^-)}_{\neq 0}\big)\big(W_{0,1}(x^+) + W_{0,1}(x^-) + \tilde{T}_{0,1}'(x)\big) = 0$$

$\forall x \in S = \bigcup_h [a_h, b_h]$

$$\Rightarrow \quad W_{0,1}(x^+) + W_{0,1}(x^-) + \tilde{T}_{0,1}'(x) = 0 \qquad \forall x \in S$$



$$x(z) = \frac{a+b}{2} + \frac{a-b}{4}\left(z+\frac{1}{z}\right)$$

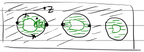


$$\left[\begin{aligned} x(z) &= z+\frac{1}{z} \\ W_{0,1}(x) &= \frac{1}{z} + \dots\dots\dots \end{aligned}\right.$$

$$W_{0,1}(z) + W_{0,1}\left(\tfrac{1}{z}\right) + \underbrace{\tilde{T}_{0,1}'(x(z))} = 0 \qquad \forall z \in U$$

$\rightsquigarrow$ analytic continuation of $W_{0,1}$ inside the unit disk (Schwarz principle).

Σ spectral curve



$$\sqrt{(x-a)(x-b)} = \frac{a-b}{4}\left(z-\frac{1}{z}\right) \qquad \sigma(z) = \frac{1}{z}.$$

$x(z) = x(\sigma(z))$

σ local involution.

$z = \pm 1 \iff x = a$ or b

endpoints

z local coord on Σ

$$dx = \frac{a-b}{4}\left(1-\frac{1}{z^2}\right)dz$$

has simple zero at $z = \pm 1$.

$*$ All the SD equations (in large N expansion) can be recursively on $2g-2+n>0$ analysed to find that. (discontinuity of SD equations)

simple or double trace matrix model

$\forall x \in S$
$x_2,\dots,x_n \in \mathbb{C} - S$

$$W_{g,n}(x^+, x_2,\dots,x_n) + W_{g,n}(x^-, x_2,\dots,x_n)$$
$$+ \oint \frac{d\xi_2}{2i\pi}\, \partial_x T_{0,2}(x,\xi_2)\, W_{g,n}(\xi_2, x_2,\dots,x_n) = -\delta_{g0}\delta_{n1} T'_{0,1}(x) - \delta_{g0}\delta_{n2}\frac{1}{(x-x_2)^2}$$

linear loop equation.

$T_{0,3}, T_{0,4}$
RHS $\neq 0$ for any (g,n)
⇝ blobs.

$\Rightarrow$ $W_{g,n}$ can be continued analytically on same Σ

* Analysing further the SD equations, brings them in the form

$$\tilde{W}_{g,n+1}(z, \sigma(z), z_2,\dots,z_n) + \sum_{\substack{J\cup J' = \{z_2,\dots,z_n\} \\ h+h'=g}}' \tilde{W}_{h,1+|J|}(z,J)\,\tilde{W}_{h',1+|J'|}(\sigma(z),J')$$

quadratic loop equation. $=$ holomorphic when $z \to$ endpoint

whereas $W_{g,n}(z_1,\dots,z_n)$ are meromorphic with poles $z_i \to$ endpoint

② Abstract loop equations

Setting Σ complex curve
$x : \Sigma \to \mathbb{P}^1$ meromorphic function.

A = zeroes of dx = ramification points
(assume simple, finitely many)

$\underset{z\to a}{x} = x + z^2$
$\sigma(z) = -z$
defined locally near a

$\omega_{0,1}$ holo 1-form on Σ so that $d\left(\frac{\omega_{0,1}}{dx}\right)$ is not zero at $z \to A$

$\omega_{0,2}(z_1,z_2)$ fundamental bidiff of 2nd kind
(meromorphic in z_1, in $z_2 \in \Sigma$
double pole at $z_1 = z_2$

$$\omega_{0,2}(z_1,z_2) \underset{z_1\to z_2}{\sim} \frac{dz_1\,dz_2}{(z_1-z_2)^2}$$

in matrix model

spectral curve
$x : \Sigma \to \mathbb{P}^1$ branched cover.

endpoints of segments

$$\omega_{0,1} \overset{\text{anac}}{=} W_{0,1}(x)\cdot dx$$

$$\omega_{0,2} \overset{\text{anac}}{=} \left[W_{0,2}(x_1,x_2) + \frac{1}{(x_1-x_2)^2}\right]dx_1 dx_2$$

has indeed the ppT

$\omega_{g,n}(z_1,\dots,z_n)$ $g \geq 0$, $n \geq 1$, $2g-2+n>0$)
meromorphic forms in each z_i $i = 1,\dots,n$
poles $z_i \to A$

satisfies abstract loop equations

if $\forall g,n$

- linear loop equation $\omega_{g,n}(z, z_2,\dots,z_n) + \omega_{g,n}(\sigma(z), z_2,\dots,z_n) = \underset{z\to A}{\text{simple zero}}$
- quadratic loop equation

$$\omega_{g-1,n+1}(z,\sigma(z),z_2,\dots,z_n) + \sum_{\substack{h+h'=g \\ J\cup J'=\{z_2,\dots,z_n\}}}' \omega_{h,1+|J|}(z,J)\,\omega_{h',1+|J'|}(\sigma(z),J') = \underset{z\to A}{\text{double zero}}$$

in matrix model: $\omega_{g,n} = \left(W_{g,n}(x_1,\dots,x_n) + \delta_{g0}\delta_{n2}\frac{1}{(x_1-x_2)^2}\right)dx_1\cdots dx_n$ satisfies abstract loop equations.

③ Solution using topological recursion.

- f meromorphic 1-form on Σ, poles at A

$$\mathcal{P}[f](z_0) = \sum_{a\in A}' \operatorname*{Res}_{z=a}\left[\left(\int_a^z \omega_{0,2}(\cdot, z_0)\right) f(z)\right]$$

$$H_0 = \mathrm{Id} - \mathcal{P}$$

- The ppT of $\omega_{0,2}$ imply that $\underset{\text{polar part}}{\mathcal{P}}, \underset{\text{holo part}}{\mathcal{H}}$ are projectors.

$\mathcal{H}[f]$ is holomorphic on Σ

- In the matrix model
  - 1-cut situation
  - multicut situation, fix # eigenvalue per cut

then $\mathcal{H}_z[\omega_{g,n}(\cdot, z_2, \dots, z_n)] = 0$ ← (normalized)

in other words $\omega_{g,n}(z_1, \dots, z_n) = \mathcal{P}_{z_1}[\omega_{g,n}(\cdot, z_2, \dots, z_n)]$

"Cauchy residue formula" adapted to our problem

(reconstruct $\omega_{g,n}$ from its local behavior at poles)

Core computation $(\omega_{g,n})_{2g-2+n>0}$ solution of abstract loop equations + normalised

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{a\in A}' \operatorname*{Res}_{z=a} G_a(z, z_1)\, \omega_{g,n}(z, z_2, \dots, z_n) \qquad G(z, z_1) = \int_a^z \omega_{0,2}(\cdot, z_1)$$

$$= \sum_{a\in A}' \operatorname*{Res}_{z=a} \frac{1}{2} \Big(\underbrace{G_a(z, z_1)\, \omega_{g,n}(z, \underbrace{z_2, \dots, z_n}_{I}) + G_a(\sigma(z), z_1)}_{S_z[G_a(z,z_1)\,\omega_{g,n}(z,I)]}\, \omega_{g,n}(\sigma(z), I) \Big)$$

$$\Delta[f](z) = f(z) - f(\sigma(z))$$
$$S[f](z) = f(z) + f(\sigma(z))$$

$$S[f_1 \cdot f_2] = \frac{1}{2}\left(Sf_1 \cdot Sf_2 + \Delta f_1\, \Delta f_2\right)$$

$$** \quad \omega_{g,n}(z_1, I) = \sum_{a\in A}' \operatorname*{Res}_{z=a} \frac{1}{4}\Big(\cancel{S_z G_a(z, z_1)\, \underbrace{S_z\, \omega_{g,n}(z, I)}_{\substack{\text{linear loop equation:}\\ \text{holomorphic}}}} + \Delta_z G_a(z, z_1)\, \boxed{\Delta_z\, \omega_{g,n}(z, I)} \Big)$$

$$f_1(z) f_2(\sigma(z)) = \frac{1}{2}\left(Sf_1\, Sf_2 - \Delta f_1\, \Delta f_2\right)$$

Quadratic loop equation can be written

$$\omega_{g,n}(z, I)\, \omega_{0,1}(\sigma(z)) + \omega_{g,n}(\sigma(z), I)\, \omega_{0,1}(z) + \mathcal{E}_{g,n}(z, I) = \overset{Q_{g,n}(z,I)}{\text{double zero at } z \to A}$$

$$\mathcal{E}_{g,n}(z, I) = \omega_{g-1,n+1}(z, \sigma(z), I) + \sum_{\substack{h+h'=g \\ J\cup J'=I}}^{\text{no } \omega_{0,1}} \omega_{h,1+|J|}(z, J)\, \omega_{h',1+|J'|}(\sigma(z), J)$$

$$\Rightarrow \frac{1}{2} S_z\, \omega_{g,n}(z, I)\, S_z\, \omega_{0,1}(z) - \frac{1}{2} \boxed{\Delta_z\, \omega_{g,n}(z, I)}\, \Delta_z \omega_{0,1}(z) + \mathcal{E}_{g,n}(z, I) = \underbrace{\text{double zero } z \to A}_{Q_{g,n}(z,I)}$$

$$** \Rightarrow \omega_{g,n}(z_1, I) = \sum_{a\in A}' \operatorname*{Res}_{z=a} \frac{1}{2} \frac{\overset{\text{holo}}{\boxed{\Delta_z G_a(z, z_1)}}}{\boxed{\Delta_z\, \omega_{0,1}}} \Big[\mathcal{E}_{g,n}(z, I) + \overset{0}{\cancel{\frac{1}{2} \underbrace{S_z\, \omega_{g,n}(z, I)\, S_z\, \omega_{0,1}(z)}_{\substack{\text{linear loop equations} \\ \text{double zero } z \to A}}}} - \overset{0}{\cancel{\underbrace{Q_{g,n}(z, I)}_{\substack{\text{quadratic} \\ \text{loop equations:} \\ \text{double zero.}}}}} \Big]$$

in matrix model

$$\Delta_z\, \omega_{0,1} = \omega_{0,1}(z) - \omega_{0,1}(\sigma(z))$$
$$= \underbrace{(W_{0,1}(z) - W_{0,1}(\sigma(z)))}_{\sqrt{x - x(a)}}\, dx(z) \quad \text{double zero.}$$

$$\Rightarrow \boxed{\omega_{g,n}(z_1, I) = \sum_{a\in A}' \operatorname*{Res}_{z=a} \frac{1}{2} \frac{\int_{\sigma(z)}^{z} \omega_{0,2}(\cdot, z_1)}{\omega_{0,1}(z) - \omega_{0,1}(\sigma(z))}\, \mathcal{E}_{g,n}(z, I)}$$

Topological recursion formula. (Eynard - Orantin 07)

$$\mathcal{E}_{g,n}(z,I) = \omega_{g-1,n+1}(z,\sigma(z),I) + \sum_{\substack{h+h'=g \\ J\cup J'=I}}^{\text{no } \omega_{0,1}} \omega_{h,1+|J|}(z,J)\,\omega_{h',1+|J'|}(\sigma(z),J)$$

only contains $\omega_{g',n'}$ $2g'-2+n' < 2g-2+n$

④ General solution

$\Sigma, x, \omega_{0,1}, \omega_{0,2}$ fixed.

$$\left\{\begin{array}{l}\text{solutions of abstract} \\ \text{loop equations}\end{array}\right\} \xrightarrow{\ \Phi\ } \bigoplus_{2g-2+n>0} \left\{\begin{array}{l}\text{holo forms in } n \text{ variables} \\ \text{on } \Sigma^n\end{array}\right\}$$

$$(\omega_{g,n})_{g,n} \longmapsto \left(\phi_{g,n} = H_{z_1} \cdots H_{z_n}\, \omega_{g,n}(z_1,\dots,z_n)\right)_{g,n} \quad \text{blobs.}$$

Theorem (B., Shadrin)

Φ is a bijection

$\Phi^{-1}(0)$ is the normalized solution given by TR.

$\Phi^{-1}((\phi_{g,n})_{g,n})$ can be written as a sum over bipartite graphs

Φ vertices (blobs) $\omega^0_{g,n}$ (usual TR)

Multitrace matrix models $\longrightarrow$ Spectral curve

$(T_{h,k})_{h,k}$ $\qquad$ $\Sigma, x, \omega_{0,1}, \omega_{0,2}$

if only $T_{0,1}, T_{0,2}$: $\phi \equiv 0$ (no blobs)

otherwise $\left[\begin{array}{l}\phi_{g,n} \neq 0 \text{ for all } g,n \\ \text{obtained recursively from} \\ \omega_{g',n'} \quad 2g'-2+n' < 2g-2+n \\ T_{h,k} \quad 2h-2+k < 2g-2+n\end{array}\right.$

large N asymptotic expansion (1-cut) $\downarrow$ $\qquad$ formal matrix model $\downarrow$ $\qquad$ $\downarrow$ TR (blobbed TR)

$$\boxed{W_n = \sum_{g\geq 0} N^{2-2g-n}\, W_{g,n}} \qquad \boxed{(\omega_{g,n})_{g,n}.}$$

$$\omega_{g,n}(z_1,\dots,z_n) = W_{g,n}(x(z_1),\dots,x(z_n))\,dx(z_1)\cdots dx(z_n) + \delta_{g,0}\delta_{n,2}\frac{dx(z_1)\,dx(z_2)}{(x(z_1)-x(z_2))^2}.$$

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