The freeness alternative to thin sets in Manin's conjecture

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Manin's conjecture

Let X be a smooth projective Fano variety (i.e. the anticanonical bundle K_X^{-1} is ample) defined over $\mathbb{Q}$.

We can define the height of $x \in X(\mathbb{Q})$ by choosing an embedding $i\colon X \to \mathbb{P}^N_{\mathbb{Q}}$ and setting $h_i(x) = \max(|x_0|, \dots, |x_N|)$ when $i(x) = (x_0 : \dots : x_N)$ for $x_0, \dots, x_n \in \mathbb{Z}$ with no common factors.

Manin's conjecture predicts that the number of $x \in X(\mathbb{Q})$ of height $h(X) < T$ has for form $cT^a(\log T)^b$ for a, b, c explicit constants depending on the geometry and arithmetic of X.

Some versions also give predictions for the distribution of those x in $X(\mathbb{Q}_p)$ and $X(\mathbb{R})$.

But for these predictions to come true, we must first remove a thin set of rational points.

Thin sets and their discontents

A thin set of $X(\mathbb{Q})$ is a finite union of subsets of $X(\mathbb{Q})$ of the following two types:

- $Y(\mathbb{Q})$ for a subvariety $Y \subset X$ of X.
- The image of $Z(\mathbb{Q})$ for $f : Z \to X$ a generically finite map of degree ≥ 2.

Using cutting-edge algebraic geometry tools, Lehmann, Sengupta, and Tanimoto found a good thin set to remove.

But this is all very strange. To decide whether a rational point is good or bad, you first have to go looking for bad subvarieties of your variety, or bad coverings.

Is there a way to tell whether a rational point is good or bad by looking at just that point?

Rational points and rational curves

There is a deep analogy between the fields $\mathbb{Q}$ and $\mathbb{F}_q(T)$, $\mathbb{F}_q$ a finite field.

Let X be a variety over $\mathbb{F}_q$. The $\mathbb{F}_q(T)$-points on X are analogous to $\mathbb{Q}$-points. We have analogues of height, Manin's conjecture, etc.

But we also have a geometric structure. Why? A $\mathbb{F}_q(T)$-point of X gives a map $\mathbb{P}^1_{\mathbb{F}_q} \to X$. These are parameterized by a moduli space $\mathrm{Mor}(\mathbb{P}^1, X)$. Finite-type analogue: The moduli space $\mathrm{Mor}_\beta(\mathbb{P}^1, X)$ of maps $\mathbb{P}^1 \to X$ of numerical class β.

(Can mod out $\mathrm{Mor}_\beta(\mathbb{P}^1, X)$ by PGL_2 to get $\mathcal{M}_{0,0}(X, \beta)$, but no reason to do this here.)

The geometry of rational curves

We can interpret Manin's conjecture in this setting as a statement about the geometry of this moduli space $\mathrm{Mor}_\beta(\mathbb{P}^1, X)$.

We use results relating the geometry of a space to the number of its $\mathbb{F}_q$ points.

- Lang-Weil: varieties of dimension N have $\approx q^N$ $\mathbb{F}_q$-points.
- (Harder: Lefschetz fixed point formula in étale cohomology.)

One consequence: $\mathrm{Mor}_\beta(\mathbb{P}^1, X)$ should have dimension $n - (\beta \cdot K_x)$ once we remove a thin set.

Why?

- For the best (anticanonical) height function, curves of class β have height $q^{-\beta \cdot K_x}$.
- For the anticanonical height function, Manin predicts the count should be proportional to the height, so proportional to $q^{-\beta \cdot K_X}$.
- Can check that the leading term in the constant of proportionality is $\approx q^n$, so $\mathrm{Mor}_\beta(\mathbb{P}^1, X)$ should have $\approx q^{n-\beta \cdot K_X}$ $\mathbb{F}_q$-points.

Dimension and the tangent bundle

Manin's conjecture predicts: $\mathrm{Mor}_\beta(\mathbb{P}^1, X)$ should have dimension $n - (\beta \cdot K_x)$.

We can make the same prediction a different way!

Deformation theory: The tangent space to $\mathrm{Mor}_\beta(\mathbb{P}^1, X)$ at the point corresponding to $f\colon \mathbb{P}^1 \to X$ is $H^0(\mathbb{P}^1, f^*\mathcal{T}_X)$.

Riemann-Roch:

$$\dim H^0(\mathbb{P}^1, f^*\mathcal{T}_X) = n - (\beta \cdot K_x) + \dim H^1(\mathbb{P}^1, f^*\mathcal{T}_X).$$

Can deduce: $\mathrm{Mor}_\beta(\mathbb{P}^1, X)$ is smooth of dimension $n - (\beta \cdot K_x) + \dim H^1(\mathbb{P}^1, f^*\mathcal{T}_X)$ at the point corresponding to f if and only if $H^1(\mathbb{P}^1, f^*\mathcal{T}_X) = 0$.

Freeness in the geometric setting

Rather than removing $f: \mathbb{P}^1 \to X$ lying in a thin set, which is hard to compute explicitly in general, why not simply remove f where $H^1(\mathbb{P}^1, f^*\mathcal{T}_X) \neq 0$?

This ensures the dimension is what we want, and removes the singularities.

Concretely: Vector bundles on $\mathbb{P}^1$ can be writen as $\bigoplus_{i=1}^n \mathcal{O}(a_i)$ for $a_i \in \mathbb{Z}$. We keep the f where $f^*\mathcal{T}_X$ has all $a_i \geq -1$.

This is slightly weaker than the concept of a "free curve", meaning all $a_i \geq 0$, or "very free curve", meaning all $a_i \geq 1$. All important concepts in algebraic geometry!

Freeness in an arithmetic setting

Let X be a smooth projective Fano variety over $\mathbb{Q}$. Suppose we can spread X out to a proper scheme $\mathcal{X}$ of dimension n over $\mathbb{Z}$.

A rational point $x \in X(\mathbb{Q})$ extends to a map $f_x \colon \operatorname{Spec} \mathbb{Z} \to \mathcal{X}$.

Then $f_x^* \mathcal{T}_{\mathcal{X}}$ is a vector bundle of rank n on $\operatorname{Spec} \mathbb{Z}$, i.e. a free abelian group of rank n. Seemingly no invariants to work with...

If we fix in addition a Riemannian metric on $X(\mathbb{R})$, then $f_x^* \mathcal{T}_{\mathcal{X}}$ is a free abelian group of rank n with a symmetric bilinear form. A lattice! These have interesting invariants...

The right ones for us are the *successive minima*.

Successive minima

We now define the successive minima of a lattice Λ.

Let λ_r be the least λ such that Λ contains r linearly independent vectors of length $\leq \lambda$. So λ_1 is the length of the shortest nonzero vector v_1, λ_2 is the length of the shortest vector v_2 not a multiple of v_1, etc. We call λ_r the rth successive minimum.

The $\lambda_1, \ldots, \lambda_n$ of a lattice behave like $e^{-a_1}, \ldots, e^{-a_n}$ of a vector bundle, in particular if we order the a_i so $a_1 \geq \cdots \geq a_n$.

Following Peyre, we say x is *free* if $\lambda_n(f_x^* \mathcal{T}_{\mathcal{X}}) < H(x)^{-\epsilon}$.

(This happens unless $f_x^* \mathcal{T}_{\mathcal{X}}$ admits a distance-decreasing map to a relatively sparse lattice of rank 1, analogous to a curve free unless $f^* \mathcal{T}_X$ admits a map to $\mathcal{O}(-2)$.)

Q (Peyre): Can removing the non-free rational points replace the removal of the thin set in Manin's conjecture?

Concrete characterization for projective varieties

Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n defined by homogeneous polynomials $f_1, \ldots, f_k$. Let $x \in X(\mathbb{Q})$ be an rational point with projective coordinates $(x_0 : \cdots : x_N)$. Let

$$\Lambda_x = \{y \in \mathbb{Z}^{N+1}/\langle(x_0, \ldots, x_n)\rangle \mid y \cdot \nabla f_1(x) = \cdots = y \cdot \nabla f_k(x) = 0\}.$$

Here the norm of a vector y is the length of the projection of y to the orthogonal complement of $(x_0, \ldots, x_n)$, equivalently is $\min\{||y + t(x_0, \ldots, x_n)||_2 \mid t \in \mathbb{R}\}$.

We say x is free if

$$\lambda_n(\Lambda_x) < \max(|x_0|, \ldots, |x_N|)^{1-\epsilon}.$$

Positive example: The cubic surface case

Let $X \subseteq \mathbb{P}^3$ be defined by a cubic equation, for concreteness $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$.

The anticanonical line bundle on X is $\mathcal{O}(1)$, so the height is simply $\max(|x_0|, |x_1|, |x_2|, |x_3|)$. Manin's conjecture predicts the number of points of height $< T$ is proportional to T times a power of $\log T$.

The thin set in this case consists of at most 27 lines on the cubic surface X, for example $(x_0 : x_1 : x_2 : x_3) = (a : -a : b : -b)$. The number of points of height $< T$ on a line is proportional to T^2, much too big.

We want to check that removing unfree points can substitute for removing the thin set. In particular, we need to check that almost every point on the line is not free.

Positive example: The cubic surface case

Let $X \subseteq \mathbb{P}^3$ be defined by $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$. Let $x = (a : -a : b : -b)$.

Then $\Lambda_x = \{(y_1, y_2, y_3, y_4) \in \mathbb{Z}^4/x \mid a^2(y_1 + y_2) + b^2(y_3 + y_4) = 0.\}$ The sublattice defined by $y_1 + y_2 = y_3 + y_4 = 0$ contains only one linearly independent vector (mod x). Outside this sublattice, we must have $y_1 + y_2$ a multiple of b^2 and $y_3 + y_4$ a multiple of a^2, so the minimum length is

$$\sqrt{\frac{a^4 + b^4}{2}} \approx \max(|a|, |b|)^2 > \max(|a|, |b|)^{1-\epsilon}.$$

So x is not free.

The generator of this sublattice is $(c : -c : d : -d)$ where $ad - bc = 1$, which has length

$$\lambda_1 = \sqrt{\frac{2}{a^2 + b^2}} \approx \max(|a|, |b|)^{-1}.$$

Positive example: Hypersurfaces (Browning-S)

Birch: Let X be a smooth hypersurface of degree d in $\mathbb{P}^N$. If $N \geq 2^d(d-1)$, then the number of points in $X(\mathbb{Q})$ of height $< T$ is proportional to a constant times T^{N+1-d}. That is, Manin's conjecture is true, with empty thin set.

Browning-S: Let X be a smooth hypersurface of degree d in $\mathbb{P}^N$. If $N \geq 3 \cdot 2^{d-1}(d-1)$, then the number of free points in $X(\mathbb{Q})$ of height $< T$ is proportional to the same constant times T^{N+1-d}.

Method of proof: After Birch, suffices to upper bound the number of unfree points. If x is unfree, then there are many $y \in \Lambda_x$ with $||y|| < ||x||$. So it suffices to upper bound the number of solutions (x, y) to the system of equations $f(x) = 0, y \cdot \nabla f = 0$, which we do with a circle method argument, following closely the strategy of Birch.

Negative example: Hilbert schemes of projective space (S)

Let $X = \mathrm{Hilb}_2(\mathbb{P}^n)$. This is a resolution of the singularities of $\mathrm{Sym}^2(\mathbb{P}^n)$. Abstractly, it is the moduli space of ideal sheaves on $\mathbb{P}^n$ with quotient of length 2. Concretely, it is the quotient of the blow-up $Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n)$ of the diagonal Δ of $\mathbb{P}^n \times \mathbb{P}^n$ by the involution switching the two copies of $\mathbb{P}^n$.

In Manin's conjecture for $\mathrm{Hilb}_2(\mathbb{P}^n)$, the thin set is the image of $Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n)(\mathbb{Q})$.

Can removing unfree points substitute for removing this thin set? No! In fact, most points in this thin set are free.

Geometric idea of the proof: A generic very free curve in $\mathbb{P}^n \times \mathbb{P}^n$ will not intersect the diagonal. So its lift to $Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n)$ will remain equally free. The same will be true after we project to $\mathrm{Hilb}_2(\mathbb{P}^n)$. So there are plenty of free curves on the thin set.

Negative example: Arithmetic proof

We show that a positive proportion of rational points on $Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n)(\mathbb{Q})$, outside the diagonal, map to free points on $\mathrm{Hilb}^2(\mathbb{P}^n)$.

Starting point (Peyre): Most points in $(\mathbb{P}^n \times \mathbb{P}^n)(\mathbb{Q})$ are free.

We want to check that this property is preserved as we pullback along the blow-up map $Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n) \to \mathbb{P}^n \times \mathbb{P}^n$ and pushforward along the degree two covering $Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n) \to \mathrm{Hilb}_2(\mathbb{P}^n)$. Both are generically finite maps.

General question: Let $\pi : X \to Y$ be a generically finite map of smooth varieties. Let $x \in X(\mathbb{Q})$ be a rational point of X, not in the branch divisor. How can we compare $f_x^*\mathcal{T}_X$ to $f_{\pi(x)}^*\mathcal{T}_Y$?

How do successive minima of the tangent lattice change under generically finite maps?

We have a map of lattices $f_x^*\mathcal{T}_X \to f_{\pi(x)}^*\mathcal{T}_Y$, which because π is differentiable can't increase the length of vectors too much. The successive minima can only change a lot if either

- the image of this map is a high-index sublattice of $f_{\pi(x)}^*\mathcal{T}_Y$, or
- the length of vectors decreases a lot.

These phenomena can only happen if the determinant of the map $f_x^*\mathcal{T}_X \to f_{\pi(x)}^*\mathcal{T}_Y$ either

- vanishes mod p to a high power, for many p, or for large p, or
- is small as a real number.

Because the determinant of this map measures the discrepancy between the anticanonical bundles of X and Y, and this discrepancy is given by the branch divisor, these happen exactly when x is either

- p-adically close to the branch divisor, or
- close in a real sense to the branch divisor.

Negative example: Arithmetic proof

We know that most points in $(\mathbb{P}^n \times \mathbb{P}^n)(\mathbb{Q})$ are free.

We know that they remain free as we pullback along $Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n) \to \mathbb{P}^n \times \mathbb{P}^n$ and pushforward along $Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n) \to \mathrm{Hilb}_2(\mathbb{P}^n)$, as long as they stay away from the branch divisor in p-adic and real senses.

In both cases, the branch divisor is the exceptional divisor, i.e. the inverse image of the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$.

We can check that a positive proportion rational points stay away from the diagonal using a sieve.

So a positive proportion of rational points in the thin set are free. This causes a failure of the modified Manin's conjecture, at least in its stronger equidistribution form.

Where does this leave freeness?

It is possible that unfree points do substitute for the special subvarieties in Manin's conjecture, but do not substitute for the degree ≥ 2 covers.

Peyre has another proposal, the "all the heights" approach, which should suffice to substitute for the degree ≥ 2 thin set.

It's possible that combining the two gives a good alternative formulation of Manin's conjecture.