The Grothendieck ring of varieties

k field

$$K_0(Var_k) = \mathbb{Z}\langle \text{iso classes of alg vars}/k \rangle \Big/ X - Z - U$$

$(X \text{ var}/k,\ Z \hookrightarrow X$
$U = X \setminus Z)$

Product: $[X].[Y] = [X \times Y]$ $\quad 1 = [\operatorname{Spec} k]$

Element $\mathbb{L}$: $\mathbb{L} = [\mathbb{A}^1_k]$

ex: $[\mathbb{A}^n_k] = \mathbb{L}^n$

$$[\mathbb{P}^n_k] = \mathbb{L}^n + \mathbb{L}^{n-1} + \dots + \mathbb{L} + 1$$

Localised Grothendieck ring:

$$\mathcal{M}_k = K_0(Var_k)[\mathbb{L}^{-1}]$$

($\mathbb{L}$ is a zero divisor in $K_0(Var_k)$, Borisov 2014)

Dimensional filtration:

$\operatorname{Fil}^d \mathcal{M}_k$ = subgroup of $\mathcal{M}_k$ generated by elements

$$[X]\mathbb{L}^{-n}, \qquad \dim X - n \leq -d$$

Decreasing, exhaustive filtration $\mathcal{M}_k$

Completion: $\widehat{\mathcal{M}_k} = \varprojlim \mathcal{M}_k / \operatorname{Fil}^d \mathcal{M}_k$

Ex: $\sum_{i\geq 0} \mathbb{L}^{-i}$ convergent series

$$\sum_{i\geq 0} \mathbb{L}^{-i} = \frac{1}{1-\mathbb{L}^{-1}}$$

Motivic measures

Euler characteristic

$$\begin{aligned} K_0(Var_{\mathbb{C}}) &\to \mathbb{Z} \\ X &\mapsto \sum_{i=0}^{2\dim X} (-1)^i H_c^i(X(\mathbb{C}); \mathbb{Q}) \end{aligned}$$

Counting measure

$k = \mathbb{F}_q$

$$\begin{aligned} \#_{\mathbb{F}_q} : K_0(Var_{\mathbb{F}_q})[\mathbb{L}^{-1}] &\to \mathbb{Z}[q^{-1}] \\ X &\mapsto |X(\mathbb{F}_q)| \end{aligned}$$

Zeta functions:

X var $/\mathbb{F}_q$

$$Z_X(t) = \exp\left(\sum_{n\geq 1} \frac{|X(\mathbb{F}_{q^n})|}{n} t^n\right)$$

$$= \sum_{n\geq 0} \left| \left\{ \begin{array}{c} \text{effective zero-cycles} \\ \text{of degree } n \text{ on } X \end{array} \right\} \right| t^n$$

$\sum_{x\in X_{cl}} n_x x \quad n_x \geq 0$ s.t $\sum n_x \deg(x) = n$

$= |S^n X(\mathbb{F}_q)|$

$S^n X := X^n / S_n$ n^{th} symmetric power of X

X var $/ k$ quasi-proj

$$Z_X^{Kap}(t) = \sum_{n \geq 0} [S^n X] \, t^n \in K_0(Var_k)[[t]]$$

Kapranov's zeta function

Rem: $X / \mathbb{F}_q$ $\quad \#_{\mathbb{F}_q} Z_X^{Kap}(t) = Z_X(t)$

Weil conjectures: $Z_X(t)$ rational $\quad n = \dim X \quad X$ sm proj

$$Z_X(t) = \frac{P_1(t) \cdots P_{2n-1}(t)}{P_0(t) \cdots P_{2n}(t)} \qquad P_i(t) \in \mathbb{Z}[t]$$

$$P_0(t) = 1 - t \qquad P_{2n}(T) = 1 - q^n t$$

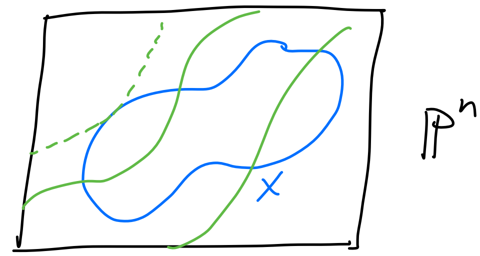
$$P_i(t) = \prod_j (1 - \alpha_{ij} t) \qquad 1 \leq i \leq 2n-1$$

$$|\alpha_{ij}| = q^{i/2}$$

(theory of weights of Deligne)

Bertini theorems:

$X \subset \mathbb{P}^n$ smooth proj alg variety



$$\underset{\substack{\| \\ \text{open subset} \\ \text{of hypersurfaces} \\ \text{intersecting } X \text{ transversely}}}{U_d} \subset \underbrace{\Gamma(\mathbb{P}^n, \mathcal{O}(d))}_{\text{homogeneous degree-}d\text{ polys in } n+1 \text{ variables}}$$

Quest°: What is the "density" of U_d ?

Thm: (Poonen '04) $k = \mathbb{F}_q$

$$\frac{|U_d(\mathbb{F}_q)|}{|\Gamma(\mathbb{P}^n, \mathcal{O}(d))(\mathbb{F}_q)|} \xrightarrow[d\to\infty]{} Z_X\left(q^{-\dim X - 1}\right)^{-1}$$

Thm: (Vakil-Wood '15) $\widehat{\mathcal{M}_k}$ ⚠ modified Groth ring

$$\frac{[U_d]}{[\Gamma(\mathbb{P}^n, \mathcal{O}(d))]} \xrightarrow[\substack{d\to\infty \\ (\text{dim. top.})}]{} Z_X^{Kap}\left(\mathbb{L}^{-\dim X - 1}\right)^{-1}$$

vs $= [U_d]\, \mathbb{L}^{-\dim \Gamma(\mathbb{P}^n, \mathcal{O}(d))}$

Problem: $\#_{\mathbb{F}_q}$ is not continuous w.r.t dim. top.

$$\left(\text{ex}: \quad q^{2n}\, \mathbb{L}^{-n} \xrightarrow[\text{dim.top.}]{} 0 \qquad \text{but} \quad \#_{\mathbb{F}_q}\left(q^{2n}\, \mathbb{L}^{-n}\right) = q^n \to \infty \right)$$

Question: $(a_n)_n$ seq of elements of $\mathcal{M}_{\mathbb{F}_q}$
How to compare:

1. $a_n \xrightarrow[\text{dim.top}]{} \text{limit } (\in \widehat{\mathcal{M}_k})$

2. $\#_{\mathbb{F}_q}(a_n) \longrightarrow \text{limit } (\in \mathbb{R})$

From now on: k finite, $= \mathbb{F}_q$.

Slogan: pass to zeta functions!
apply the zeta measure

$$\begin{aligned}
\text{zeta}: \mathcal{M}_k &\longrightarrow R_1 = \{ f \in \mathbb{C}(t),\ f(0) = 1 \} \\
X &\longmapsto Z_X(t) \\
\mathbb{L} &\longmapsto \frac{1}{1-qt} \\
\mathbb{L}^n &\longmapsto \frac{1}{1-q^n t}
\end{aligned}$$

Ring structure:

$$\begin{aligned}
R_1 &\longrightarrow \mathbb{Z}[\mathbb{C}^\times] \\
f = \prod_a (1-at)^{-k_a} &\longmapsto \sum_a k_a [a] \qquad (-\operatorname{div} f(t^{-1})) \\
k_a \in \mathbb{Z} \qquad \frac{1}{1-qt} &\longmapsto [q]
\end{aligned}$$

Topologies on R_1:

- coefficient top:

$$R_1 \xrightarrow[\text{power series exp at } 0]{} 1 + t\,\mathbb{C}[[t]] \simeq \mathbb{C}^{\mathbb{N}} \text{ with product top}$$

- weight top:

$$\Big\| \sum_a k_a [a] \Big\|_\infty = \sup_{k_a \neq 0} |a|$$

- Hadamard top:

$$\Big\| \sum_a k_a [a] \Big\|_H = \sum_a |k_a|\,|a|$$

Hadamard norm

incompatible

Refines two previous topologies

Prop: the completion of R_1 w.r.t. $\|\cdot\|_H$ is:

$$\left\{ \sum_a k_a [a] \ (\text{discretely supported}) \text{ s.t } \sum_a |k_a|\,|a| < \infty \right\}$$

$$\ni \sum k_a [a] \longmapsto \prod (1-at)^{-k_a}$$

Hadamard factorisation $\Big\downarrow S$

$$\mathcal{H}_1 = \left\{ \frac{f}{g} \quad f, g \text{ entire functions of genus } 0,\ f(0) = g(0) = 1 \right\}$$

space of Hadamard functions

Principle: if $a_n \in \mathcal{M}_{\mathbb{F}_q}$ a "natural" sequence of classes s.t

$$Z_{a_n}(t) \longrightarrow f \in \mathcal{H}_1$$

both in the coeff and in the weight topology, then it converges also in the Hadamard top.

Back to Bertini:

Conjecture: $X \subset \mathbb{P}^n_{\mathbb{F}_q}$ sm proj variety

$$U_d \subset \Gamma(\mathbb{P}^n, \mathcal{O}(d))$$

hypersurfaces intersecting X transversely

$$Z_{U_d}\left(q^{-\dim \Gamma(\mathbb{P}^n, \mathcal{O}(d))} t\right) \xrightarrow[\text{Hadamard top}]{} Z^{\mathrm{Kap}}_{X, \mathrm{zeta}}\left(\mathbb{L}_{\mathrm{zeta}}^{-\dim X - 1}\right)^{-1}$$

Pattern-avoiding zero-cycles

Motivation from number thy:

$a_1, \dots, a_m$ are relatively n-prime if there does not exist $b \geq 2$ s.t $\forall i$ $b^n | a_i$

($\Leftrightarrow$ $\gcd(a_1, \dots, a_m)$ n^{th} power free)

Thm: Given $m, n \geq 1$

$$\lim_{d \to \infty} \frac{\# \{ (a_1, \dots, a_m) \in \{1, \dots, d\}^m, \ a_1, \dots, a_m \text{ rel } n\text{-prime} \}}{d^m}$$

exists, and equals $\zeta(mn)^{-1}$.

Zero-cycles: X quasi-proj $/ k$

Def: $X_n^{(d_1, \dots, d_m)} = \left\{ \begin{array}{l} (C_1, \dots, C_m) \in \mathrm{Sym}^{d_1} X \times \dots \times \mathrm{Sym}^{d_m} X \\ \text{s.t } \forall x \in X, \text{ multiplicities of } x \text{ in} \\ C_1, \dots, C_m \text{ are } \underline{\text{not}} \text{ all } \geq n \end{array} \right\}$

Obs: $X_2^d = \mathrm{Conf}^d X \subset \mathrm{Sym}^d X$

Thm: $k = \mathbb{F}_q$

$$\lim_{d_1, \dots, d_m \to \infty} \frac{[X_n^{(d_1, \dots, d_m)}]_{zeta}}{[\mathrm{Sym}^{d_1} X \times \dots \times \mathrm{Sym}^{d_m} X]_{zeta}} \underset{\substack{\text{Hadamard} \\ \text{top}}}{=} Z^{Kap}_{X, zeta} \left(\mathbb{L}_{zeta}^{-mn \dim X} \right)^{-1}$$

Remarks:

- More generally, works for motivic measure

$$\varphi : \mathcal{M}_k \to R \qquad \text{(normed rg)}$$

satisfying some natural conditions.

- lifts / generalizes results by
  Farb - Wolfson - Wood 2019

- Proof: generating function argument

$$\frac{\sum_{\underline{d}} [X_n^{\underline{d}}]\, \underline{t}^{\underline{d}}}{\sum_{\underline{d}} [\mathrm{Sym}^{\underline{d}} X]\, \underline{t}^{\underline{d}}} = Z_X^{Kap}\left((t_1 \cdots t_m)^n\right)^{-1}$$