Isomonodromic deformations: confluence, reduction & quantisation

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Isomonodromic deformations of systems of rank *m* with poles of any order on $\mathbb{P}^1$

- Algebraic approach to “irregular times”
- Hamiltonian formulation in terms of flows on co-adjoint orbits
- Confluence
- Quantisation
- Universality and τ-functions

Fuchsian case

$$\frac{dY}{d\lambda} = A(\lambda)Y, \qquad A(\lambda) = \sum_{i=1}^{n} \frac{A^{(i)}}{\lambda - u_i},$$

Isomonodromic deformations

$$dA^{(i)} = \sum_{i \neq j} [A^{(i)}, A^{(j)}] \frac{du_i - du_j}{u_i - u_j},$$

Phase space:

$$\mathcal{O}_1^* \times \mathcal{O}_2^* \times \dots \mathcal{O}_n^* \times \mathcal{O}_\infty^*, \qquad \left\{ A^{(i)} \underset{,}{\otimes} A^{(j)} \right\} = \delta_{ij} \left[A^{(i)} \otimes \mathbb{I}, \Pi \right]$$

Hamiltonians

$$H^{(i)} = \frac{1}{2} \operatorname{Res}_{\lambda = u_i} \operatorname{Tr} \left(A(\lambda)^2 \right),$$

Quantise to KZ for $\mathfrak{g} = \mathfrak{gl}_m$

$$d\psi = \sum_{i \neq j} \Pi^{ij} \frac{du_i - du_j}{u_i - u_j} \psi, \qquad B \times U(\mathfrak{gl}_m(\mathbb{C}))^{\otimes n} \to B,$$

$\Pi^{ij} \in \operatorname{End}(U(\mathfrak{gl}_m(\mathbb{C})^{\otimes n})$ [Reshetikhin '92]

Irregular case

$$\frac{dY}{d\lambda} = A(\lambda)Y, \qquad A(\lambda) = \sum_{i=1}^{n}\sum_{k=0}^{r_i} \frac{A_k^{(i)}}{(\lambda - u_i)^{k+1}} + \sum_{k=1}^{r_\infty} A_k^{(\infty)}\lambda^{k-1},$$

Phase space:

$$\widehat{\mathcal{O}}^*_{r_1} \times \widehat{\mathcal{O}}^*_{r_2} \times \dots \widehat{\mathcal{O}}^*_{r_n} \times \widehat{\mathcal{O}}^*_{r_\infty}, \qquad \left\{A_k^{(i)} \overset{\otimes}{,} A_l^{(j)}\right\} = \begin{cases} \delta_{ij}[A_{k+l}^{(i)} \otimes \mathbb{I}, \Pi], & k+l \leq r_i \\ 0 & k+l > r_i \end{cases}$$

Hamiltonians? $\Leftarrow$ confluence

Confluent KZ?

State of the art

- Quantisation of Gaudin

$$dA^{(i)} = \sum_{i\neq j}[A^{(i)}, A^{(j)}]\frac{du_i - du_j}{\widetilde{u}_i - \widetilde{u}_i},$$

  - simple poles: Mukhin, Tarasov, Varchenko '06 (see also Talalaev '11)
  - non semisimple order two pole at infinity: Feigin, Frenkel, Rybnikov '10
  - generalisation of Gaudin corresponding to non-highest weight reps of any simple Lie algebra: Feigin, Frenkel, Toledano–Laredo '10
  - non semisimple with irregular singularities: Vicedo, Young '18
- Deform quantum Gaudin to give standard KZ:
  - A_n root system de Concini Procesi '95
  - any lie algebra: Millson, Toledano–Laredo '05, Felder, Markov, Tarasov, '00
- KZ equations with irregular singularities
  - irregular singular points of arbitrary Poincaré rank for $\mathfrak{sl}_2$: Jimbo, Nagoya and Sun '08
  - one irregular singularity of Poincaré rank 2 and several other simple poles: Rembado '19
  - Feigin, Frenkel, Toledano–Laredo quantum integrable systems are expected to give rise to *confluent KZ equations*- not been explicitly written.

Phase space

$$A(\lambda)=\sum_{i=1}^{n}\sum_{k=0}^{r_i}\frac{A_k^{(i)}}{(\lambda-u_i)^{k+1}}+\sum_{k=1}^{r_\infty}A_k^{(\infty)}\lambda^{k-1},$$

$$\sum_{k=0}^{r_i}\frac{A_k^{(i)}}{(\lambda-u_i)^{k+1}}\in\widehat{\mathcal{O}}^*_{r_i}\subset\hat{\mathfrak{g}}^\star_{r_i},\qquad \sum_{k=1}^{r_\infty}A_k^{(\infty)}z^{k-1}\in\widehat{\mathcal{O}}^*_{r_\infty}\subset\hat{\mathfrak{g}}^\star_{r_\infty}$$

for $z=\lambda-u_i$ or $z=\frac{1}{\lambda}$:

$$\hat{\mathfrak{g}}_r=\mathfrak{g}[z]^+/z^{r+1}\mathfrak{g}[z]^+,\quad \mathfrak{g}[z]^+\simeq\mathfrak{g}[[z]]\Big/\mathfrak{g}[z]^-,\quad \mathfrak{g}[z]^-=\left\{f\in\mathfrak{g}[[z]]:\ \lim_{z\to\infty}f(z)=0\right\}$$

Example

$$\frac{dY}{d\lambda}=\left(U+\frac{V}{\lambda}\right)Y,\qquad U\text{ diagonal, } V\in\mathfrak{so}_n.$$

$U+\frac{V}{\lambda}\notin\hat{\mathcal{O}}^\star_1\times\hat{\mathcal{O}}^\star_0\Rightarrow$ it is not in our phase space.

$$\frac{dY}{d\lambda}=\left(u\Lambda+\frac{V}{\lambda}\right)Y,\qquad \Lambda\text{ constant.}$$

Isomonodromic deformations in $\widehat{\mathcal{O}}^*_{r_1}\times\widehat{\mathcal{O}}^*_{r_2}\times\dots\widehat{\mathcal{O}}^*_{r_n}\times\widehat{\mathcal{O}}^*_{r_\infty}$

Theorem

$$\sum_{i=1}^n\sum_{k=0}^{r_i}\frac{A_k^{(i)}}{(\lambda-u_i)^{k+1}}+\sum_{k=1}^{r_\infty}A_k^{(\infty)}\lambda^{k-1},\quad \sum_{i=1}^n\sum_{k=0}^{r_i}\frac{B_k^{(i)}}{(\lambda-u_i)^{k+1}}+\sum_{k=1}^{r_\infty}B_k^{(\infty)}\lambda^{k-1}.$$

are related by a linear Poisson automorphism iff

$$B_k^{(i)}=\sum_{j=k}^{r_i}A_j^{(i)}\mathcal{M}_{k,j}^{(r_i)}(t_1^{(i)},t_2^{(i)},\dots t_{r_i}^{(i)}),$$

$$\mathcal{M}_{k,j}^{(r)}=\frac{1}{j!}\frac{d^j}{d\varepsilon^j}P_r^{(i)}(t^{(i)},\varepsilon)^k\Big|_{\varepsilon=0},\quad P_r^{(i)}(t^{(i)},\varepsilon)=\sum_{k=1}^{r}\varepsilon^k t_k^{(i)}.$$

Example

$$\hat{\mathcal{O}}_0^\star\times\hat{\mathcal{O}}_3^\star:\quad \frac{A_0^{(1)}}{\lambda-u_1}+t_1^3A_3^{(\infty)}\lambda^2+(t_1^2A_2^{(\infty)}+2t_1t_2A_3^{(\infty)})\lambda+t_1A_1^{(\infty)}+t_2A_2^{(\infty)}+t_3A_3^{(\infty)}$$

Isomonodromic deformations in $\widehat{\mathcal{O}}^*_{r_1} \times \widehat{\mathcal{O}}^*_{r_2} \times \dots \widehat{\mathcal{O}}^*_{r_n} \times \widehat{\mathcal{O}}^*_{r_\infty}$

$$A(\lambda) = \sum_{i=1}^{n} \sum_{k=0}^{r_i} \frac{\sum_{j=k}^{r_i} A_j^{(i)} \mathcal{M}_{k,j}^{(r_i)}(t_1^{(i)}, \dots t_{r_i}^{(i)})}{(\lambda - u_i)^{k+1}} + \sum_{k=1}^{r_\infty} \sum_{j=k}^{r_\infty} A_j^{(\infty)} \mathcal{M}_{k,j}^{(r_\infty)}(t_1^{(\infty)}, \dots t_{r_i}^{(\infty)}) \lambda^{k-1}$$

Theorem

The isomonodromic deformation equation is given by

$$\Omega_j^{(i)} = \int \frac{\partial A}{\partial t_j^{(i)}} d\lambda, \qquad \Omega_{u_j} = \int \frac{\partial A}{\partial u_j} d\lambda.$$

Examples

Example

$A(\lambda) \in \hat{\mathcal{O}}_1^\star \times \hat{\mathcal{O}}_1^\star$:

$$A = \frac{A_0^{(0)}}{\lambda} + \frac{t_1^{(0)} A_1^{(0)}}{\lambda^2} + t_1^\infty A_1^{(\infty)}$$

Fix $t_1^{(\infty)} = 1 \Rightarrow \Omega_1^{(0)} = -\frac{A_1^{(0)}}{\lambda}$.

Example

$A(\lambda) \in \mathcal{O}_0^\star \times \hat{\mathcal{O}}_3^\star$:

$$A = \frac{A_0^{(1)}}{\lambda - u} + \frac{t_1^3 A_3^{(0)}}{\lambda^4} + \frac{t_1^2 A_2^{(0)} + 2 t_1 t_2 A_3^{(0)}}{\lambda^3} + \frac{t_1 A_1^{(0)} + t_2 A_2^{(0)} + t_3 A_3^{(0)}}{\lambda^2} + \frac{A_0^{(0)}}{\lambda}$$

Fix $t_1 = 1$ and $t_2 = 0$, $\Omega_3^{(\infty)} = A_3^{(\infty)}\lambda$ and $\Omega_0^{(1)} = -\frac{A_0^{(1)}}{\lambda - u_1}$. $\Rightarrow$ PDE in u_1, t_3.

Confluence: $1+1$

$$\frac{dY}{d\lambda}=A(\lambda)Y,\qquad A(\lambda)=\sum_{i=1}^{n}\frac{A^{(i)}}{\lambda-u_i},$$

Confluence u_{n-1} and u_n:

$$u_n=w,\qquad u_{n-1}=w+\varepsilon t_1.$$

$$A(\lambda)=\sum_{i=1}^{n-2}\frac{A^{(i)}}{\lambda-u_i}+\frac{B}{\lambda-w}+\frac{C}{\lambda-w-\varepsilon t_1},\quad B=A^{(n-1)},\quad C=A^{(n)}$$

$$\frac{B}{\lambda-w}+\frac{C}{\lambda-w-\varepsilon t_1}\sim\frac{C+B}{\lambda-w}+\frac{\varepsilon t_1}{(\lambda-w)^2}C+O(\varepsilon^2).$$

$$C=\frac{1}{\varepsilon}A_1^{(n-1)}+C_0+O(\varepsilon),\quad B=-\frac{1}{\varepsilon}A_1^{(n-1)}+B_0+O(\varepsilon),\quad C_0+B_0=A_0^{(n-1)}.$$

$$\tilde{A}(\lambda):=\lim_{\varepsilon\to 0}A(\lambda)=\sum_{i=1}^{n-2}\frac{A^{(i)}}{\lambda-u_i}+t_1\frac{A_1^{(n-1)}}{(\lambda-w)^2}+\frac{A_0^{(n-1)}}{\lambda-w}$$

Confluence: $1+1$

Theorem

The 1+1 confluence procedure gives a Poisson morphism:

$$\mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \dots \mathcal{O}_n^\star \times \mathcal{O}_\infty^\star \xrightarrow{\text{confluence}} \mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \dots \mathcal{O}_{n-2}^\star \times \hat{\mathcal{O}}_{2,n-1}^\star \times \mathcal{O}_\infty^\star.$$

Namely, if $A^{(i)}$, B, C satisfy the the standard Lie Poisson brackets then

$$\left\{A_{1,\alpha}^{(n-2)}, A_{1,\beta}^{(n-2)}\right\} = 0, \quad \left\{A_{1,\alpha}^{(n-2)}, A_{0,\beta}^{(n-2)}\right\} = -\chi_{\alpha\beta}^{\gamma} A_{1,\gamma}^{(n-2)},$$
$$\left\{A_{0,\alpha}^{(n-2)}, A_{0,\beta}^{(n-2)}\right\} = -\chi_{\alpha\beta}^{\gamma} \left(A_{0,\gamma}^{(n-2)}\right).$$

Confluence: $r+1$

Theorem

Consider an r–parameter family of connections of the following form:

$$A=\sum_{k=0}^{r}\frac{B_k(t_1,t_2\dots t_{r-1})}{(\lambda-u)^{k+1}}+\frac{C}{\lambda-v}+\dots \tag{1}$$

Assume

$$v=u+\sum_{i=1}^{r}t_i\varepsilon^i=u+P_r(t,\varepsilon), \tag{2}$$

then

$$\tilde{A}=\sum_{i=0}^{r+1}\frac{\tilde{B}_i(t_1,t_2\dots t_r,t_{r+1})}{(\lambda-u)^{i+1}}+\textit{holomorphic terms},$$

where $\tilde{B}_i$'s are given by

$$\tilde{B}_i(t_1\dots,t_{r+1})=\sum_{k=i}^{r}\tilde{A}_k\mathcal{M}_{i,k}^{(r+1)}(t_1\dots t_{r+1}). \tag{3}$$

Hamiltonians

Theorem

The confluent Hamiltonians $H_1^{(i)},\dots,H_{r_i}^{(i)}$ which correspond to the times $t_1^{(i)},\dots t_{r_i}^{(i)}$ are defined as follows:

$$\begin{pmatrix} H_1^{(i)}\\ H_2^{(i)}\\ \dots\\ H_{r_i}^{(i)}\end{pmatrix} = \left(\mathcal{M}^{(r)}\right)^{-1}\begin{pmatrix} S_1^{(i)}\\ S_2^{(i)}\\ \dots\\ S_{r_i}^{(i)}\end{pmatrix}, \tag{4}$$

where

$$S_k^{(i)} = \frac{1}{2}\oint_{\Gamma_{u_i}} (\lambda-u_i)^k \mathrm{Tr} A^2 \mathrm{d}\lambda \tag{5}$$

The Hamiltonian H_u corresponding to the time u is instead given by the standard formula

$$H_{u_i} = \frac{1}{2}\mathop{\mathrm{Res}}_{\lambda=u_i}\mathrm{Tr} A(\lambda)^2.$$

Examples

Example

$A(\lambda) \in \hat{\mathcal{O}}_1^\star \times \hat{\mathcal{O}}_1^\star$ (fix $t_1^{(\infty)} = 1$): $\quad A(\lambda) = \frac{A_0^{(0)}}{\lambda} + \frac{t_1^{(0)} A_1^{(0)}}{\lambda^2} + A_1^{(\infty)}$

$$H = \mathrm{Tr}\left(\frac{(A_0^{(0)})^2}{2 t_1^{(0)}} + t_1^{(\infty)} A_0^{(1)} A_1^{(\infty)} \right).$$

Example

$A(\lambda) \in \mathcal{O}_0^\star \times \hat{\mathcal{O}}_3^\star$ (fix $t_1 = 1$ and $t_2 = 0$)

$$A = \frac{A_0^{(1)}}{\lambda - u} + \frac{A_3^{(0)}}{\lambda^4} + \frac{A_2^{(0)}}{\lambda^3} + \frac{A_1^{(0)} + t_3 A_3^{(0)}}{\lambda^2} + \frac{A_0^{(0)}}{\lambda}$$

$$H_u = \mathrm{Tr} A_0^{(1)} \left(\frac{A_3^{(0)}}{u^4} + \frac{A_2^{(0)}}{u^3} + \frac{A_1^{(0)} + t_3 A_3^{(0)}}{u^2} + \frac{A_0^{(0)}}{u} \right)$$

$$H_3 = \mathrm{Tr} \left(-\frac{A_0^{(1)} A_3^{(0)}}{u} + \frac{2 A_0^{(0)} A_2^{(0)} + \left(A_1^{(0)} \right)^2}{2} + t_3 A_1^{(0)} A_3^{(0)} \right)$$

Quantisation

Theorem

The confluent KZ Hamiltonians $\widehat{H}_{u_i}$, and $\widehat{H}_1^{(i)},\dots,\widehat{H}_r^{(i)}$ are the following elements of the universal enveloping algebra $U\left(\hat{\mathfrak{g}}_{r_1}\oplus\cdots\oplus\hat{\mathfrak{g}}_{r_\infty}\right)$:

$$\hat{H}_{u_i}=\frac{1}{2}\operatorname*{Res}_{\lambda=u_i}\mathrm{Tr}_0\hat{A}(\lambda)^2,$$

$$\mathcal{M}^{(r_i)}\begin{pmatrix}\hat{H}_1^{(i)}\\ \hat{H}_2^{(i)}\\ \dots\\ \hat{H}_{r_i}^{(i)}\end{pmatrix}=\begin{pmatrix}\hat{S}_1^{(u_i)}\\ \hat{S}_2^{(u_i)}\\ \dots\\ \hat{S}_{r_i}^{(u_i)}\end{pmatrix},\qquad \hat{S}_k=\frac{1}{2}\oint_{\Gamma_{u_i}}(\lambda-u_i)^k\mathrm{Tr}_0\hat{A}(\lambda)^2\mathrm{d}\lambda,$$

where

$$\hat{A}(\lambda)=\sum_i^n\left(\sum_{j=0}^{r_i}\frac{\hat{B}_j^{(i)}\left(t_1^{(i)},t_2^{(i)}\dots t_{r_i}^{(i)}\right)}{(\lambda-u_i)^{j+1}}\right),$$

$$\hat{B}_j^{(i)}(t_1^{(i)},\dots t_{r_i}^{(i)})=\sum_{k=j}^{r}\hat{A}_k^{(i)}\mathcal{M}_{j,k}^{(r_i)}(t_1^{(i)},t_2^{(i)}\dots t_{r_i}^{(i)}),\quad \hat{A}_k=\sum_\alpha e_\alpha^{(0)}\otimes e_\alpha^{(i)}\otimes z_i^k.$$

Universal phase space

$$A(\lambda)=\sum_{i=1}^{n}\sum_{k=0}^{r_i}\frac{A_k^{(i)}}{(\lambda-u_i)^{k+1}}+\sum_{k=1}^{r_\infty}A_k^{(\infty)}\lambda^{k-1}\in\mathcal{O}_1^*\times\mathcal{O}_2^*\times\dots\mathcal{O}_n^*\times\mathcal{O}_\infty^*$$

Marsden–Weinstein reduction of the standard Poisson structure on

$$\oplus_{i=1}^{n+1}(T^\star\mathfrak{gl}_m)^{r_i+1}=\oplus_{k=1}^{d}T^\star\mathfrak{gl}_m,\qquad d=\sum_{i=1}^{n+1}r_i+n+1$$

w.r.t. the *inner group action:*

$$\forall(g^{(1)},\dots,g^{(\infty)})\in\otimes_{i=1}^{n+1}G,\quad g^{(i)}\underset{\text{inner}}{\times}(P_k^{(i)},Q_k^{(i)})=(g^{(i)}P_k^{(i)},Q_k^{(i)}(g^{(i)})^{-1}),$$

where we understand $P_k^{(i)},Q_k^{(i)}$ as lying in the $r_1+\dots+r_{i-1}+i+k$ copy of $T^\star\mathfrak{gl}_m$.

Universal phase space

Example

$A(\lambda) \in \mathcal{O}^\star \times \hat{\mathcal{O}}_3^\star$, $d = 1 + 4 = 5$,

$$(\underbrace{P_1, Q_1}_{u}, \underbrace{P_2, Q_2, \dots, P_5, Q_5}_{0}) \in \oplus_{k=1}^5 T^\star \mathfrak{gl}_m$$

$$A = \frac{A_0^{(1)}}{\lambda - u} + \frac{A_3^{(0)}}{\lambda^4} + \frac{A_2^{(0)}}{\lambda^3} + \frac{A_1^{(0)} + t_3 A_3^{(0)}}{\lambda^2} + \frac{A_0^{(0)}}{\lambda}$$

$$A_0^{(1)} = Q_1 P_1, \quad A_3^{(0)} = Q_2 P_5, \quad A_2^{(0)} = Q_2 P_4 + Q_3 P_5,$$

$$A_1^{(0)} = Q_2 P_3 + Q_3 P_4 + Q_4 P_5, \quad A_0^{(0)} = Q_2 P_2 + Q_3 P_3 + Q_4 P_4 + Q_5 P_5$$

Moment map w.r.t. $(g^{(1)}, g^{(0)}) \in SL_m \otimes SL_m$:

$$\Lambda_0^{(1)} = P_1 Q_1, \quad \Lambda_3^{(0)} = P_5 Q_2, \quad \Lambda_2^{(0)} = P_4 Q_2 + P_5 Q_3,$$

$$\Lambda_1^{(0)} = P_3 Q_2 + P_4 Q_3 + P_5 Q_3, \quad \Lambda_0^{(0)} = P_2 Q_2 + P_3 Q_3 + P_4 Q_4 + P_5 Q_5$$

Universal phase space

Example

Hamiltonians

$$H_u = \mathrm{Tr} A_0^{(1)} \left(\frac{A_3^{(0)}}{u^4} + \frac{A_2^{(0)}}{u^3} + \frac{A_1^{(0)} + t_3 A_3^{(0)}}{u^2} + \frac{A_0^{(0)}}{u} \right)$$

$$H_3 = \mathrm{Tr} \left(-\frac{A_0^{(1)} A_3^{(0)}}{u} + \frac{2A_0^{(0)} A_2^{(0)} + \left(A_1^{(0)}\right)^2}{2} + t_3 A_1^{(0)} A_3^{(0)} \right)$$

$$A_0^{(1)} = Q_1 P_1, \quad A_3^{(0)} = Q_2 P_5, \quad A_2^{(0)} = Q_2 P_4 + Q_3 P_5,$$

$$A_1^{(0)} = Q_2 P_3 + Q_3 P_4 + Q_4 P_5, \quad A_0^{(0)} = Q_2 P_2 + Q_3 P_3 + Q_4 P_4 + Q_5 P_5$$

The Hamiltonians are homogeneous in P, Q of degree 2.

Isomonodromic τ-function

$$\mathrm{d}\ln(\tau) = \sum_i \left(H^{(i)}_{u_i} \mathrm{d}u_i + \sum_{k=1}^{r_i} H^{(i)}_k \mathrm{d}t^{(i)}_k \right).$$

Lemma

Homogeneous degree 2 $\Rightarrow \sum_{k=1}^{d} P_k \mathrm{d}Q_k = 2 \sum_i \left(H^{(i)}_{u_i} \mathrm{d}u_i + \sum_{k=1}^{r_i} H^{(i)}_k \mathrm{d}t^{(i)}_k \right)$

$\Rightarrow$

$$\mathrm{d}S = \sum_{k=1}^{d} P_k \mathrm{d}Q_k - \sum_i \left(H^{(i)}_{u_i} \mathrm{d}u_i + \sum_{k=1}^{r_i} H^{(i)}_k \mathrm{d}t^{(i)}_k \right) = \mathrm{d}\ln(\tau).$$

Theorem

The semi-classical solution of the confluent KZ system $\Psi \sim \exp\left(\frac{i}{\hbar}\mathcal{S}\right)$, *evaluated along solutions of the classical isomonodromic Hamiltonians is given by the isomonodromic* τ*-function:*

$$\Psi(Q(t), t) \sim \tau^{\frac{i}{\hbar}}.$$