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29.03.2021

PART I. RATIONAL POINTS ON CUBIC SURFACES AND MOUFANG LOOPS

PART II. DISTRIBUTION OF RATIONAL POINTS ON ALGEBRAIC VARIETIES

PART I. INTRODUCTION

First Scene:
An open place. Thunder and lightning.
Enter three witches.

Shakespeare. Macbeth, Act I.

This epigraph was introducing the reader to the Chapter I of the book

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where a problem was stated (Problem 11.11) , that was remaining unsolved for more than 50 years.

PART I. INTRODUCTION

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where a problem was stated (Problem 11.11) , that was remaining unsolved for more than 50 years.

The first part of this talk is dedicated to the recent breakthrough: the solution of this problem by Dimitri Kanevsky. It is based on his article (as yet unpublished)

[Ka21] Dimitri Kanevsky. *An example of a non–associative Moufang loop of point classes on a cubic surface.* **Preprint 2021.**

The second part puts his constructions in a wider framework.

I will start with a brief presentation of basic definitions and problems.

THREE WITCHES: COMMUTATIVE MOUFANG LOOPS

• **DEFINITION 1.** *A symmetric quasigroup is a set E with binary composition law $\circ : E \times E \to E$ satisfying the following condition:*

The triple relation $L(x, y, z) : x \circ y = z$ is $\mathbf{S}_3$–invariant.

• **EXAMPLE 1. Consider an irreducible cubic curve in a projective plane over a field k, and let E be the set of its non–singular points.**

Then the triple relation $L(x, y, z)$: *(x, y, z) are collinear* **(taking in account multiplicities of intersections) defines on E a structure of symmetric quasigroup.**

• **(NON-)EXAMPLE 2. Consider now an irreducible cubic hypersurface of dimension ≥ 2 in a projective space over k. One can try define the triple collinearity relation in the same way as above, but then we will have to consider some points in members of such a relation with** *infinite multiplicity* **.**

The following trick will allow us to (partially) extend the statement of Example 1 in multidimension case.

• **DEFINITION 2.** *A symmetric quasigroup E is called Abelian one, if for any element $u \in E$ the composition law $xy := u \circ (x \circ y)$ turns E into an Abelian group.*

• **PROPOSITION 1.** *In the setup of (non-)Example 2, the collinearity relation $L(x, y, z)$ induces the structure of abelian quasigroup on any subset of E, consisting of non–singular k–points of an irreducible intersection of the hypersurface with plane.*

Generalising this remark, we will introduce our next Definition.

• **DEFINITION 3. Let E be a symmetric quasigroup. It is called a CH–quasigroup, if any three elements of E generate in it an Abelian subquasigroup.**

(Of course, "CH" comes from "Cubic Hypersurface")

• **THEOREM 0.** *Let* $(E, \circ)$ *be a symmetric quasigroup. For any* $x \in E$*, denote by* $t_x : E \to E$ *the map* $E \to E$ *sending each* y *to* $x \circ y$*. Then the following properties are equivalent:*

(i) $(E, \circ)$ *is an Abelian symmetric quasigroup.*

(ii) The group $T^0(E)$ *with its natural left action upon* E*, generated by products of even number of reflections* t_x *is abelian.*

(iii) For any $x, y, z \in E$*, we have* $(t_x t_y t_z)^2 = 1$ *(identical map* $E \to E$*).*

(iv) For any $u \in E$*, the composition law* $xy := u \circ (x \circ y)$ *defines on* E *the structure of abelian group with identity* u*.*

(v) The same as (iv) for some element u*.*

In [Ma86], the proof of this Theorem takes pages 12–13. It is organised as a series of implications

$$(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (i)$$

and

$$(ii) \Longrightarrow (iv) \Longrightarrow (v) \Longrightarrow (i).$$

• **SKETCH OF PROOF.** <u>(i) $\Longrightarrow$ (ii)</u> **: In this case** $x \circ y = ux^{-1}y^{-1}$**, and** $t_x t_y(z) = (x^{-1}y)z$ **in terms of an abelian group structure on** E**. Therefore product of any even number of reflections is a group translation, and they commute.**

<u>(ii) $\Longrightarrow$ (iii)</u> **: Since** $T^0(E)$ **is abelian, pairwise products of reflections commute, hence**

$$(t_x t_y t_z)^2 = t_z t_y (t_z t_x) t_y t_z = (t_z t_x) t_z t_y t_y t_z = 1.$$

<u>(iii) $\Longrightarrow$ (i)</u> **: First of all, accepting (iii) one easily sees, that for any** $x, y, z \in E$ **we have**

$$t_x t_y t_z = t_{y \circ (x \circ z)}.$$

Now, again accepting (iii), consider the map $E \to T^0(E) : x \mapsto \overline{x} := t_u t_x$**. Obviously, it is an embedding. To check that it is surjective, one checks that**

$$t_x t_y = \overline{x \circ (u \circ y)}.$$

Finally, commutativity of $T^0(E)$ **from a combination of these identities.**

By that time, we established that (i), (ii), and (iii) are equivalent, so in place of the implication $(ii) \Longrightarrow (iv)$**, we may check the following two:**

<u>**(iv)** $\Longrightarrow$ **(v)**</u> **: obvious.**

<u>**(v)** $\Longrightarrow$ **(i)**</u> **: by assumption,** $xy := u \circ (x \circ y)$ **is the composition law of an Abelian group.**

We have $(xy)(x \circ y) = u \circ [(u \circ (x \circ y)) \circ (x \circ y)] = u \circ u$**. This implies commutativity of the quasigroup** E **via intermediate step** $x \circ y = (u \circ u)x^{-1}y^{-1}$**.**

The last step of Proof follows from the remark, made during the check that $(i) \Longrightarrow (ii)$**.**

• **DEFINITION 4. (i) A** *Commutative Moufang Loop* **(CML for brevity) is a set** E **with commutative binary composition law** $\star : E \times E \to E : (x,y) \mapsto x \star y$**, having identity** $u \in E$ **and left inverse map** $E \to E : x \mapsto x^{-1}$**, satisfying the following additional identities (called "weak associativity" in "Cubic Forms", Def. 1.4):**

$$x \star (x \star y) = (x \star x) \star y, \quad (x \star y) \star (x \star z) = (x \star x) \star (y \star z),$$

$$(x \star y) \star (x \star z) = ((x \star x) \star y) \star z.$$

A $\star$**–product of** $n \geq 1$ **copies of one element** $x \in E$ **will be denoted simply** x^n**, because it does not depend on positions of brackets. Similarly, we denote by** x^n **the inverse element to** x^{-n}**, if** $n < 0$**.**

(ii) A CML E **is called** *non–associative* **one, if there exists a triple of its elements** (x,y,z) **such that** $x \star (y \star z) \neq (x \star y) \star z$**.**

(iii) *Associative centre* $Z(E)$ **of a CML** $(E, \star)$ **is the subset**

$$Z(E) := \{x \in E \,|\, x \star (y \star z) = (x \star y) \star z \; for\, all \; y, z \in E\}.$$

The associative centre of a CML E is an associative subloop (and therefore an abelian group). The quotient loop of CML w.r.t. its centre is a CML of exponent 3: for any $x \in E/Z(E)$ we have $x^3 = 1$.

Using modern language, one could say that, what we will be studying from now on, are *Symmetric Quasigroup Operad, Moufang Operad,* **and** *Algebras* **over them.**

Computations, shown on several pages further on, actually prove some operadic identities, expressed in terms of monomials in generic operadic algebras.

FROM CH–QUASIGROUPS TO CMLs AND BACK

• **THEOREM 1.** *Let $(E, \circ, u)$ be a CH–quasigroup, endowed with some fixed element $u \in E$. Then E with composition law $(x, y) \mapsto x \star y := u \circ (x \circ y)$ is CML with identity u. For different choices of u, we get isomorphic CML's.*

• **THEOREM 2.** *Let $(E, \star, c)$ be a CML, endowed with some fixed element of its associative center $c \in Z(E)$. Then the composition law $x \circ y := cx^{-1} \star y^{-1}$ turns E into CH–quasigroup, denoted E_c.*

How these constructions can be extended from basic structures to morphisms between them, is described in the following Theorem. We start with two triples $(E, \star, c)$ and $(F, \star, d)$, as in Theorem 2.

The common notation $\star$ for multiplication laws in two different CML's should not lead to a misunderstanding.

• **THEOREM 3.** *Let $f: E_c \to F_d$ be a morphism of quasigroups, as above, and* 1 *denotes the identity in E_c. Then $d = f(c)f(1)^2$, and the map $g = f(1)^{-1}f : E \to F$ is a CML–morphism.*

Conversely, let $g : E \to F$ be a morphism of two CMLs, and $c \in E; d, b \in F$ are such three elements that $g(c) = db^3$. Then the map $f := b^{-1}g$ is a morphism of CH–quasigroups.

• **SKETCH OF PROOF. If $f : E_c \to E_d$ is a morphism of quasigroups, then $f(cx^{-1} \star y^{-1}) = df(x)^{-1} \star f(y)^{-1}$.**

Substituting here $x = y = 1$, we get $f(c) = df(1)^{-2}$. Similar substitutions show that $f(y)^{-1} = f(1)^{-2}f(y^{-1})$, $f(x)^{-1} = d^{-1}f(1)f(cx^{-1})$, etc., and then, changing variables and using the last "weak associativity" formula (Def. 4(i)), we come to

$$g(xy) = f(1)^{-1}f(xy) = f(1)^{-2}(f(x)f(y)) = g(x)g(y).$$

Conversely, consider a morphism of CMLs $g : E_c \to F_d$**, and an element** $b \in F$**, such that** $g(c) = db^3$**.**

We must check that $f(cx^{-1}y^{-1}) = df(x)^{-1}f(y)^{-1}$**.**

We have

$$f(cx^{-1}y^{-1}) = b^{-1}g(cx^{-1}y^{-1}) = b^{-1}((db^3)(g(x)^{-1}g(y)^{-1})).$$

Using the fact that d **belongs to the centre of** F_d**, we deduce from here**

$$f(cx^{-1}y^{-1}) = d(b^2(g(x)^{-1}g(y)^{-1})).$$

Now again, because of weak associativity (Def. 4),

$$df(x)^{-1}f(y)^{-1} = d(bg(x)^{-1})(bg(y)^{-1}) = d(b^2(g(x)^{-1}g(y)^{-1}). \qquad \blacksquare$$

MOUFANG LOOPS FROM CUBIC HYPERSURFACES

• **COLLINEARITY. Let k be a field, V a cubic hypersurface in $\mathbf{P}^{r+1}$, $r \geq 2$, over k.**

A family of three k–points $P_1, P_2, P_3 \in V(k)$ is called *collinear one*, **if all points lie on a projective line $\mathbf{L} \subset \mathbf{P}^{r+1}$ defined over k, and moreover, one of the following conditions holds:**

(a) *$P_1 + P_2 + P_3$ is the intersection cycle of V and L, with correct multiplicities;*

or else

(b) $L \subset V$**.**

• **ADMISSIBLE EQUIVALENCE RELATIONS. An equivalence relation $\mathcal{A}$ on a Zariski dense subset $V(k)^0 \subset V(k)$ as above is called** *admissible one*, **if the following condition is satisfied:**

Let P_i and P_i' be two collinear families of points in $V(k)^0$. If pairs (P_i, P_i') for $i = 1, 2$ belong to the same equivalence classes $\bmod \mathcal{A}$, *the same is true for (P_3, P_3').*

One can check directly that if $\mathcal{A}$ is an admissible equivalence relation as above, then the partial composition law $P_3 := P_1 \circ P_2$ induced by collinearity defines on $E := V(k)^0/\mathcal{A}$ the structure of a commutative symmetric quasigroup, and therefore the composition law $X \star Y := U \circ (X \circ Y)$ induces the structure of CML .

The following result refers to admissible equivalence relations upon some subsets $V(k)^0$ in the case, when geometry of V itself is *"sufficiently general"*.

The latter restriction means that *not all points in $V(k)$ are singular, and each line $\mathbf{L} \subset \mathbf{P}^{r+1}$ defined over k intersects V at 2 or 3 different geometric points* **(not 1 or ∞.)**

• **THEOREM 3.** *If geometry of V is sufficiently general, then any admissible equivalence relation on the set $V(k)^0 \subset V(k)$ of smooth k–points has equivalence classes dense in Zariski topology, and the resulting CMLs with identity satisfy the relation $X^6 = 1$* **.**

NON–ASSOCIATIVE CML FROM A CUBIC SURFACE:

PRESENTATION OF AN EXAMPLE

• **GROUND RING. It will be $\mathbf{Z}_3[t]/(t^2+t+1)$: extension of the 3–adic completion of $\mathbf{Z}$ by a primitive cubic root of 1. We will put $\theta := t \bmod (t^2+t+1)$.**

• **CUBIC SURFACE. Our cubic surface $V \subset P^3$ over $k := \mathbf{Q}_3(\theta)$ will be given by the equation in homogeneous coordinates $T_0^3 + T_1^3 + T_2^3 + \theta T_3^3 = 0$.**

We will represent its k–points by equivalence classes of quadruples $p = (t_0 : \cdots : t_3) \in \mathbf{Z}_3[\theta]^4$ defined up to a common multiplication by invertible elements of the ground ring.

• **THEOREM 4.** *Define the equivalence relation $\mathcal{A}_3$ on the set $V(k)$ by*

$$(t_0 : \cdots : t_3) \sim (t_0'\varepsilon : \cdots : \tau_3'\varepsilon),$$

where $\varepsilon \equiv 1 \bmod (1-\theta)^3$. This equivalence relation is admissible one, and the respective CML is non–associative.

SKETCH OF PROOF OF THEOREM 4

• **ADMISSIBILITY OF $\mathcal{A}_3$. We must check that if $P_1, P_2, P_3 \in V(k)$ and $P'_1, P'_2, P'_3 \in V(k)$ are two collinear families, such that $P_i \sim P'_i \bmod \mathcal{A}_3$ for $i = 1, 2$, then also $P_3 \sim P'_3 \bmod \mathcal{A}_3$.**

This was checked by direct computations in Sec. 6 of [Ka21]. The verification was subdivided there into three cases:

1. $P_1 = P_2$ **and** $P'_1 = P'_2$.
2. P_1 **and** P_2 **are different modulo** $1 - \theta$.
3. $P_1 \equiv P_2 \bmod (1 - \theta)$.

• **NON–ASSOCIATIVITY OF COMPOSITION** $\mathrm{mod}\,\mathcal{A}_3$**. Consider the following family of three points in** $V(k)$**, given by their homogeneous coordinates:**

$$Q_1 := (1 : 0 : -1 : 0), \quad Q_2 := (0 : 1 : -\theta : 0),$$

and

$$Q_0 := \text{a lift of } (1 : -1 - 3\theta : 1 - \theta : -1 + \theta) \bmod (1 - \theta)^3 \text{ to } V(k).$$

Finally, put $U_0 := (1 : -1 : 0 : 0)$**. According to Theorem 1 above, we may define** $P \star Q := U_0 \circ (P \circ Q)$**, if** $(U_0, P \circ Q, P \star Q)$ **are collinear, and in this triple there are** ≤ 2 **coinciding points.**

Then we have

$$(Q_0 \star Q_1) \star Q_2 \neq Q_0 \star (Q_1 \star Q_2) \bmod (1 - \theta)^3.$$

This crucial statement was checked in Subsections 5.4.2 and 5.4.3 of [Ka21] by direct computation of both sides. ■

• **CONJECTURE ([Ka21]).** *The admissible equivalence relation* $\mathcal{A}_3$ *is universal in the sense of [Ma86], pp. 43, 44, 54, 69.*

Roughly speaking, this means that it is the finest equivalence relation in the setup described above.

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• **FROM PART I to PART II.**

The first Part of this talk was dedicated to the description and study of algebraic structures, induced upon sets of "typical" (e. g. non–singular) algebraic points of varieties by various projective embeddings, or more generally, ample line bundles.

In the second Part, we focus upon the fact that *distribution* **of such points might be highly** *nonhomogeneous***, even if we consider only points on smooth cubic surfaces: generally, there can much more such points on the set of lines that on the remaining Zariski open subset.**

Part II. INTRODUCTION:

HEIGHTS AND DISTRIBUTION OF RATIONAL POINTS

• **HEIGHTS.** **Let** U, V **be two quasiprojective varieties over a number field** K**,** $[K : \mathbf{Q}] < \infty$**. We want to give a precise meaning to an intuitive feeling that the set of** $K-$**points of** U **is** *considerably less* **than that of** $K-$**points of** V**, or is** *approximately of the same size***.**

Start with the ground field $K = \mathbf{Q}$ **and projective space** $\mathbf{P}^n$**. If in addition we choose and fix a projective coordinate system** $(x_0 : x_1 : \cdots : x_n)$ **on this space, we might start with estimating the number of points with** $\max(|x_i) \leq H$**, when** $H \in \mathbf{R}$ **tends to infinity. It is easy to see that this number grows approximately as** H^{r+1}**, and so the bigger is the dimension of a projective space, the more points it contains.**

To be more precise in a more general environment, we must introduce ample line bundles on our varieties, and respective generalisations of homogeneous coordinates and heights.

In order to define heights over general algebraic number fields, we need the following preparations.

Let K be a number field, $\Omega_K = \Omega_{K,f} \sqcup \Omega_{K,\infty}$ the set of its places v represented as the union of finite and infinite ones. K_v denotes the respective completion of K.

For $v \in \Omega_{K,f}$, denote by $\mathcal{O}_v$, resp. m_v, the ring of integers of K_v, resp. its maximal ideal. The Haar measure dx_v on K_v is normalised in such a way that the measure of $\mathcal{O}_v$ becomes 1. Moreover, for an archimedean v, the Haar measure will be the usual Lebesgue measure, if v is real, and for complex v it will be induced by Lebesgue measure on **C**, for which the unit square $[0,1]+[0,1]i$ has volume 2.

Let the map $|.|_v : K_v \to \mathbf{R}^*_{\geq 0}$ **be defined by the condition** $d(\lambda x)_v = |\lambda|_v dx_v$**.**

Then for any $\lambda \in K^*$ **we have the following product formula:** $\prod_v |\lambda|_v = 1$**.**

Now let $\mathbf{P}^n$ **be a projective space over** K **with a chosen system of homogeneous coordinates as above, that is a basis of sections in** $\Gamma(P^n_K, \mathcal{O}(1)$**. Then we can define** *the exponential Weil height* **of a point** $p = (x_0(p) : \cdots : x_n(p)) \in \mathbf{P}^n(K)$ **as**

$$h(p) := \prod_{v \in \Omega_K} \max\{|x_0(p)|_v, \ldots, |x_n(p)|_v\}.$$

Because of the product formula, the height does not change, if we replace coordinates $(x_0 : \cdots : x_n)$ **by** $(\lambda x_0 : \cdots : \lambda x_n)$**,** $\lambda \in K$**.**

However, it can change, if we change the coordinate system. If h'_{L_V} **is another height, corresponding to a different choice of the basis of sections, then there exist two positive real constants** C, C' **such that for all** x**,**

$$C h_{L_V}(x) \leq h'_{L_V}(x) \leq C' h_{L_V}(x).$$

We must find ways to compare sizes of sets of points not too sensitive to such coordinate changes.

We describe below our main constructions.

• **HEIGHT ZETA FUNCTIONS. Let** (U, L_U) **be a pair consisting of a projective variety** U **over** K **and an ample line bundle** L_U **on it. Then we can define the height function** $h_{L_v}(p)$ **on** $p \in U(K)$ **using the same formula as above, but this time interpreting** (x_i) **as a basis of sections in** $\Gamma(U, L_U)$**.**

Now define the height zeta–function

$$Z(U, L_U, s) := \sum_{x \in V(K)} h_{L_U}(x)^{-s}.$$

• **CONVERGENCE BOUNDARY. Denote by** $\sigma(U, L_U) \in \mathbf{R}$ **the lower limit of positive reals** σ **for which** $Z(U, L_U, s)$ **absolutely converges if** $Re\, s \geq \sigma$**.**

We will call $\sigma(U, L_U)$ **the respective** *convergence boundary***.**

Clearly, it is finite (because this is so for projective spaces), and non–negative whenever $U(K)$ **is infinite.**

Intuitively, we may say that V **contains "considerably less"** K**–points than** U**, if**

$$\sigma(V, L_V) < \sigma(U, L_U).$$

and "aproximately the same" amount of K**–points, if**

$$\sigma(V, L_V) = \sigma(U, L_U).$$

• **EXAMPLE: ACCUMULATING SUBVARIETIES. Let V be a Zariski closed subvariety of U over K. If**

$$\mathrm{card}\,\{x\in V(K)\,|\,h_{L_V}(x)\le H\}\cdot \mathrm{card}\,\{x\in (V\setminus U)(K)\}\,|\,h_{L_V}(x)\le H\}^{-1}\to 0$$

as $H\to\infty$, then V is called an *accumulating subvariety* **in U in the sense of**

[BaMa90] V. Batyrev, Yu. Manin. *Sur le nombre des points rationnels de hauteur bornée des variétés algébriques.* **Math. Ann. 286 (1990), pp. 27–43.**

[FrMaTsch89] J. Franke, Yu. Manin, Yu. Tschinkel. *Rational points of bounded height on Fano varieties.* **Invent. Math., 95, no. 2 (1989), pp. 421–435.**

Clearly, then $\sigma(V,L_V)=\sigma(U,L_U)$.

We will now describe a categorical setup appropriate for describing various versions of *accumulation* **and connecting combinatorics of accumulating subvarieties with** *spectra* **in the sense of homotopical algebra.**

RATIONAL POINTS, SIEVES, AND ASSEMBLERS

• SIEVES AND GROTHENDIECK TOPOLOGIES. Let $\mathcal{C}$ be a category. A *sieve* **in $\mathcal{C}$ is its full subcategory $\mathcal{C}'$ such that if $f : V \to U$ is a morphism in $\mathcal{C}'$, and $g : W \to V$ is any morphism in $\mathcal{C}$, then their composition $f \circ g : W \to U$ is a morphism in $\mathcal{C}'$.**

This notion is convenient in order to define a *Grohendieck topology* **on $\mathcal{C}$: it is a collection of sieves $\mathcal{J}(U)$, one for each object U of $\mathcal{C}$, satisfying three axioms: see**

[MaMar18] Yu. Manin, M. Marcolli. *Homotopy types and geometries below* $\operatorname{Spec}\mathbb{Z}$. **In: Dynamics: Topology and Numbers. Conference to the Memory of Sergiy Kolyada. Contemp. Math., AMS 744 (2020). arXiv:math.CT/1806.10801.**

for details and some additional notions. In particular, each object U of a category with Grothendieck topology is endowed with *covering families*: **collections of morphisms $\{f_i : U_i \to U \,|\, i \in I\}$ such that the full subcategory of $\mathcal{C}$ containing all morphisms in $\mathcal{C}$ factoring through f_i belongs to the initial collection of sieves.**

• **ASSEMBLERS. An** *assembler* **is a small category $\mathcal{C}$ endowed with a Grothendieck topology and initial object $\emptyset$. All morphisms in it must be** *monomorphisms*, **and any two disjoint finite covering families must admit a common refinement which is also a finite disjoint covering family.**

Assemblers themselves form a category, in which a morphism is a functor continuous with respect to their Grothenieck topologies, sending initial object to initial object, and disjoint morphisms to disjoint morphisms.

In order to use assemblers related to the distribution of rational points, we will first of all define formally certain *sieves* **via point distribution.**

For many more details, see

[Za17a] I. Zakharevich. *The K–theory of assemblers.* **Adv. Math. 304 (2017), pp. 1176–1218 .**

[Za17b] I. Zakharevich. *On K_1 of an assembler.* **J. Pure Appl. Algebra 221, no.7 (2017), pp. 1867–1898 .**

• **CATEGORIES $\mathcal{C}(U, L_U)$. Let U be a projective variety over K and L_U an ample rank 1 vector bundle on U over K.**

By definition, objects of $\mathcal{C}(U, L_U)$ are locally closed subvarieties $V \subset U$ also defined over K, and morphisms are the structure embeddings $i_{V,U}$, or simply i_V: $V \to U$. Here we did not mention L explicitly, but it is natural to endow each V by $L_V := i_V^*(L_U)$.

Structure embeddings are compatible with these additional data so that we have in fact structure functors $\mathcal{C}(V, L_V) \to \mathcal{C}(U, L_U)$ which make of each $\mathcal{C}(V, L_V)$ a full subcategory of $\mathcal{C}(U, L_U)$ closed under precomposition, that is, *a sieve* **.**

We will call such categories $\mathcal{C}(U, L_U)$ *geometrical sieves*, **and now introduce the** *arithmetical sieves* $\mathcal{C}^{ar}(U, L_U)$ **in the following way.**

• **LEMMA.** *The family of those morphisms* $i_{V,U}$ *as above, together with their sources and targets, for which*

$$0 < \sigma(V, L_V) < \sigma(U, L_U),$$

forms a sieve in $\mathcal{C}(U, L_U)$ *denoted* $\mathcal{C}^{ar}(U, L_U)$*.*

This statement is fairly obvious.

Notice, that if $V(K)$ **is a finite set, then** $\sigma(V, L_V) = 0$**, but the converse is not true:** $\sigma(V, L_V) = 0$ **for any abelian variety** V/K **and for many other classes of** V**. A complete geometric description of this class of varieties seemingly is not known.**

• **ARITHMETIC ASSEMBLERS. Using sieves** $\mathcal{C}^{ar}(U, L_U)$**, we can easily introduce the respective arithmetic assemblers** $\mathcal{C}_U$**: the relevant Grothendieck topology is simply the Zariski topology over** K**, and** $\emptyset$ **is the empty scheme.**

ANTICANONICAL HEIGHTS AND POINTS COUNT

• ANTICANONICAL HEIGHTS: DIMENSION ONE. Let (U, L_U) **be as above a pair consisting of a variety and ample line bundle defined over** K, $[K:\mathbf{Q}] < \infty$. **Choose a exponential height function** h_L, **and set for** $B \in \mathbf{R}_+$

$$N(U, L_U, B) := \mathrm{card}\{x \in U(K) \,|\, h_L(x) \leq B\}.$$

On page XXII, we based the definition of an arithmetical sieve upon an intuitive idea that $i_V : V \to U$ **belongs to this sieve,** *if the number of* K*–points on* U *is "considerably less" that such number on* V. **To make this idea precise, we used convergence boundaries.**

Below, we will use considerably more precise count of points in order to define subtler sieves on a more narrow class of varieties U, **using** *counting functions themselves* $N(U, L_U, B)$ **in place of convergence boundaries.**

Start with one–dimensional U.

If U is a smooth irreducible curve of genus g, *with nonempty set* $U(K)$**, we have the following basic alternatives:**

$$g = 0: \quad U = \mathbf{P}^1, \quad L_U = -K_V, \quad N(U, L_U, B) \sim cB.$$

$$g = 1: \quad U = \text{ an elliptic curve, with rank of Picard groupe } r, \quad N(U, L_U.B) \sim c(\log B)^{r-1}.$$

$$g > 0: \quad N(U, L_U, B) = const, \text{ if } B \text{ is big enough.}$$

A survey of expected typical behaviours of multidimensional analogs can be found in the Introduction to [FrMaTsch89].

Below, our attention will be focussed upon *Fano varieties***, that is, varieties with ample anticanonical bundle ω_V^{-1}, as a wide generalisation of the one–dimensional case $g = 0$.**

• **ANTICANONICAL HEIGHTS FOR FANO VARIETIES. The most precise** *conjectural* **asymptotic formula for Fano varieties (or Zariski open subsets of them) with dense $U(K)$ has the form**

$$N(U, \omega_U^{-1}, B) \sim cB(\log B)^t, \quad t := \mathrm{rk}\, Pic\, U - 1.$$

It certainly is *wrong* **for many subclasses of Fano varieties. On the other hand, it is**

(i) *stable under the direct products;*

(ii) *compatible with predictions of Hardy–Littlewood for complete intersections;*

(iii) *true for quotients of semisimple algebraic groups modulo parabolic subgroups: see [FrMaTsch89], Sections 1–2***.**

HEIGHT COUNT WRT MORE GENERAL LINE BUNDLES L_U ON FANO VARIETIES: CONJECTURES.

In the paper

[BaMa90] V. Batyrev, Yu. Manin. *Sur le nombre des points rationnels de hauteur bornée des variétés algébriques.* **Math. Ann. 286 (1990), pp. 27–43.**

it was suggested that only a slight generalisation of formula on page XXX should be "typical" (although valid in a much more restricted set of cases):

$$N(U, L_U, B) \sim cB^{\beta}(\log B)^t, \quad t := \mathrm{rk}\, Pic\, U - 1.$$

As was argued in [BaMa90], β **should be defined by the relative positions of** L_U **and** $-K_U$ **in the cone of pseudo–effective divisors of** U**: see precise conjectures there.**

Sh. Tanimoto in the paper

[Ta19] Sh. Tanimoto. *On upper boundaries of Manin type.* **arXiv:1812.03423v2, 29 pp.**

provided arguments proving various inequalities for these numbers related to the conjectures in [BaMa90].

Finally, the subtlest information about such asymptotics is given by several conjectures and proofs regarding exact value of the constant c.

• FROM ASYMPTOTIC FORMULAS TO SIEVES. If we restrict ourselves by those V for which we can define a *Grothendieck topology, objects of which satisfy strong asymptotic formulas discussed above***, or their weaker versions, then we can try to define sieves in it by some inequalities weakening earlier ones, such as**

$$N(V, L_V, B)/N(U, L_U, B) = o(1)$$

or even, for $\beta(U, L_U) = \beta(V, L_V)$,

$$t(V, L_V) < t(U, L_U).$$

where $t(V, L_V)$ and $\beta(V, L_V)$ refer to formula on page XXXI.

SIEVES "BEYOND HEIGHTS" ?

• THIN SETS AND TAMAGAWA MEASURES. Here we survey recent attempts to define geometry of subsets of rational points of $V(K)$ containing *"considerably less"* **points than V.**

From our viewpoint, these definitions should also be tested on compatibility with the philosophy of sieves and assemblers.

Below we adopt the framework of

[Sa20] W. Sawin. *Freeness alone is insufficient for Manin–Peyre.* **arXiv:2001.06078. 7 pp.**

in which $K = \mathbf{Q}$.This paper starts with Conjecture 1.1 called "Modern formulation of Manin's conjecture", and involves the following shifts from our earlier setup based upon heights.

(i) *Summation* **over points x of height $\leq B$ is replaced by the** *averaging* **of the measures δ_x of the same points embedded into the adelic space $V(\mathbf{A_Q})$.**

(ii) Such an averaging means *integration* **with respect to a certain measure. The respective class of measures consists of the called** *Tamagawa measures* τ**.**

Let V **be a geometrically integral smooth projective Fano variety,** r **rank of Picard group. Let** $\mathcal{V}$ **be its proper integral model over Z, and** $L(s, Pic\,V_{\overline{\mathbf{Q}}})$ **and the respective local zetas** L_v **are** *zetas of lattices***.**

Then

$$\tau := (\lim_{s\to 1}(s-1)^r L(s, Pic\,V_{\overline{\mathbf{Q}}}))\prod_v L_v(s, Pic\,V_{\overline{\mathbf{Q}}})^{-1}\omega_v.$$

Here ω_v **is defined by the natural measure on local non–archimedean points of** V **or archimedean volume form.**

(iii) Finally, define the numbers $\alpha(V)$ and $\beta(V)$ by

$$\alpha(V) := r \operatorname{vol}\{y \in ((\operatorname{Pic}(V) \otimes \mathbf{R})^{eff})^\vee \mid K_V \cdot y \leq 1\},$$

and

$$\beta(V) := \operatorname{card} H^1(\operatorname{Gal}(\overline{\mathbf{Q}/\mathbf{Q}}), \operatorname{Pic} V_{\overline{\mathbf{Q}}}).$$

Then the modern formulation of the conjecture on the number of rational points of bounded height on a Fano variety, according to [Sa20], can be stated as follows.

Let $f : U \to V$ be a morphism of geometrically integral smooth projective varieties. Call it a *thin morphism*, **if the induced map $U \to f(U)$ is generically finite of degree $\neq 1$.**

• **CONJECTURE.** *There exists the complement W to the union of a finite family of thin morphisms such that we have an exact formula for weak limit of the form*

$$\lim_{B\to\infty} \frac{1}{B(\log B)^{r-1}} \sum_{\substack{x\in W(\mathbf{Q})\\ H(x)<B}} \delta_x = \alpha(V)\beta(V)\tau^{Br},$$

where τ^{Br} is the restriction of Tamagawa measure on the subset of $V(\mathbf{A_Q})$, on which the Brauer–Manin obstruction vanishes.

Regarding Brauer–Manin obstruction, see the monograph and many references therein.

[CThSk19] **J.-L. Colliot-Thélene, A. Skorobogatov.** *The Brauer–Grothendieck group.* **imperial.ac.uk 2019, 360 pp.**

For a class of varieties for which this conjecture is valid, we obtain interesting new possibilities for defining sieves.

THANK YOU FOR YOUR ATTENTION !