$\mathbb{Q}$-Points on Algebraic Groups & Spectrum

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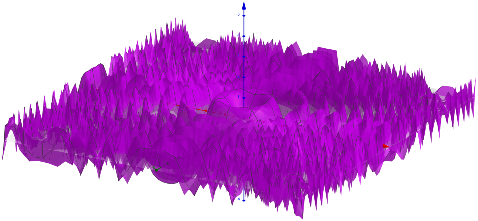
Question (Katz)

Can one "hear" the shape of a space?

$$\boxed{\begin{array}{c}\text{Spectral Information}\\ \text{(eigenvalues of the Laplace Operator)}\end{array}} \Longrightarrow \boxed{\text{Geometry}}$$



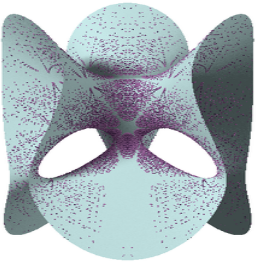
- Weyl Law
  (asymptotics of eigenvalues),
- Selberg Trace Formula
  (Spectrum $\leftrightarrow$ closed geodesics),
- Quantum Unique Ergodicity
  (curvature $< 0 \rightarrow$ asymptotics of eigenfunctions),
- ...

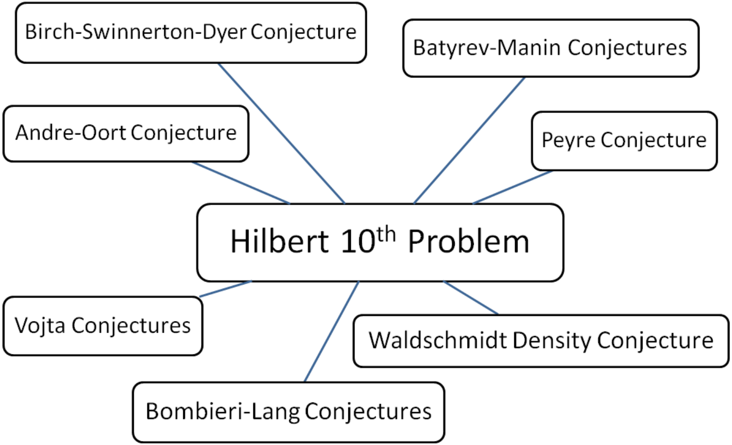
Question

Does Geometry determine Arithmetic?

For an algebraic variety X over $\mathbb{Q}$,

$$\boxed{\text{Geometry of } X(\mathbb{C})} \quad \Longrightarrow \quad \boxed{\text{Distribution of } X(\mathbb{Q})}$$




Birch-Swinnerton-Dyer Conjecture
Batyrev-Manin Conjectures
Andre-Oort Conjecture
Peyre Conjecture
Hilbert 10th Problem
Vojta Conjectures
Waldschmidt Density Conjecture
Bombieri-Lang Conjectures

Question

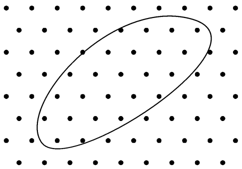
Can one "count" the spectrum of a space?

Arithmetic Information (Distribution of $\mathbb{Q}$-solutions)	$\Longrightarrow$	Spectral Information (Authomorphic Reps)

Simple Example: Counting Lattice Points in $\mathbb{R}^d$

Λ – a lattice in $\mathbb{R}^d$

Counting Problem: For domains $\Omega_T \subset \mathbb{R}^d$,



$$|\Lambda \cap \Omega_T| \approx \frac{\mathrm{vol}(\Omega_T)}{\mathrm{vol}(\mathbb{R}^d/\Lambda)}$$

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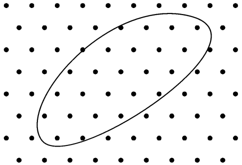

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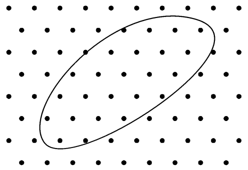

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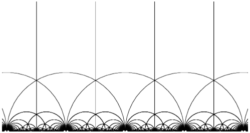
Here the connection is provided by classical Fourier Analysis.

Hyperbolic Surfaces

Γ – an arithmetic subgroup of $SL_2(\mathbb{R})$ (e.g. $\Gamma = SL_2(\mathbb{Z})$).



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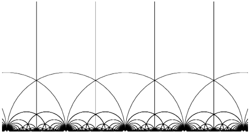

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<u>**Spectral Problem:**</u>

For the Laplace operator $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$. Describe

$$\mathrm{Spec}\Big(\Delta \curvearrowright L^2(\Gamma \backslash \mathbb{H}^2)\Big),$$

or more generally,

$$SL_2(\mathbb{R}) \curvearrowright L^2(\Gamma \backslash SL_2(\mathbb{R})).$$

Ramanujan Conjecture

For $z \in \mathbb{H} = \{\mathrm{Im}(z) > 0\}$,

$$\Delta(z) = e^{2\pi i z} \prod_{k=1}^{\infty} (1 - e^{2\pi i k z})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}.$$

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Δ is holomorphic, with $\Delta(i\infty) = 0$, and satisfing

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \Delta(z) \quad \text{for} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

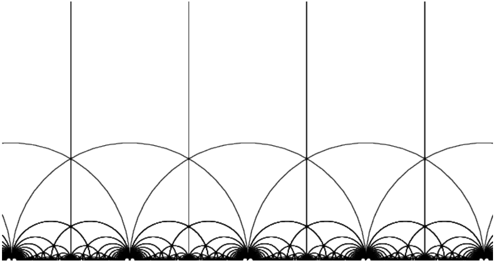
with $k = 12$. It is an example of a *holomorphic cusp form*.

Maass Forms

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ (or a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$).

A *Maass form* of weight k is functions $f : \mathbb{H} \to \mathbb{C}$ satifying:

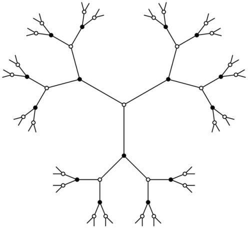
- $f\left(\frac{az+b}{cz+d}\right) = \left(\frac{cz+d}{|cz+d|}\right)^k f(z)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,
- an eigenfunction of $\Delta_k = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + iky\frac{\partial}{\partial x}$,
- a growth condition at infinity.



Hecke operators

Hecke Tree: for $z \in \mathbb{H}$ and a prime p,

$$S_p(z) = \{pz, (z+i)/p : \ i = 0, \ldots, p-1\}.$$



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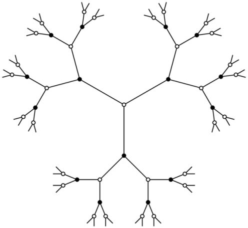

Hecke Operators: for $f : \mathbb{H} \to \mathbb{C}$,

$$T_p(f)(z) = \frac{1}{p+1} \sum_{w \in S_p(z)} f(w).$$

.

Ramanujan–Petersson Conjecture

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The Hecke operators T_p have uniform "spectral gap": all eigenvalues $\lambda \neq 1$ of T_p on Maass forms satisfy

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The best known bound:

Deligne: Yes, for holomorphic forms

Kim-Sarnak: $|\lambda| \leq \frac{2p^{\frac{39}{64}}}{p+1}$, in general.

Laplace operator on hyperbolic surfaces

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Question

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Question

$$\lambda_1(S_n) \to ? \textit{ as } n \to \infty.$$

$\exists$ examples when $\lambda_1(S_n) \to 0$.

Selberg Conjecture

$\Gamma_n = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma = I \bmod n\}$ – congruence lattices

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$$\textbf{Kim-Sarnak:} \qquad \lambda_1 \geq \frac{975}{4096}.$$

Spectral Gap

It turns out that both

Ramanujan–Petersson Conjecture and Selberg Conjecture

are better to understand in terms

$\boxed{\text{Spectral Gap Property}}$

for group actions.

Spectral Gap

Let $\pi : G \to \mathcal{U}(H)$ be a unitary representation of a group G.

Def. An *almost invariant vector* is sequence of unit vectors v_n with

$\|\pi(g)v_n - v_n\| \to 0$ as $n \to \infty$, uniformly on compact sets of G.

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  $\phi_{n_k}(x) = e^{2\pi i n_k x}$ (for suitable n_k) gives a.i. vectors,

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  $\phi_{n_k}(x) = e^{2\pi i n_k x}$ (for suitable n_k) gives a.i. vectors,
- Also for more general (non-atomic) actions $T \curvearrowright L^2_0(X)$ ...

Spectral Gap and Averaging Operators

Given a probability measure β on G, one defines the operator:

$$\pi(\beta) : H \to H : \; v \mapsto \int_G \pi(g) v \, d\beta(g)$$

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It turns out that

Selberg Conjecture and Ramanujan–Petersson Conjecture

are, in fact, about establishing

explicit bounds on the norms $\|\pi(\beta)\|$.

Selberg Conjecture (equivalent formulation)

We consider the action of $\mathsf{SL}_2(\mathbb{R})$ on the space $X_n = \Gamma_n \backslash \mathsf{SL}_2(\mathbb{R})$ and the averaging operators $\pi_{n,\infty}(\beta_r) : L^2_0(X_n) \to L^2_0(X_n)$:

$$\pi_{n,\infty}(\beta_r)\phi(x) = \frac{1}{\mathrm{vol}(B_r)} \int_{B_r} \phi(xg)\, dg,$$

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Conjecture

The operators $\pi_{n,\infty}(\beta_r)$ have uniform "spectral gap":

$$\|\pi_{n,\infty}(\beta_r)\| \leq C_{n,\epsilon}\, vol(B_r)^{-\frac{1}{2}+\epsilon} \quad \textit{for all } \epsilon > 0.$$

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Let

$$\Gamma_{n,p} = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}[1/p]) :\ \gamma = I \bmod n\}, \quad (n,p) = 1$$
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A generalization: Property (τ)

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Theorem (. . . , Kazhdan, . . . , Buger–Sarnak, . . . , Clozel)

The actions

$$\pi_{n,\infty} : G(\mathbb{R}) \curvearrowright L^2_0(X_{n,p}) \quad \textit{and} \quad \pi_{n,p} : G(\mathbb{Q}_p) \curvearrowright L^2_0(X_{n,p})$$

have uniform "spectral gap":

$\exists$ uniform $\delta > 0$ such that

$$\|\pi_{n,*}(\beta_r)\| \le C_{n,*}\, vol(B_r)^{-\delta}.$$

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Let L be a reductive algebraic subgroup of G over $\mathbb{Q}$.

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For sufficiently large r,

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Example: For $G = \mathrm{SL}_d$ with $d \geq 3$,

$$\sup_n \|\pi_{n,p}(\beta_r)\| \geq C_p \, vol(B_r)^{-\frac{2}{d}}.$$

Questions

We discuss

- What spectral information tells us about rational points?
- What distributions of rational points tells about spectrum?

Theorem (Franke–Manin-Tschinkel)

For the flag varieties $X = G/P$

$$\left|\{x \in X(\mathbb{Q}) : \mathrm{H}_{\mathcal{L}}(x) \leq T\}\right| \sim c(X, \mathcal{L}) T^{a(\mathcal{L})} (log T)^{b(\mathcal{L})-1}.$$

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Other homogeneous varieties . . . (cf. Batirev–Manin Conjecture)

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We further explore <u>spacial distribution</u> of rational points.

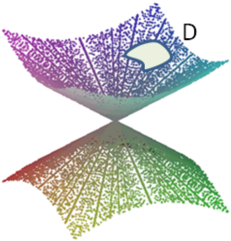
Discrepancy of distribution

Let $S(r)$ be locally finite subsets of real algebraic variety X.

$S(r)$'s is *equidistributed* w.r.t. a measure m on X if

$$\exists v(r) \to \infty : \frac{|S(r) \cap D|}{v(r)} \to m(D)$$

for all "nice" compact domains D.

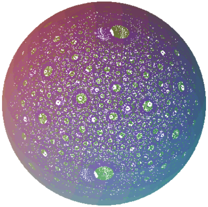


We introduce the *discrepancy*:

$$\mathcal{D}(S(r), D) = \left| \frac{|S(r) \cap D|}{v(r)} - m(D) \right|.$$

Example

$X = \{x \in \mathbb{R}^d : \ Q(x) = 1\}$ – a rational ellipsoid



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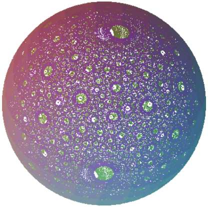

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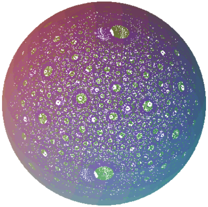

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Schmidt, Beck: For any finite set $R \subset S^{d-1}$,

$$\left(\int_0^{\pi} \int_{S^{d-1}} \mathcal{D}(R, B(x, \eta))\, dx d\eta \right)^{1/2} \gg |R|^{\frac{1}{2} - \frac{1}{2(d-1)}}.$$

Discrepancy for rational points

G – a simply connected simple algebraic group over $\mathbb{Q}$ ($G = \mathrm{SL}_n$)

$\Gamma_{n,p} = \{\gamma \in G(\mathbb{Z}[1/p]) :\ \gamma = I \bmod n\}, \quad (n,p) = 1$

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For $D \subset G(\mathbb{R})$, we set:

$$B_p(r) = \{b \in G(\mathbb{Q}_p) : \|b\| \leq r\},$$

$$v_p(r) = vol(B_r),$$

$$\mathcal{D}(S_{n,p}(r), D) = \left| \frac{|S_{n,p}(r) \cap D|}{v_p(r)} - \frac{vol(D)}{|\Gamma_{1,p}/\Gamma_{n,p}|} \right|.$$

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D – a "nice" compact domains in $G(\mathbb{R})$

Theorem (G.–Nevo)

For any $\varepsilon > 0$ and a.e. $g \in G(\mathbb{R})$,

$$\mathcal{D}(S_{n,p}(r), Dg) \ll_{n,p,D,\varepsilon} v_p(r)^{-\delta+\varepsilon}$$

(here δ is the exponent in the norm estimate).

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Clozel–Oh–Ullmo: estimates using bounds on matrix coefficients

Ghosh–G.–Nevo: different approach using averaging operators, gives weaker estimate $v_p(r)^{-\delta/(d+1)}$

Converse Problem

We have:

$$\boxed{\text{Spectral Gap}} \Rightarrow \boxed{\text{Estimates on Arithmetic Discrepancy}} \ ?$$

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Question:

$$\boxed{\text{Estimates on Arithmetic Discrepancy}} \Rightarrow \boxed{\text{Spectral Gap}} \ ?$$

Sarnak–Xue approach to Selberg Conjecture

Theorem (Sarnak–Xue)

For congruence lattices Γ_n in $SL_2(\mathbb{R})$, eigenvalues $\lambda \neq 0$ of the Laplacian on $\Gamma_n \backslash \mathbb{H}$ satisfy $\lambda \geq \frac{5}{36}$.

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Conjecture (Sarnak–Xue)

For $B_r = \{g \in G : \|g\| \leq r\}$,

$$|\Gamma_n \cap B_r| \ll \frac{vol(B_r)^{1+\epsilon}}{vol(G/\Gamma_n)} + vol(B_r)^{1/2}, \quad \epsilon > 0.$$

Arithmetic approach to p-adic case

"Spectral Gap" for $\pi_{n,p} : G(\mathbb{Q}_p) \curvearrowright L^2_0\big(\Gamma_{n,p}\backslash(G(\mathbb{R}) \times G(\mathbb{Q}_p)\big)$?

We assume that $G(\mathbb{R})$ is not compact.

Let $B(x,\eta)$ denote balls in $G(\mathbb{R})$ w.r.t. invariant metric.

We set:

$$\Delta_{p,n}(x,\eta;r) = \mathcal{D}\big(S_{n,p}(r), B(x,\eta)\big)$$
$$E(\eta,r) = \|\Delta_{p,n}(\cdot,\eta;r)\|_{L^2(G(\mathbb{R})/G(\mathbb{Z}))}$$

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Theorem (G.–Nevo)

For any irreducible subrepresenation σ of $\pi_{n,p}$,

$$\|\sigma(\beta_r)\| \ll_{n,p} \eta^{-\dim(G)} E(\eta,r)$$

for $\eta \in (0,\eta_0)$.

Applications

Starting with the formula

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- Get explicit estimates on the discrepancy $E(\eta, r)$ to get upper operator norm bounds
- Get explicit estimates on the norm $\|\sigma(\beta_r)\|$ to get lower bounds on discrepancy

Application I: Spectral Gap for forms of SL_2

Let G be a form of SL_2 over $\mathbb{Q}$.

(Eq., the group of norm one elements in a division algebra over $\mathbb{Q}$)

Theorem (G.–Nevo)

When $G(\mathbb{R})$ is non-compact, the representions

$$\pi_{n,p} : G(\mathbb{Q}_p) \curvearrowright L_0^2(\Gamma_{n,p} \backslash (G(\mathbb{R}) \times G(\mathbb{Q}_p))$$

satisfy

$$\|\pi_{n,p}(\beta_r)\| \leq C_{n,p}\, vol(B_r)^{-\delta}$$

with explicit uniform $\delta > 0$.

Here $\delta = 1/4$ if G is anisotropic over $\mathbb{Q}$ and $\delta = 1/16$ in general.

Application II: lower bounds on discrepancy

Let $G = \mathrm{SL}_d$.

Theorem (G.–Nevo)

For all $\eta \in (0, \eta_0)$ *und* $r \geq r_0$,

$$\left\| \mathcal{D}\big(S_{n,p}(r), B(\cdot, \eta)\big) \right\|_{L^2(G(\mathbb{R})/G(\mathbb{Z}))} \geq C_{n,p}\, \eta^{d^2-d} v_p(r)^{-\frac{1}{2}}.$$

In comparison, the best upper bound (G.–Nevo) is

$$\left\| \mathcal{D}\big(S_{n,p}(r), B(\cdot, \eta)\big) \right\|_{L^2(G(\mathbb{R})/G(\mathbb{Z}))} \leq C'_{n,p}\, \eta^{(d^2-d)/2} v_p(r)^{-\frac{1}{2(d-1)}}.$$