4D/2D duality and representation theory

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physical theories

3D N=4 gauge theory

invariants
(observables)

relation/duality

math. objects

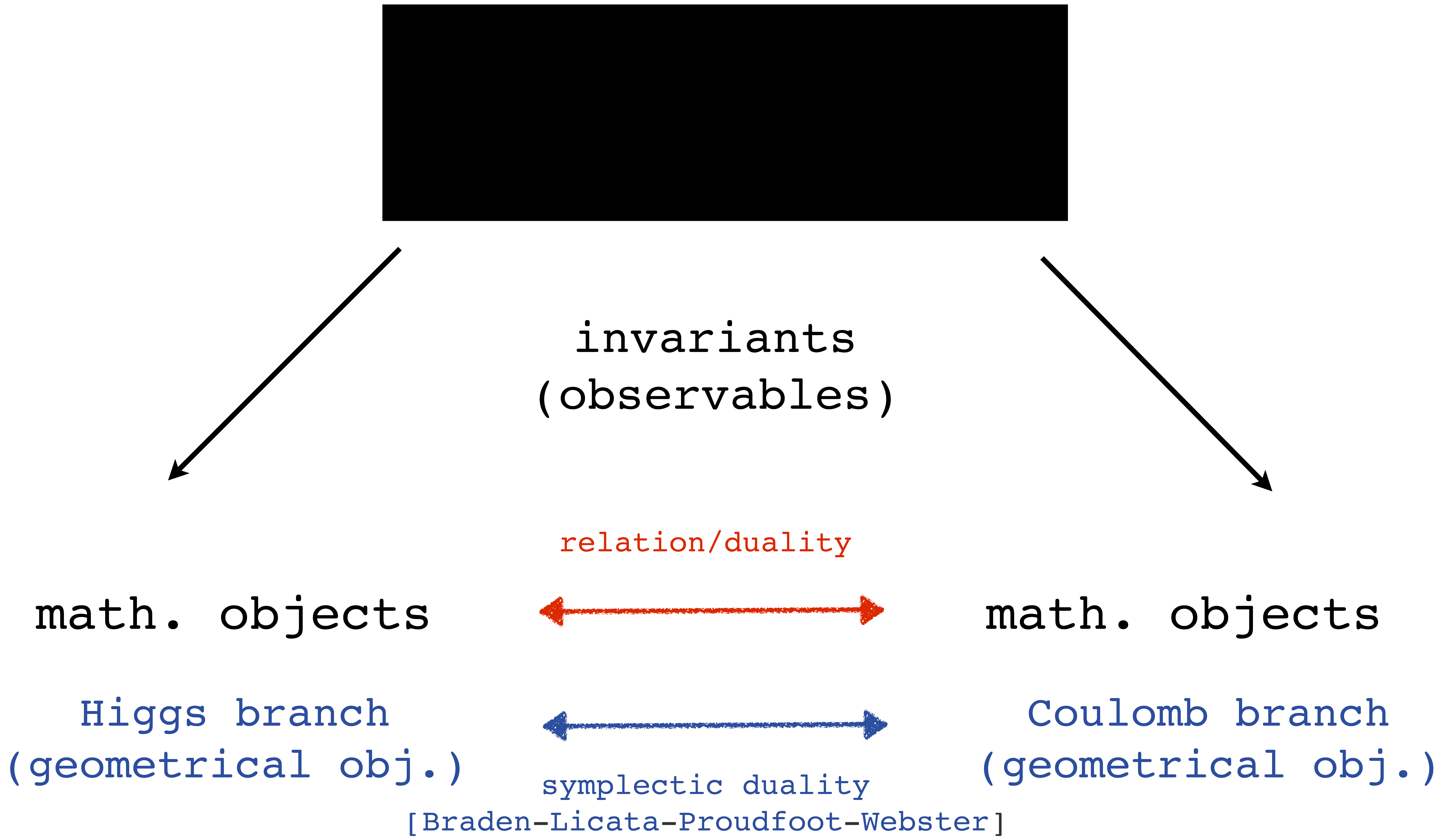
Higgs branch
(geometrical obj.)

math. objects

Coulomb branch
(geometrical obj.)

Remark

At the moment there is no rigorous mathematical definition of quantum field theories in dimension higher than 2.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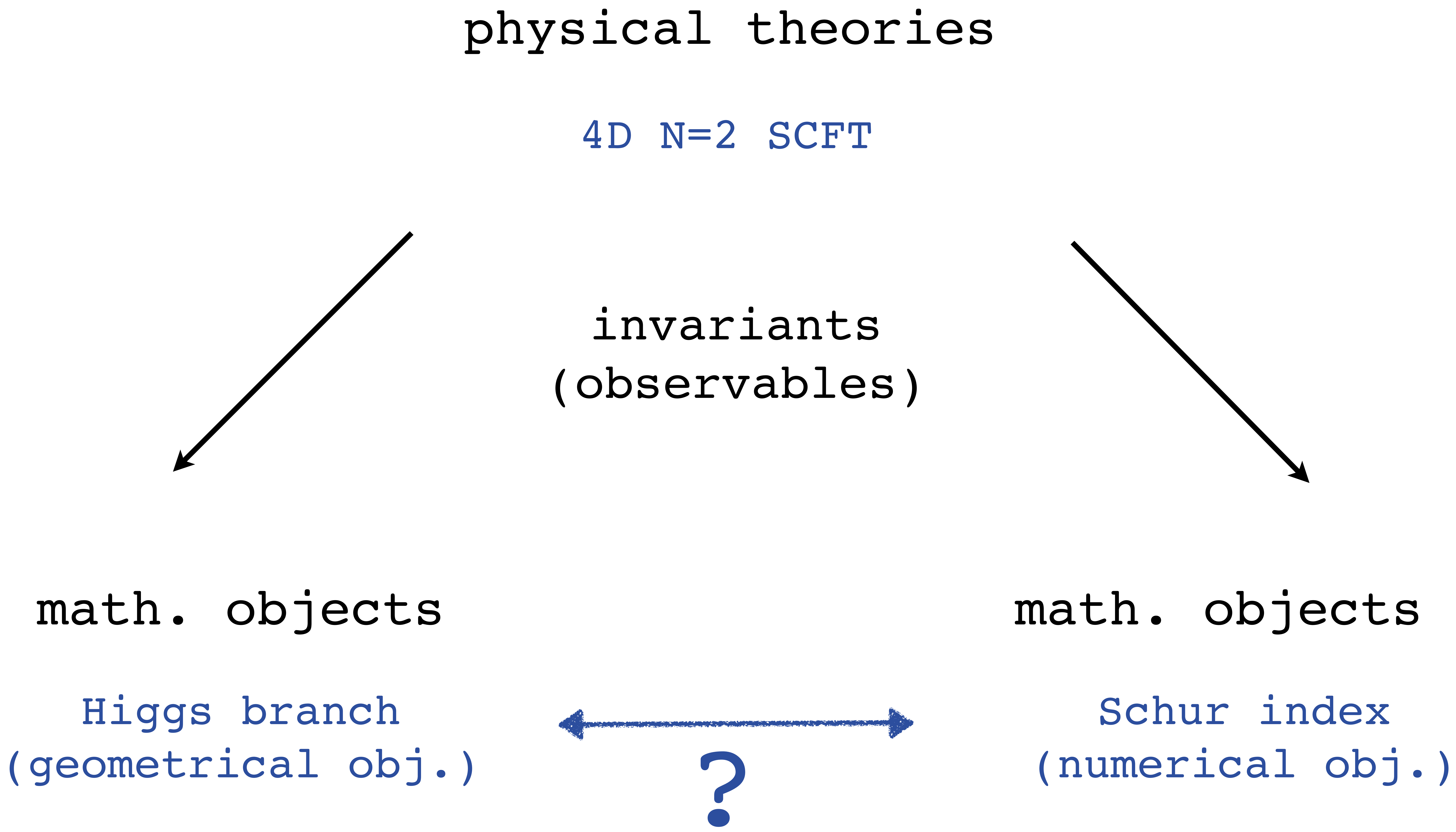
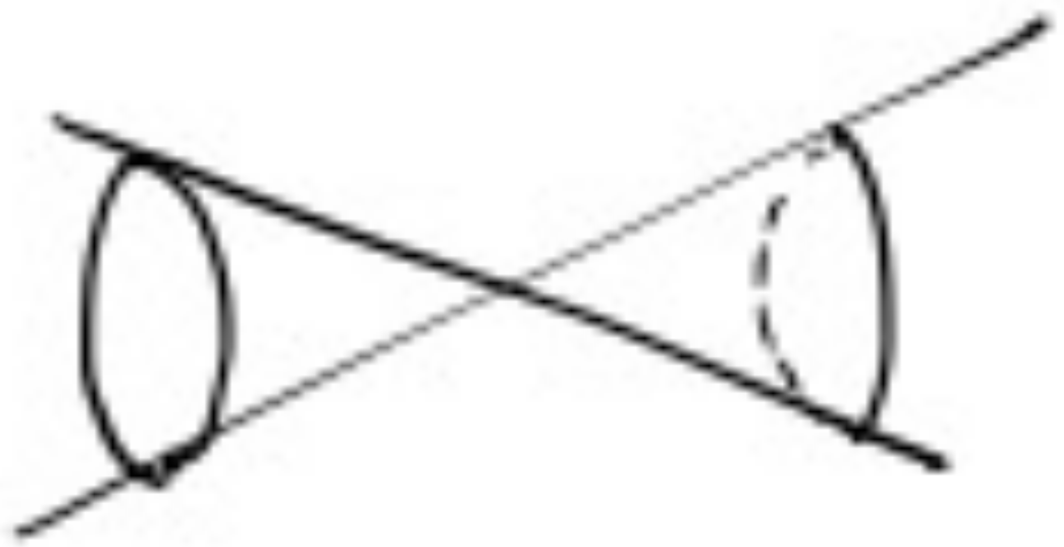
(geometrical obj.)
symplectic duality
[Braden-Licata-Proudfoot-Webster]


physical theories
4D N=2 SCFT
invariants
(observables)
math. objects
Higgs branch
(geometrical obj.)
?
math. objects
Schur index
(numerical obj.)

Example

$T_{III} =$ 4D theory obtained from type III elliptic fibration via F-theory

$$\mathrm{Higgs}(T_{III}) = \{(x,y,z) \in \mathbb{C}^3 \mid x^2 + yx = 0\} =: \mathcal{N}$$



$$\mathrm{Schur}(T_{III}) = q^{1/4}(1 + 3q + 9q^2 + 19q^3 + 60q^4 + \dots)$$
$$= \eta(q^3)^3/\eta(q)^3,$$

where $\eta(q) = q^{1/24} \prod_{j \geq 1} (1 - q^j).$

Higgs branch $\mathcal{N}$

$$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid \mathrm{tr}(A) = a + d = 0\}$$

Lie algebra by $[A, B] = AB - BA$, $\quad \dim \mathfrak{g} = 4 - 1 = 3$

$$\mathcal{N} \cong \{A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g} \mid \det(A) = -(a^2 + bc) = 0\} \subset \mathfrak{g}$$

$= \{A \in \mathfrak{g} \mid A^2 = 0\}$, the nullcone of $\mathfrak{g}$.

important object in representation theory

How about the Schur index $q^{1/4}(1 + 3q + 9q^2 + 19q^3 + 60q^4 + \dots)$?

$\dim \mathfrak{g}$

$$V(\mathfrak{g}) = S(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \qquad \text{symmetric algebra of } \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$$

$$= \mathrm{span}_{\mathbb{C}}\{(x_1 \otimes t^{-n_1}) \dots (x_r \otimes t^{-n_r}) \mid x_i \in \mathfrak{g},\ n_i > 0\}$$

$$\text{degree} = n_1 + \cdots + n_r$$

$$= \bigoplus_{d \geq 0} V(\mathfrak{g})_d, \qquad \dim V(\mathfrak{g})_d < \infty.$$

$$\mathrm{ch}\, V(\mathfrak{g}) = \sum_{d \geq 0} (\dim V(\mathfrak{g})_d) q^d$$

$$V(\mathfrak{g}) = S(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}]) \quad = \mathrm{span}_{\mathbb{C}}\{(x_1\otimes t^{-n_1})\dots(x_r\otimes t^{-n_r}) \mid x_i\in\mathfrak{g},\ n_i>0\}$$

$$\text{degree } 0:\ 1, \qquad \dim V_0 = 1,$$

$$\text{degree } 1:\ x\otimes t^{-1}, \qquad \dim V_1 = 3,$$

$$\text{degree } 2:\ x\otimes t^{-2},\quad (x\otimes t^{-1})^2,\quad (x\otimes t^{-1})(y\otimes t^{-1}), \qquad \dim V_2 = 9,$$

$$\mathrm{ch}\, V(\mathfrak{g}) = \sum_{d\geq 0}(\dim V(\mathfrak{g})_d)q^d \ = 1+3q+9q^2+22q^3+\dots$$

$$\vee$$

$$q^{1/4}(1+3q+9q^2+19q^3+\dots)$$

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$ affine Kac-Moody algebra `[Kac, Moody]`

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\, \mathrm{tr}(xy)\delta_{m+n,0}K,$$

$$[K, \widehat{\mathfrak{g}}] = 0$$

$$[D, x \otimes t^m] = mx \otimes t^m$$

$$\widehat{\mathfrak{g}} = (\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus (\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D)$$

Lie subalgebras

PBW Theorem

For $k \in \mathbb{C}$, $V^k(\mathfrak{g}) := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D)} \mathbb{C}_k \cong U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \cong V(\mathfrak{g})$.

$\mathfrak{g}[t] = 0, \ K = k\, \mathrm{id}\,, D = 0$

Facts

1) $V^k(\mathfrak{g})$ is not irreducible as an $\widehat{\mathfrak{g}}$-module in general.

2) $V^k(\mathfrak{g})$ admits a unique simple quotient $L_k(\mathfrak{g})$, which is graded.

$$\mathrm{ch}\, L_k(\mathfrak{g}) = \sum_{d \in \mathbb{Z}_{\geq 0}} \dim L_k(\mathfrak{g})_d q^d$$

depends on k

Kac-Wakimoto (1988)

For $k = -4/3$, $\mathrm{ch}\, L_k(\mathfrak{g}) = 1 + 3q + 9q^2 + 19q^3 + \dots$

$$= q^{-1/4} \frac{\eta(q^3)^3}{\eta(q)^3}$$

★ The factor $q^{1/4}$ has a representation theoretic meaning.

and $q^{1/4}\, \mathrm{ch}\, L_k(\mathfrak{g})$ is called the normalized character of $L_k(\mathfrak{g})$

We have seen both Higgs branch and Schur index are related with $\mathfrak{g} = \mathfrak{sl}_2$

$$\begin{array}{ccc} \| & & \nwarrow \\ \mathcal{N} & & L_{-4/3}(\mathfrak{g}) \end{array}$$

Question Can we relate $L_{-4/3}(\mathfrak{g})$ with $\mathcal{N}$?

$$S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*] \twoheadrightarrow \ L_k(\mathfrak{g})/(\mathfrak{g} \otimes t^{-2}\mathbb{C}[t^{-1}])L_k(\mathfrak{g}) \ =: R_{L_k(\mathfrak{g})}$$

$$x_1 x_2 \dots x_r \mapsto (x_1 \otimes t^{-1}) \dots (x_r \otimes t^{-1}) + (\mathfrak{g} \otimes t^{-2}\mathbb{C}[t^{-1})L^k(\mathfrak{g})$$

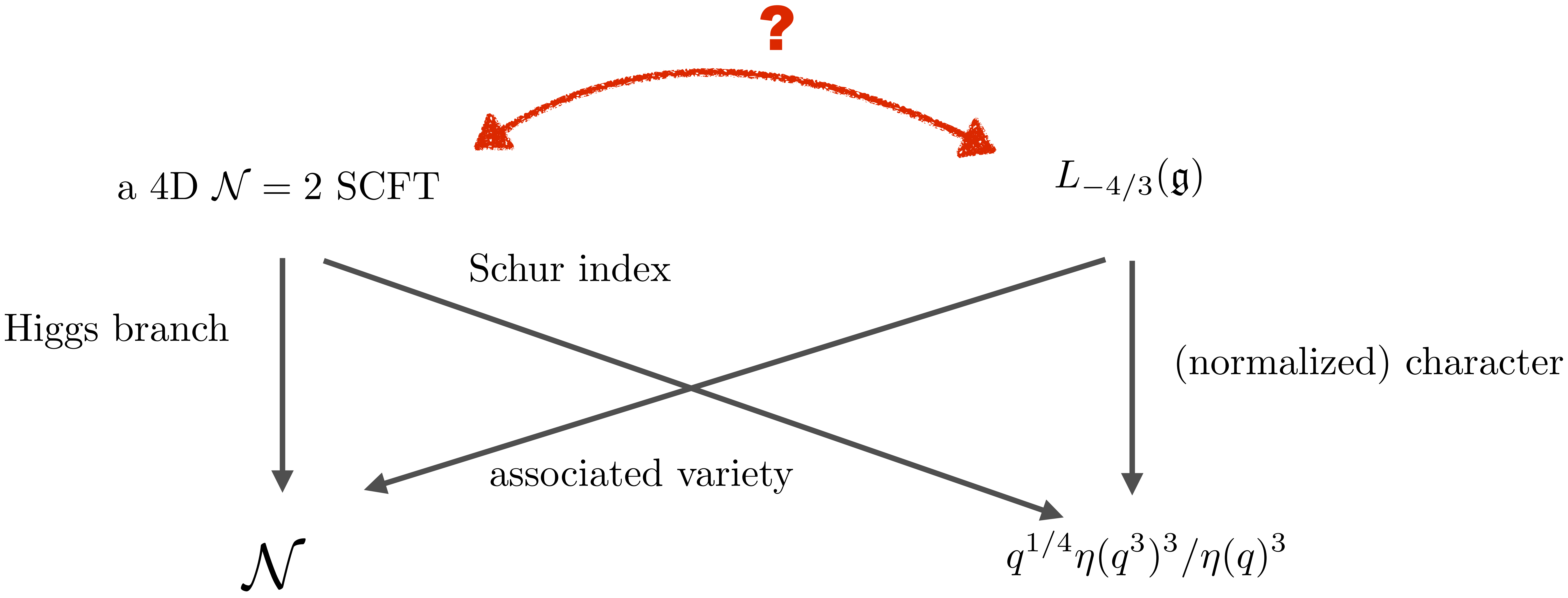
$$R_{L_k(\mathfrak{g})} \cong \mathbb{C}[\mathfrak{g}^*]/I_k \qquad I_k \subset \mathbb{C}[\mathfrak{g}^*], \qquad \text{an ideal}$$

$$X_{L_k(\mathfrak{g})} := \{\lambda \in \mathfrak{g}^* \mid f(\lambda) = 0 \ \forall f \in I\} \subset \mathfrak{g}^* = \mathfrak{g}$$

the associated variety of $L_k(\mathfrak{g})$.

Feigin-Malikov 1997

For $k = -4/3$, we have $X_{L_k(\mathfrak{g})} = \mathcal{N}$.



Is this a coincidence?

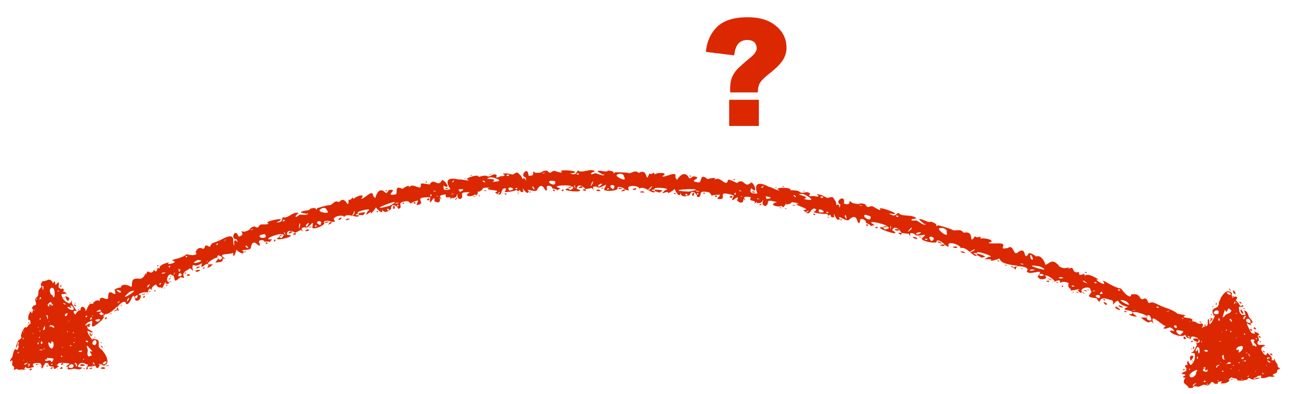
VOA (vertex operator algebra/vertex algebra)

1) Vertex algebras were introduced by Borcherds as a mathematical framework of TWO-DIMENSIONAL conformal field theories.

2) Typical examples of VOA/VA are $V^k(\mathfrak{g})$ and $L_k(\mathfrak{g})$.

So a VOA can be regarded as a generalization of affine Kac-Moody algebras.

4) For any VA V, one can define its associated variety X_V, which is a finite-dimensional Poisson variety ([A.2012]).

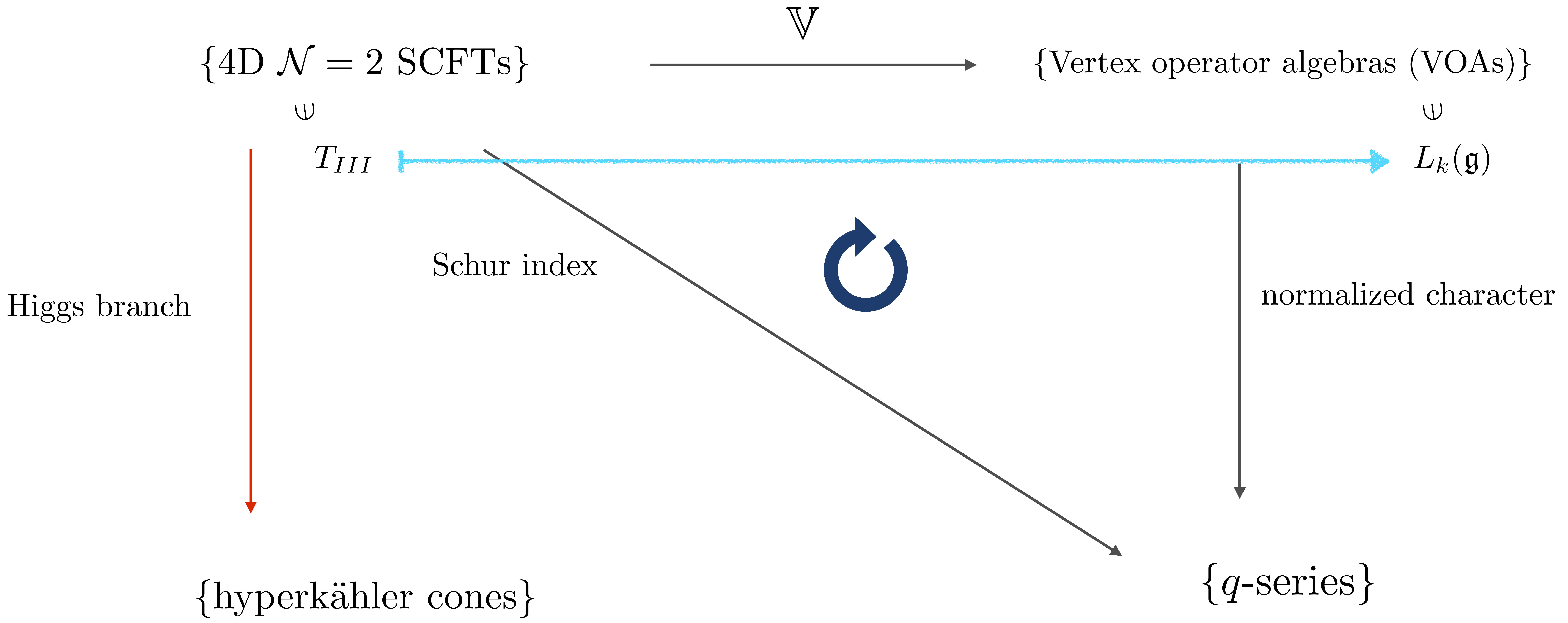


a 4D $\mathcal{N} = 2$ SCFT

$L_{-4/3}(\mathfrak{g})$

VOA $\leftrightarrow$ 2D CFT

Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees2015



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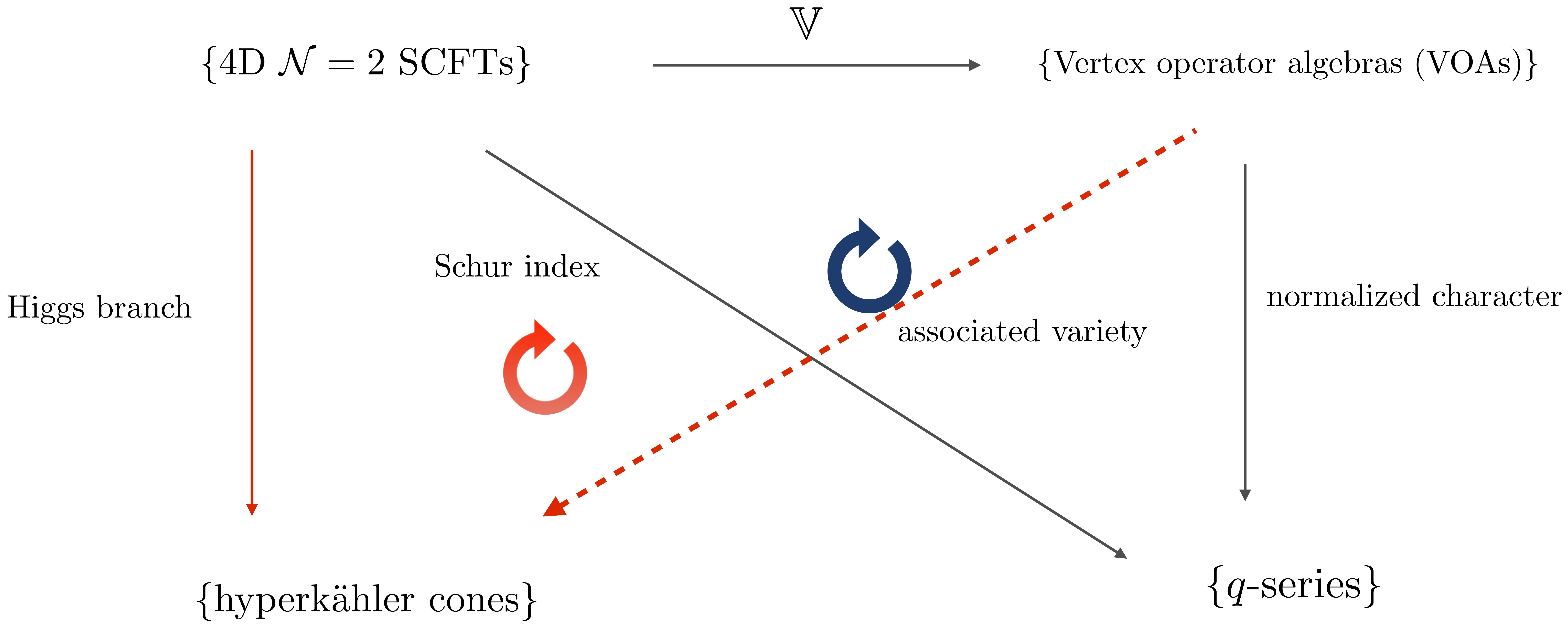


1) $\mathbb{V}(\mathcal{T})$ is NEVER unitary. In particular, $\mathbb{V}$ is not surjective.

2) $\mathbb{V}$ is expected to be injective.

In other words, $\mathbb{V}(\mathcal{T})$ is expected to be a complete invariant of $\mathcal{T}$ **!!**

Vertex algebras

A vertex algebra is a vector space V equipped with

$|0\rangle \in V$ (the vacuum vector),

$T \in \mathrm{End}(V)$ (the translation operator),

a linear map $Y : V \to (\mathrm{End}(V))[[z, z^{-1}]]$, $a \mapsto a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{n-1}$,

such that

$$a(z)b \in V((z)),$$

$$|0\rangle(z) = \mathrm{id}, \quad a(z)|0\rangle \in a + zV[[z]],$$

$$[T, a(z)] = (Ta)(z) = \frac{d}{dz}a(z),$$

$$(z-w)^n [a(z), b(w)] = 0 \text{ for } n \gg 0 \text{ in } (\mathrm{End}(V)[[z^{\pm}, w^{\pm}]] \quad \text{(locality)}.$$

Example

$$V = V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K\oplus\mathbb{C}D)} \mathbb{C}_k$$

$$|0\rangle = 1 \otimes 1, \qquad [T, x \otimes t^n] = -n(x \otimes t^{n-1}),\ T|0\rangle = 0,$$

$$x(z) = \sum_{n\in\mathbb{Z}} (x \otimes t^n) z^{-n-1} \quad \text{for } x \in \mathfrak{g} \hookrightarrow V^k(\mathfrak{g}) \ni (x \otimes t^{-1})|0\rangle.$$

A vertex algebra V

$\longrightarrow$ A Poisson algebra $R_V = V/\operatorname{span}_{\mathbb{C}}\{a_{(-2)}b \mid a, b \in V\}$, (Zhu's C_2-algebra)

$$\bar{a} \cdot \bar{b} = \overline{a_{(-1)}b}$$

$$\{\bar{a}, \bar{b}\} = \overline{a_{(0)}b}.$$

$$X_V = \operatorname{Spec} R_V.$$

Example

$$V = V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D)} \mathbb{C}_k$$

$R_V = V/t^{-2}\mathfrak{g}[t^{-1}]V \cong \mathbb{C}[\mathfrak{g}^*]$, and so $X_{V^k(\mathfrak{g})} = \mathfrak{g}^*$.

The surjection $V^k(\mathfrak{g}) \to L_k(\mathfrak{g})$ induces a surjection $R_{V^k(\mathfrak{g})} \to R_{L_k(\mathfrak{g})}$,

and so $X_{L_k(\mathfrak{g})} \subset X_{V^k(\mathfrak{g})} = \mathfrak{g}^*$, G-invariant, conic.

An associated variety of a VOA is a Poisson variety.

From Beem-Rastelli conjecture it follows that if a VOA V comes from a 4D theory,

then X_V should have a finitely many symplectic leaves.

A.-Kawasetsu 2018

If X_V has many symplectic leaves,

then the normalized character $\chi_V(q)$ satisfies a modular linear differential equation.

$\Rightarrow$ certain modularity of Schur indices (important in physics)

Beem-Rastelli conjecture

Examples of VOAs coming from 4D N=2 SCFTs

~~integrable representations of $\widehat{\mathfrak{g}}$~~

(boundary) admissible representations $L_k(\mathfrak{g})$ of $\widehat{\mathfrak{g}}$, $k + h^\vee \in \mathbb{Q}_{>0}$, [Song-Xie-Yan2015]

$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}}_k$, $\exists$nilpotent orbit $\mathbb{O}_k$ [A.2015]

admisinle W-algebras $\mathcal{W}_k(\mathfrak{g}, f)$ [Wang-Xie-2018, Xie-Yan2019]

$X_{\mathcal{W}^k(\mathfrak{g},f)} = \overline{\mathbb{O}}_k \cap \mathcal{S}_f$, nilpotent Slodowy slice [A.2015]

$\mathcal{S}_f = e + \mathfrak{g}^f$, $\{e, h, f\}$ $\mathfrak{sl}_2$-triple.

Beem-Rastelli conjecture is a physical conjecture in general,

because 4D $\mathcal{N}=2$ SCFT is not mathematically defined.

theory of class $\mathcal{S}$ [Gaiotto 2012]

$\begin{array}{c} \mathcal{S} \\ \parallel \\ \text{six} \end{array}$

obtained by "compatifying" a 6D theory on a punctured Riemann surface Σ

$$S_G(\Sigma)$$

G : flavor symmetry group (complex semisimple group)

$\exists$mathematical definition of $\mathrm{Higgs}(S_G(\Sigma))$

[Moore-Tachikawa 2012,Ginzburg-Kazhdan,Braverman-Finkelberg-Nakajima 2018]

★ Enough to describe the Higgs branches for genus zero Σ.

$\mathrm{MT}_{G,r} = \mathrm{Higgs}(S_G(\mathbb{P}^1 \text{ with } r\text{-puncturres}))$,

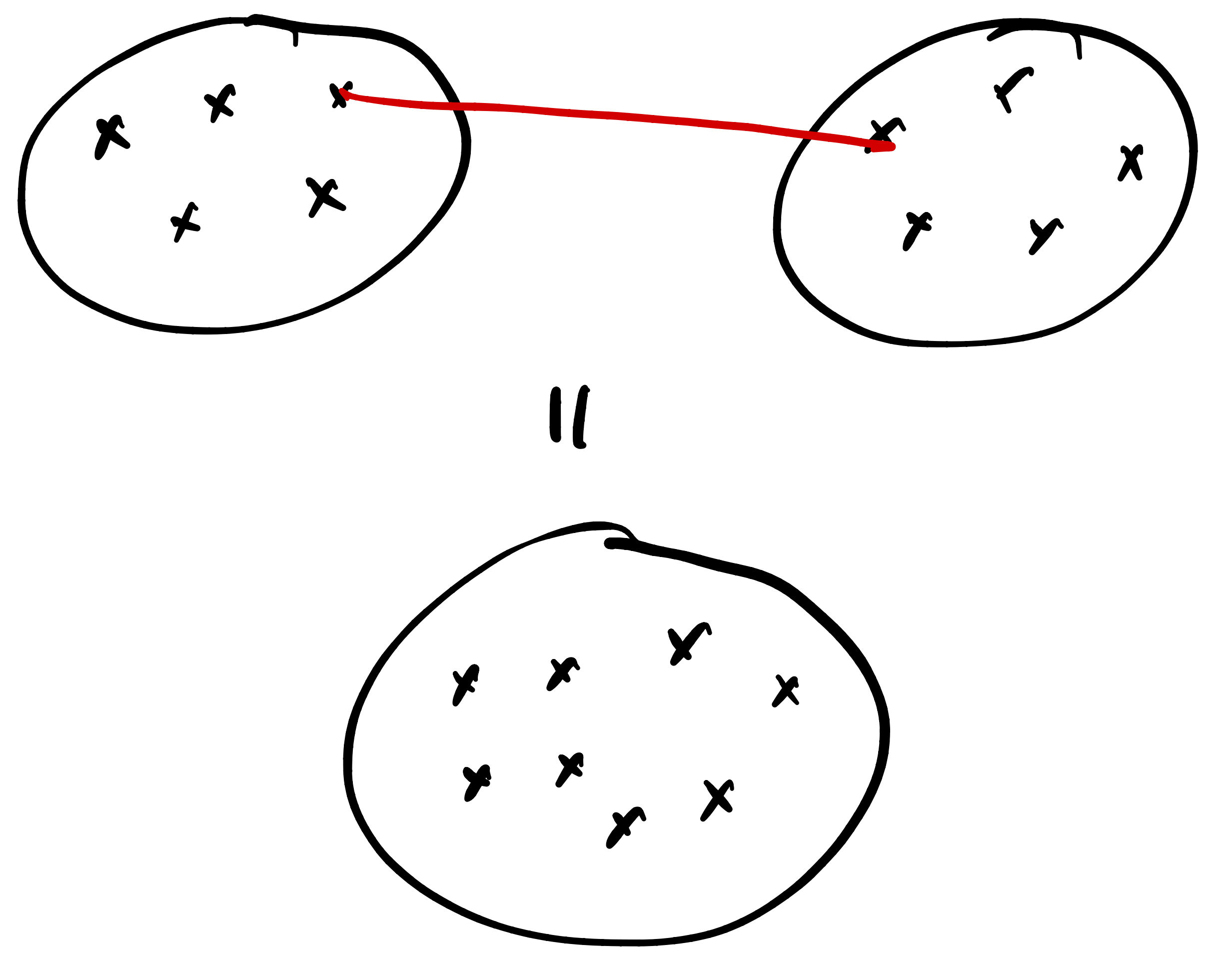
equipped with Hamiltonian action of r-copies of G.

$\mathrm{MT}_{G,2} = T^*G$,

$\mathrm{MT}_{G,1} = G \times \mathcal{S}$, $\mathcal{S} = e + \mathfrak{g}^f$, Kostant-Slodowy slice, ($\{e, h, f\}$ a regular $\mathfrak{sl}_2$-triple)

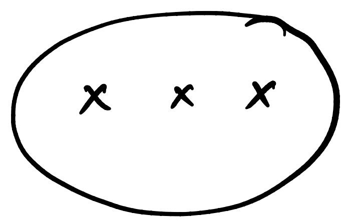
$(\mathrm{MT}_{G,r} \times \mathrm{MT}_{G,s})///\Delta(G) \cong \mathrm{MT}_{G,r+s-2}$ (associativity)

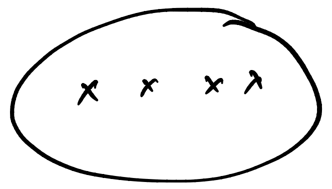
symplectic reduction



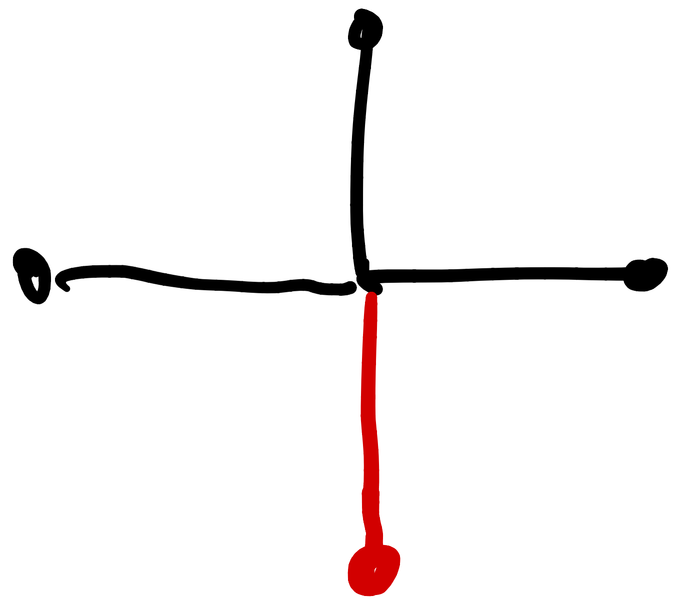
Examples

$G = SL_2(\mathbb{C}) = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid \det A = 1\}.$

$\Sigma =$  $\mathrm{MT}_{G,3} = (\mathbb{C}^2)^{\otimes 3}, \qquad G \circlearrowright \mathbb{C}^2$

$\Sigma =$ 

$\mathrm{MT}_{G,4} = \overline{\mathbb{O}}_{D_4,min},$ the minimal nilpotent orbit closure in D_4,



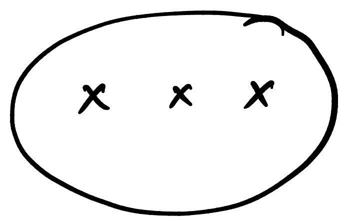
associativity for $r = s = 3$

$$((\mathbb{C}^2)^{\otimes 3} \times (\mathbb{C}^2)^{\otimes 3})///\Delta(SL_2(\mathbb{C})) \cong \overline{\mathbb{O}}_{D_4,min}$$

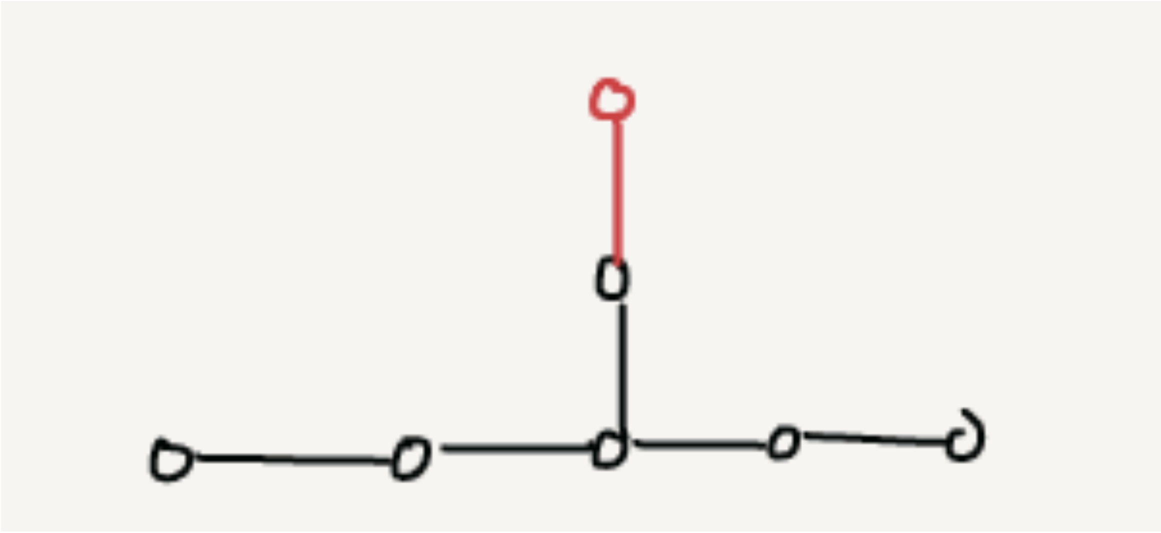
ADHM construction for $\overline{\mathbb{O}}_{D_4,min}$

[Atiyah-Drinfeld-Hitchin-Manin]

$G = SL_3(\mathbb{C})$

$\Sigma =$ 

$\mathrm{MT}_{G,3} = \overline{\mathbb{O}}_{E_6,min},$ minimal nilpotent orbit closure in E_6.



In general, there is no simple description of $\mathrm{MT}_{G,r}$.

Construction of VOAs (chiral algebras of class $\mathcal{S}$)

$G \longrightarrow \widehat{\mathfrak{g}}$,

$X \longrightarrow$ VOA V such that $X_V = X$,

$\mu : X \to \mathfrak{g}^*$ moment map $\longrightarrow$ vertex algebra homomorphism $V^k(\mathfrak{g}) \to V$
such that the induced morphism $X_V \to X_{V^k(\mathfrak{g})} = \mathfrak{g}^*$ coincides with μ,

$T^*G \longrightarrow \mathcal{D}^{ch}_{G,k}$, algebra of chiral differential operators on G at level k,
$(X_{\mathcal{D}^{ch}_{G,k}} \cong T^*G)$ [Malikov-Schechtman-Vaintrob1999, Beilinson-Drinfeld, Arhipov-Gatisgory2002]

$G \times \mathcal{S} \longrightarrow H^0_{DS}(\mathcal{D}^{ch}_{G,k})$ $(X_{H^0_{DS}(\mathcal{D}^{ch}_{G,k})} \cong G \times \mathcal{S})$,

$(X \times Y)///\Delta(G) \longrightarrow H^{\infty/2+\bullet}(\widehat{\mathfrak{g}}, \mathfrak{g}, V \otimes W)$ $(X_V \cong X,\ X_W \cong Y)$,
$X_{H^{\infty/2+\bullet}(\widehat{\mathfrak{g}}, \mathfrak{g}, V \otimes W)} \cong (X_V \times X_W)///\Delta(G)$ in nice cases.

$k = -h^\vee =$ the critical level

the large center (the Feigin-Frenkel center) of $\tilde{U}_{-h^\vee}(\widehat{\mathfrak{g}})$ exists.

A. 2018

There exists a unique family of VOAs $\{\mathbf{V}_{G,r}\}$ such that

1) $\exists$ VA homomorphism $V^{-h^\vee}(\mathfrak{g})^{\otimes r} \to \mathbf{V}_{G,r}$,

and the action of $(\mathfrak{g} \otimes \mathbb{C}[t])^{\otimes r}$ on $\mathbf{V}_{G,r}$ integrates to the action of $G[[t]]^r$;

2) $\mathbf{V}_{G,2} = \mathcal{D}^{ch}_{G,-h^\vee}, \quad \mathbf{V}_{G,1} = H^0_{DS}(\mathcal{D}^{ch}_{G,-h^\vee})$,

3) $H^{\infty/2+i}(\widehat{\mathfrak{g}}, \mathfrak{g}, \mathbf{V}_{G,r} \otimes \mathbf{V}_{G,s}) \cong \delta_{i,0}\mathbf{V}_{G,r+s-2}$.

Moreover,

4) Each $\mathbf{V}_{G,r}$ is simple, and its central charge is $\dim \mathrm{MT}_{G,r} - 24(r-2)(\rho|\,\rho^\vee)$,

5) $$\mathrm{tr}_{\mathbf{V}_{G,r}}(q^{L_0} z_1 z_2 \dots z_r) = \sum_{\lambda \in P_+} \left(\frac{q^{\langle \lambda, \rho^\vee \rangle} \prod_{j=1}^{\infty} (1-q^j)^{\mathrm{rk}\,\mathfrak{g}}}{\prod_{\alpha \in \Delta_+} (1 - q^{\langle \lambda+\rho, \alpha^\vee \rangle})} \right)^{r-2} \prod_{k=1}^{r} \mathrm{tr}_{\mathbb{V}_\lambda}(q^{-D} z_k),$$

6) $X_{\mathbf{V}_{G,r}} \cong \mathrm{MT}_{G,r}$.

★ In the above, 1)-5) are the properties that $\mathbb{V}(S_G(\Sigma))$ should have.

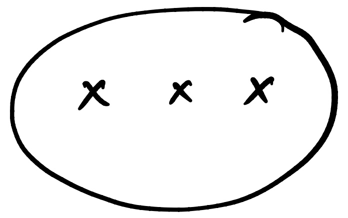
[Beem-Peelaers-Rastelli-van-Rees 2015]

$\Longrightarrow$ $\mathbf{V}_{G,r} = \mathbb{V}(S_G(\mathbb{P}^1 \text{ with } r\text{-punctures}))$.

$\Longrightarrow$ Beem-Rastelli conjecture is true for class $\mathcal{S}$-theory 👍

Examples

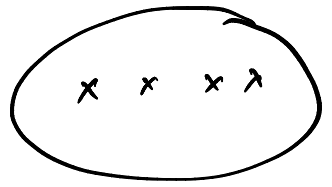
$G = SL_2(\mathbb{C})$

$\Sigma =$ 

$\mathrm{Higgs}(S_G(\Sigma)) = (\mathbb{C}^2)^{\otimes 3}, \qquad G \circlearrowright \mathbb{C}^2$

$\mathbb{V}(S_G(\Sigma)) =$ the $\beta\gamma$ system associated with the symplectic vector space $(\mathbb{C}^2)^{\otimes 3}$.

affinization of the Weyl algebra

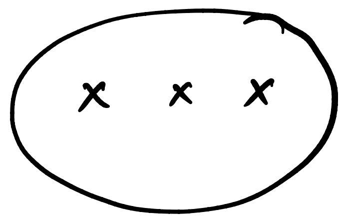
$\Sigma =$ 

$\mathrm{Higgs}(S_G(\Sigma)) = \overline{\mathbb{O}}_{D_4,min},$

$\mathbb{V}(S_G(\Sigma)) = L_{-2}(D_4).$

Conjectured in [Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees2015]

$G = SL_3(\mathbb{C})$

$\Sigma =$ 

$$\mathrm{MT}_{G,3} = \overline{\mathbb{O}}_{E_6,min},$$

$$\mathbb{V}(S_G(\Sigma)) = \mathbf{V}_{G,3} = L_{-3}(E_6).$$

Conjectured in [Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees2015]

The isomorphism $X_{L_{-2}(D_4)} \cong \overline{\mathbb{O}}_{D_4,min}$ and $X_{L_{-3}(E_6)} \cong \overline{\mathbb{O}}_{E_6,min}$ were previously shown in [A.-Moreau2016].

In general $\mathbf{V}_{G,r}$ is a W-algebras in the sense that it is generated by a Lie algebra,

and there is no simple description in general.

A 4D $\mathcal{N} = 2$ SCFT $\mathcal{T}$ also has a Coulomb branch, which is the moduli space of G-Higgs bundles on Σ.

It is expected that the representation theory of $\mathbb{V}(\mathcal{T})$ is closely connected with the Higgs branch.

[Dedushenko-Gukov-Nakajima-Pei-Ye2018]

Thank you for your attention!