Limits of complex integrals and non-archimedean geometry

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§1 Motivation

X_t, $t \in D^\times$ (punctured disk), family of complex varieties

Philosophy: asymptotics of such a family as $t \to 0$ can often be expressed in terms of NA geometry.

Example: X_t family of CY varieties with max degeneration, metric behavior described conjecturally by Kontsevich–Soibelman (A-side) + B-side resembles a lot Berkovich's retraction in NA geometry.

Why NA geometry?

$X_t \rightsquigarrow X|\mathcal{M}er$

$\mathcal{M}er$ field of germs of meromorphic functions at 0.

$\mathrm{Mer} \hookrightarrow \mathbb{C}((t))$ t-adic completion.

$| \, |_b$ on $\mathbb{C}((t))$ by setting $|t|_b = \tau$ fixed τ in $(0,1)$.

<u>Example:</u> Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function with compact supports.

We consider the $(1,1)$-form

$$\omega_t := -\frac{1}{\log|t|}\,\varphi\left(-\frac{\log|z(z-t)|}{\log|t|}\right) d\log|z| \wedge \frac{d\arg z}{2\pi}$$

We are interested in

$$\int_{\mathbb{P}^1(\mathbb{C})} \omega_t$$

Fix $K > 1$ (eg $K = 2$)

$$\int_{\mathbb{P}^1(\mathbb{C})} \omega_t = I_1 + I_2 + I_3$$

with $I_1 = \int_{|z| \leq |t|/K} \omega_t$

$$I_2 = \int_{|t|/K \leq |z| \leq K|t|} \omega_t$$

$$I_3 = \int_{|z| \geq K|t|} \omega_t .$$

Then elementary computation give

$$\lim_{t \to 0} I_1 = \int_{x \leq -1} \varphi(x-1)\, dx$$

$$\lim_{t \to 0} I_2 = 0$$

$$\lim_{t \to 0} I_3 = \int_{x \geq -1} \varphi(2x)\, dx .$$

Thus $\lim_{t \to 0} \int_{\mathbb{P}^1(\mathbb{C})} \omega_t = \int_{x \leq -1} \varphi(x-1)\, dx$

$$+ \int_{x \geq -1} \varphi(2x)\, dx .$$

The RHS is non-archimedean in nature.

$\mathbb{C}((t)) \quad |t|_b = \tau \in (0,1)$

On $\mathbb{P}_1^{an}$ the Berkovich analytification there is a $(1,1)$-form in the sense of Chambert-Loir & Ducros

$$\omega_b := -\frac{1}{\log |t|_b} \varphi\left(- \frac{\log |z(z-t)|_b}{\log |t|_b} \right)$$

$$d' \log |z|_b \wedge d'' \log |z|_b$$

and $\int_{\mathbb{P}^{1,an}} \omega_b = \int_{x \leq 1} \varphi(x-1)\, dx$

$$+ \int_{x \geq -1} \varphi(2x)\, dx .$$

So

$$\lim_{t\to 0} \int_{\mathbb{P}^1(\mathbb{C})} \omega_t = \int_{\mathbb{P}^{1,an}} \omega_b$$

Non-archimedean forms

On $V = \mathbb{R}^n$ Lagerberg

for U open in V

$$\mathcal{A}^{p,q}(U) = \mathcal{C}^\infty(U) \otimes \Lambda^p(V^*) \otimes \Lambda^q(V^*)$$

$$\omega = \sum_{\substack{|I|=p \\ |J|=q}} \varphi_{I,J}\, d'x_I \wedge d''x_J$$

$$d' : \mathcal{A}^{p,q} \longrightarrow \mathcal{A}^{p+1,q}$$
$$d'' : \mathcal{A}^{p,q+1} \longrightarrow \mathcal{A}^{p,q+1}$$

$$d' : \varphi\, d'x_I \wedge d''x_J \longmapsto \sum \frac{\partial \varphi}{\partial x_i}\, d'x_i \wedge d'x_I \wedge d''x_J$$

similarly for d''.

Integration:

$$\int_U \varphi \, d'x_{\{1,\dots,n\}} \wedge d''x_{\{1,\dots,n\}}$$

$$:= (-1)^{\frac{n(n-1)}{2}} \int_U \varphi \, dx_1 \wedge \dots \wedge dx_n$$

Extended by CLD to piecewise linear setting:

n-dim piecewise linear subspace of $\mathbb{R}^N$ + additional data "calibration" i.e. for each n-face the choice of some n-vector [in the previous case $\Leftrightarrow \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n}$]

k a field with some ultrametric norm
$|k| \subset \mathbb{R}_+$ $\quad |x+y| \leq \sup(|x|,|y|)$

$\rightsquigarrow$ Berkovich spaces over k
locally arc connected + loc.-compact.

X alg. variety $/k$

$\rightsquigarrow X^{an}$ in the Berkovich sense

so if $X = \operatorname{Spec} A$

$X^{an} = \{$ multiplicative semi-norms $A \to \mathbb{R}_+ \}$

+ topology generated

$f \in A$ $\{ |\ |_x \mid |f|_x \in O \}$,
$O \subset \mathbb{R}_+$
open

X proper $\Leftrightarrow X^{an}$ is compact
separated separated

X^{an} retracts strongly to a PL complex
(Hrushovski - L.).

ex $\mathbb{G}_m^{n,an}$ retracts to $\mathbb{R}^n_{>0}$

$\Sigma = \mathbb{R}^n_{>0} \hookrightarrow \mathbb{G}_m^{n,an}$

$(r_1, \dots, r_n) \mapsto$
semi-norm

$\sum a_I T^I \mapsto \max |a_I| r^I$.

Connections between PL and Berkovich

Theorem (Bieri-Groves, Ducros)

X compact Berkovich space
$f = (f_1, \ldots, f_n)$ invertible analytic functions on X
then $\log|f|(X) \subset \mathbb{R}^n$
it is a finite union of polytopes.

X Berkovich space
V open $\quad f = (f_1, \ldots, f_n) \quad f_i \in \mathcal{O}_X(V)^*$

CLD define (p,q)-forms on V
as pullbacks
$(\log|f|)^*(\alpha)$ with
α a (p,q)-form on some PL object $\subset \mathbb{R}^n$.

$\mathcal{A}^{p,q}_X \qquad d', d''$

Integration of (n,n) forms (X dim n).
May assume X compact
$\omega = \varphi \, d' \log|f_I| \wedge d'' \log|f_J|$

then $\log|f|(X)$ is $\bigcup_{1\le i\le m} P_i$
polytopes

WMA $m=1$ $P_i = P$

- if $\dim P < n$, then $\omega = 0$ and $\int_X \omega = 0$
- if $\dim P = n$

$$\omega = \psi(\log|f_1|, \dots, \log|f_n|)\; d'\log \dots \wedge d''\log \dots$$

WMA $\log|f|(\partial X) \subset \partial P$

$$\begin{tikzcd}
X \arrow[r, "f"] & \mathbb{G}_m^{n,an} \\
& \Sigma = \mathbb{R}^n_{>0} \arrow[u, phantom, "\cup" description] \\
& P \subset \mathbb{R}^n \arrow[u, phantom, "\cong" description] \arrow[uu, bend left, "\eta"']
\end{tikzcd}$$

above $\eta(\mathring{P})$ f is finite flat of degree d.

Set $\displaystyle\int_X \omega = \underline{\underline{d}}\,(-1)^{\frac{n(n-1)}{2}} \int \psi \, dx_1 \wedge \dots \wedge dx_n.$

CLD theory analogous to

complex geometry :

- Stokes theorem
- Poincaré - Lelong
- currents
- Monge - Ampère etc ...

Our result may be stated as follows

(We consider forms of the type

$\boxed{\lambda := -\log|t|}$

$$\omega = \frac{1}{\lambda^{|I|}} \sum_{I,J} \varphi\left(\frac{\log|f_1|}{\lambda}, \dots, \frac{\log|f_n|}{\lambda}\right) d\log|f_I| \wedge \frac{d\arg}{2\pi}(f_J)$$

Zariski locally.

We have to allow some f_i's to vanish but not those occuring in $d\log|f_I|$ or $d\arg|f_J|$.

similarly $\lambda_b := -\log|t|_b$

$$\omega_b = \frac{1}{\lambda_b^{|I|}} \sum \varphi\left(\frac{\log|f_1|_b}{\lambda_b}, \dots \right) d'\log|f_I|_b \wedge d''\log|f_J|_b .$$

<u>Main theorem</u>. Let X be a smooth alg variety over $\mathcal{M}_{er}$ of dim n.

Then
$$\lim_{t \to 0} \int_{X_t} \omega = \int_{X^{an}} \omega_b$$

X^{an} w. respect to $\mathcal{M}_{er} \hookrightarrow \mathbb{C}((t))$.

II Connecting complex (p,q)-forms and NA (p,q)-forms.

Connecting complex and NA worlds

- hybrid spaces (Berkovich, Boucksom, Jonsson);
- another approach here.

Work on a field $\tilde{\mathbb{C}}$ with two structures. Considered long ago by A. Robinson.

Fix $\mathcal{U}$ a non-principal ultrafilter on $\mathbb{C}$, containing all neigh. of 0.

Then $\quad {}^*\mathbb{R} := \prod_{t\in\mathbb{C}} \mathbb{R}/\mathcal{U} \qquad {}^*\mathbb{C} = \prod_{t\in\mathbb{C}} \mathbb{C}/\mathcal{U}$.

$$(x_t) \sim (x'_t) \iff \{t \mid x_t = x'_t\} \subset \mathcal{U},$$

$$|\ | : {}^*\mathbb{C} \longrightarrow {}^*\mathbb{R}_{\geq 0}$$

In ${}^*\mathbb{C}$ or ${}^*\mathbb{R}$ we

(a_t) t-bounded if $|a_t| \leq |t|^{-N}$ some $N \in \mathbb{N}$ along $\mathcal{U}$

t-negligible if $|a_t| \leq |t|^N$ all $N \in \mathbb{N}$ along $\mathcal{U}$

A_r (A_c) $= \{t\text{-bounded el of } {}^*\mathbb{R}\}$ (${}^*\mathbb{R}$)

local ring with maxideal M_r (M_c) $= \{t\text{-negligible el of } A_r\}$ (A_c)

So we may define

$$\tilde{\mathbb{R}} := A_r / m_r .$$

Similarly $\tilde{C} := A_C / m_C$.

We have $\tilde{C} = \tilde{R}(i)$

$\uparrow$ alg. closed $\quad \uparrow$ real closed

We have

$\mathcal{M}er \hookrightarrow \tilde{C}$

$$\boxed{\text{we have } |\ | : \tilde{C} \longrightarrow \tilde{R}_{\geq 0}}$$

nat. norm

If $z \in \tilde{C}$ is bounded (i.e. $|z_t| \leq N$ along $\mathcal{U}$).

then $\exists!\, \alpha \in \mathbb{C}$ s.t. $z - \alpha$ is infinitesimal i.e. smaller than $|z-\alpha| < \frac{1}{N}$ for any $N > 0$, $N \in \mathbb{N}$.

We set $\alpha := st(z)$

Fact if $z \in \tilde{C}^*$

$\dfrac{\log |z|}{\log |t|}$ is bounded

So $\alpha := st\left(\frac{\log|z|}{\log|t|}\right) \in \mathbb{R}$

Fix $\tau \in \mathbb{R}$, $0 < \tau < 1$.

Set $|z|_b := \tau^{\alpha}$

Note $|t|_b = \tau$.

We get a valuation

$$\boxed{|\ |_b : \tilde{\mathbb{C}} \longrightarrow \mathbb{R}_+}$$

Fact: There is a nice integration theory over $\tilde{\mathbb{R}}$ and $\tilde{\mathbb{C}}$ (with values in $\tilde{\mathbb{R}}$ and $\tilde{\mathbb{C}}$)

Take X an n-dim variety over $\tilde{\mathbb{C}}$.

We define Zariski sheaves $\mathcal{A}_X^{p,q}$ of (p,q) forms on X, locally of the form

$$\omega = \frac{1}{\lambda^p} \sum_{\substack{I,J \\ |I|=p \\ |J|=q}} \varphi_{I,J}\left(\frac{\log|f_1|}{\lambda}, \ldots \frac{\log|f_m|}{\lambda}\right)$$

$$d\log|f_I| \wedge d\log|f_J|$$

we have differentials

$$d : A_X^{p,q} \to A_X^{p+1,q}$$

$$d^\#: A_X^{p,q} \longrightarrow A_X^{p,q+1}$$

$$d^\# \longleftrightarrow d^c$$

Similarly we define

$\mathcal{B}_X^{p,q}$ Zariski sheaves on X:

on X^{an} (w.r.t. $|\ |_b$):

locally of the form

$$\omega_b = \frac{1}{\lambda_b^{|I|}} \sum \varphi_{I,J}\left(\frac{\log|f_1|_b}{\lambda_b}, \ldots\right)$$

$$d' \log|f_I|_b \wedge d'' \log|f_J|_b.$$

Main Theorem (Ducros, Hrushovski, L.)

There exists a unique morphism of bigraded differential algebras

$$A_X^{\bullet,\bullet} \longrightarrow B_X^{\bullet,\bullet}$$

sending ω to ω_b

$[d \mapsto d', \ d^{\#} \mapsto d'']$

which is compatible with integration in the sense that for (n,n)-forms

$$\mathrm{sh}\left(\int_{X(\tilde{C})} \omega\right) = \int_{X^{an}} \omega_b .$$

Difficult point: ω_b does not depend on the way one writes ω locally.

To prove this, compatibility with integration + fact that there is duality forms $\longleftrightarrow$ currents in the NA world