The super Mumford form and Sato Grassmannian

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Outline

The moduli space of Riemann surfaces & bosonic string theory

Let $\mathscr{M}_g$ be the moduli space of Riemann surfaces.

In bosonic string theory, the g loop contribution to the partition function is the integral of the Polyakov measure over the moduli space of genus g Riemann surfaces:

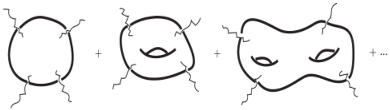
$$Z_g = \int_{\mathscr{M}_g} d\pi_g.$$

The Polyakov measure was shown by [3] Belavin and Knizhnik to be given by the modulus squared of the Mumford form μ_g:

$$d\pi_g = \mu_g \wedge \overline{\mu}_g.$$

The Mumford form μ_g is a trivializing section of the Mumford isomorphism $\lambda_2 \otimes \lambda_1^{-13}$.

Moduli space of SRSs difficulties



In [5] D'Hoker and Phong calculated the genus 2 contribution to the scattering amplitude in superstring theory. They integrated over the moduli space of SRSs by descending to an integral over the moduli space of Riemann surfaces via a natural holomorphic map, and describing the fiber using the super period matrix.

$$\mathbf{A}(1,\cdots,N)=\int_{\Gamma}\int_{\Sigma^N}\int dp_I^{\mu}\,\mathcal{F}_L(\tilde{m}_A,\tilde{z}_i,\tilde{\theta}_i;p_I^{\mu})\,\mathcal{F}_R(m_A,z_i,\theta_i;p_I^{\mu})$$

Essentially, they used the Torelli map sending a super Riemann surface to its Jacobian, in the moduli space principally polarized Abelian super varieties. This technique is more challenging for higher genus due to the Schottky problem, possible zero modes, and a more complicated parametrization of the fiber.

Perhaps the super Sato Grassmannian Gr(H) could provide an alternate strategy.

Super vector spaces

A super vector space is a $\mathbb{Z}_2$-graded vector space, $V = V_0 \oplus V_1$, called the even and odd components. If $\dim(V_0) = m$ and $\dim(V_1) = n$, we say $\dim(V) = m|n$.

Morphisms of super vector spaces must preserve the parity. $\mathrm{Hom}(V, W)$ denotes the morphisms $V \to W$.

The parity reversing operator Π swaps the even and odd components.

$$(\Pi V)_0 := V_1 \qquad\qquad (\Pi V)_1 := V_0$$

Example

We may regard the hom $\mathrm{Hom}(V, W)$ as a classical vector space, while the internal hom $\underline{\mathrm{Hom}}(V, W)$ is a super vector space:

$$(\underline{\mathrm{Hom}}(V, W))_0 := \mathrm{Hom}(V, W)$$
$$(\underline{\mathrm{Hom}}(V, W))_1 := \mathrm{Hom}(\Pi V, W) \cong \mathrm{Hom}(V, \Pi W)$$

Superalgebras

A superalgebra is a $\mathbb{Z}_2$-graded algebra. A superalgebra is supercommutative if the supercommutator vanishes.

$$[a, b] := ab - (-1)^{|a||b|} ba = 0$$

In particular, any odd element has square zero: $|\alpha| = 1 \quad \Rightarrow \quad \alpha^2 = 0$.

Let A be a superalgebra. Then M is a A-module if $|am| = |a| + |m|$.

The internal hom $\underline{\mathrm{Hom}}_A(M, N)$ are those maps T which satisfy $aT(m) = (-1)^{|a||T|} T(am)$. We can represent T as a matrix:

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad\qquad B\colon\ M_1 \to N_0,\ \text{etc.}$$

If T is invertible, the superdeterminant=Berezinian is defined as

$$\mathrm{Ber}\, T = \det(A - BD^{-1}C)\Big/\det D = \det A\Big/\det(D - CA^{-1}B)$$

Lie superalgebras and Lie superalgebroids

Definition

A Lie superalgebra is a super vector space $\mathfrak{g}$ with a morphism $[\quad,\quad]\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ satisfying

- $[x,y]+(-1)^{|x||y|}[y,x]=0$
- $(-1)^{|x||z|}[x,[y,z]]+(-1)^{|y||x|}[y,[z,x]]+(-1)^{|z||y|}[z,[x,y]]=0$

Example

The general linear Lie superalgebra $\mathfrak{gl}(V) := \underline{\mathrm{Hom}}(V,V)$, the internal endomorphism space. We have that $(\mathfrak{gl}(V))_0 \cong \mathfrak{gl}(V_0)\oplus\mathfrak{gl}(V_1)$.

A Lie (super)algebroid is the many-body generalization of a Lie (super)algebra. A Lie (super)algebroid is a vector bundle $E\to M$ with a vector bundle map $a\colon E\to TM$ (called the anchor map), an (super) alternating bracket on the sections of E, and (super) Jacobi identity.

Atiyah Lie superalgebroids

The Atiyah Lie superalgebra $\mathscr{A}_L$ on X is the Lie superalgebroid of order 1 operators on the line bundle L. It is a Lie superalgebra extension with a compatible left $\mathscr{O}_X$-module structure:

$$0 \to \mathscr{O}_X \to \mathscr{A}_L \xrightarrow{\mathrm{sym}_1} \mathscr{T}_X \to 0,$$

where the anchor map is the symbol map defined as $\mathrm{sym}_1(D)(f) = [D, f]$.

A holomorphic connection on L may be defined as a splitting of the Atiyah sequence above $\nabla\colon \mathscr{T}_X \to \mathscr{A}_L$ as $\mathscr{O}_X$-modules. When a connection is also a splitting as Lie superalgebroids, that is when $\Big[\nabla(v), \nabla(w)\Big] = \nabla\Big([v, w]\Big)$, then the connection is flat.

Action Lie superalgebroids

The action Lie superalgebroid associated to a Lie superalgebra homomorphism $\phi\colon \mathfrak{g} \to \Gamma(X, TX)$ is given by the trivial vector bundle $X \times \mathfrak{g}$ with anchor map $a\colon X \times \mathfrak{g} \to TX$ such that $a(x, X) = \phi(X)(x)$, and bracket

$$[V, W] = \mathscr{L}_{\widehat{\phi}(V)}(W) - (-1)^{|V||W|}\mathscr{L}_{\widehat{\phi}(W)}(V) + [V, W]_{\mathfrak{g}}$$

where $V, W \in \Gamma(X \times \mathfrak{g}, X)$ and $\widehat{\phi}(V)(x) = \phi(V(x))(x)$.

Lemma ([8] Kosmann-Schwarzbach and Mackenzie 2002)

There is a bijection between

- *Lie superalgebra morphisms* $\mathfrak{g} \to \Gamma(X, \mathscr{A}_L)$
- *Lie superalgebroid morphisms* $\mathscr{G} = \mathfrak{g} \ltimes X \to \mathscr{A}_L$

Holomorphic supermanifolds

Definition

A complex supermanifold of dimension $m|n$ is the data $(X, \mathscr{O}_X)$ such that

- X is a complex manifold of dimension m
- $\mathscr{O}_X$ is a sheaf on X of supercommutative $\mathbb{C}$ algebras
- $\mathscr{O}_X$ is locally isomorphic to $\mathscr{O}_{\mathbb{C}^m} \otimes \bigwedge(\zeta_1, \dots, \zeta_n)$ (holomorphic)

We usually write $\mathbb{C}^{m|n} = (\mathbb{C}^m, \mathscr{O}_{\mathbb{C}^m}[\zeta_1, \dots, \zeta_n])$, leaving implicit the anticommuting of the odd functions.

The subsheaf $\mathscr{J}_X \subset \mathscr{O}_X$ is defined as the nilpotent ideal generated by odd functions. The $m|0$-dim manifold $(X, \mathscr{O}_X/\mathscr{J}_X)$ is called the reduction of $(X, \mathscr{O}_X)$.

Functor of points

In order to capture the odd functions on a superspace, we can use S-points. An S-point of X is a morphism of superspaces $S \to X$.

Example

- For $S = \mathbb{C}^{0|0}$ a point, then an S-point of X is just a topological point of X.
- For $S = \mathbb{C}^{0|s}$ a superpoint, and X of dim $= m|n$, then an S-point of X is a topological point of X and n odd functions on S.

Example

The super Grassmannian $\mathrm{Gr}(c|d, V)$ may be defined as the functor

$$S \mapsto \left\{ \text{rank } c|d \text{ subbundles of } V \otimes_{\mathbb{C}} \mathcal{O}_S \right\}$$

The reduced space of $\mathrm{Gr}(c|d, V)$ is $\mathrm{Gr}(c, V_0) \times \mathrm{Gr}(d, V_1)$.

Super Riemann surfaces (SUSY curves)

Definition

A super Riemann surface (SRS) is a complex supermanifold of dimension $1|1$ with a maximally nonintegrable distribution $\mathscr{D}$ of rank $0|1$. That is precisely,

- $\mathscr{D}$ is an odd subbundle of the tangent bundle $\mathscr{T}$
- the induced map by Lie bracket $[\ ,\ ] : \mathscr{D}^{\otimes 2} \to \mathscr{T}/\mathscr{D}$ is an isomorphism.

We can always find local coordinates (called superconformal coordinates) $z|\zeta$ such that $\mathscr{D}$ is generated by

$$D_\zeta = \frac{\partial}{\partial \zeta} + \zeta \frac{\partial}{\partial z}.$$

The sheaf of superconformal vector fields $\mathscr{T}^s \subset \mathscr{T}$ is the sheaf whose sections S satisfy $[S, \mathscr{D}] \subset \mathscr{D}$. In other words, a superconformal vector field S preserves the SUSY structure. A section S may be written in local coordinates as $[f D_\zeta, D_\zeta]$ for some function f.

Moduli space of SRSs

The moduli space of super Riemann surfaces may be denoted $\mathfrak{M}_g$. For $g \geq 2$, it has dimension $3g-3|2g-2$ and is a Deligne-Mumford stack.

The reduction of $\mathfrak{M}_g$ is known to be the moduli space of spin curves. Without odd moduli, we have $\mathscr{D}^\vee \cong K^{1/2}$ where K is the canonical bundle of the reduced Riemann surface.

In other words, a single super Riemann surface is split: that is X is given by the total space of $\Pi K^{1/2} \to X_{\text{red}}$ where $K^{1/2}$ is a choice of spin structure. On a Riemann surface of genus g, there are 2^{2g} different spin structures.

While working with split superspaces is much nicer, the interesting part of supergeometry is working with nonsplit superspaces. For example, $\mathfrak{M}_g$ is not split for $g \geq 5$ ([6] Donagi and Witten).

A super numerical coincidence ([9] Manin 1986)

Theorem ([17] Voronov 1988; [4] Deligne)

For a smooth proper family of $1|1$ supermanifolds $\pi : X \to S$, the super Mumford isomorphism is

$$\lambda_{j/2} \cong \lambda_{1/2}^{-(-1)^j(2j-1)}, \qquad \text{in particular} \qquad \lambda_{3/2} \cong \lambda_{1/2}^5.$$

where $\lambda_{j/2} := \operatorname{Ber} R\pi_*(\omega_{X/S}^{\otimes j})$ *and* $\omega_{X/S} := \operatorname{Ber} \Omega^1_{X/S}$.

The Neveu-Schwarz algebra has a natural representation ϱ_j on the formal super Laurent series $\mathbb{C}((z))[\zeta]$ defined by Lie derivative action on $\mathbb{C}((z))[\zeta]\,[dz|d\zeta]^{\otimes j}$.

Proposition ([16] Ueno and Yamada 1988)

The pullbacks of the super Japanese cocycle η along the representations ϱ_j satisfies:

$$\varrho_j^*(\eta) = -(-1)^j(2j-1)\varrho_1^*(\eta)$$

A classical numerical coincidence ([9] Manin 1986)

Theorem ([13] Mumford 1977)

For a smooth proper family of curves $\pi : X \to S$, the Mumford isomorphism is

$$\lambda_j \cong \lambda_1^{(6j^2-6j+1)}, \qquad \textit{in particular} \qquad \lambda_2 \cong \lambda_1^{13}.$$

where $\lambda_j := \det R\pi_*(\omega_{X/S}^{\otimes j})$ *and* $\omega_{X/S} := \Omega^1_{X/S}$.

The Virasoro algebra has a natural representation ϱ_j on the formal Laurent series $\mathbb{C}((z))$ defined by Lie derivative action on $\mathbb{C}((z))dz^{\otimes j}$.

Proposition

The pullbacks of the Japanese cocycle η along the representations ϱ_j satisfies:

$$\varrho_j^*(\eta) = (6j^2 - 6j + 1)\varrho_1^*(\eta)$$

The moduli space of triples and the Witt algebra

Definition

Define the moduli stack $\mathscr{M}_{g,1^k}$ to be the moduli space of triples (C, p, z), where C is a genus g Riemann surface, p is a puncture, and z is a formal coordinate near the puncture considered as a k-jet equivalence class.

We construct the pro-Deligne-Mumford stack

$$\mathscr{M}_{g,1^\infty} = \varprojlim \mathscr{M}_{g,1^k}.$$

The Witt algebra $\mathfrak{witt} \cong \mathbb{C}((z))\frac{\partial}{\partial z}$. Its central extension is the Virasoro algebra.

Proposition ([7] Kontsevich 1987)

The Witt algebra acts on the moduli space $\mathscr{M}_{g,1^\infty}$ by vector fields.

$$T_{(C,p,z)}(\mathscr{M}_{g,1^\infty}) \cong \mathfrak{witt}\big/\Gamma(C \setminus p, \mathscr{T}_C)$$

The Sato Grassmannian

Define a subspace D of $\mathbb{C}((z))$ to be discrete if $D \cap \mathbb{C}[\![z]\!]$ and $\mathbb{C}((z))/(D + \mathbb{C}[\![z]\!])$ are finite dimensional. For example, $z^{n+1}\mathbb{C}[z^{-1}]$ is discrete.

Definition ([14] Sato 1983)

The Sato Grassmannian $\mathrm{Gr}(\mathbb{C}((z)))$ is the infinite dimensional manifold whose points are discrete subspaces $D \subset \mathbb{C}((z))$.

Slightly altering the standard definition of a determinant line bundle, we can define $\det_{\mathrm{Gr}}$ on the Sato Grassmannian.

Definition

We define the determinant line bundle on $\mathrm{Gr}(\mathbb{C}((z)))$ as

$$\det{}_{\mathrm{Gr}}(D) := \frac{\det\left(D \cap \mathbb{C}[\![z]\!]\right)}{\det\left(\mathbb{C}((z))/(D + \mathbb{C}[\![z]\!])\right)}.$$

The classical numerical coincidence explained

Roughly speaking, the moduli space of curves can be considered a subset of the Sato Grassmannian, where there is 'nice' Lie algebra action.

Theorem ([15] Segal and Wilson 1985)

The Krichever map $\mathscr{M}_{g,1^\infty} \to \mathrm{Gr}(\mathbb{C}((z)))$ *defined by*

$$\kappa_j(C, p, z) = \Gamma(C \setminus p, \omega_C^{\otimes j}) \subset \mathbb{C}((z)).$$

where $\omega_C := \Omega^1_C$, *is an injective analytic map.*

The pullback of the determinant bundle is $\kappa_j^* \det_{\mathrm{Gr}} \cong \lambda_j$.
Then the representation theory of the Virasoro algebra gives that:

Theorem ([7] Kontsevich 1987; [2] Arbarello, DeConcini, Kac, Procesi 1988)

There exists a flat holomorphic connection on the line bundle $\lambda_1^{-(6j^2-6j+1)} \otimes \lambda_j$.

The super moduli space triples

Definition

Define the moduli stack $\mathfrak{M}_{g,1^k_{\mathrm{NS}}}$ to be the moduli space of triples $(\Sigma, p, z|\zeta)$, where Σ is a genus g SUSY curve, p is the divisor which represents NS punctures, and $z|\zeta$ is a coordinate system near the punctures considered as a k-jet equivalence class.

We construct the pro-Deligne-Mumford stack

$$\mathfrak{M}_{g,1^\infty_{\mathrm{NS}}} = \varprojlim \mathfrak{M}_{g,1^k_{\mathrm{NS}}}.$$

Witt superalgebra and Neveu-Schwarz superalgebra

The super Witt algebra $\mathfrak{switt}$ is the Lie superalgebra of superconformal vector fields on a punctured formal neighborhood of a point in $\mathbb{C}^{1|1}$.

$$\mathfrak{switt} = \left\{ [fD_\zeta, D_\zeta] = 2f(z|\zeta)\frac{\partial}{\partial z} + (-1)^{|f|} D_\zeta f(z|\zeta) D_\zeta : \; f(z|\zeta) \in \mathbb{C}((z))[\zeta] \right\}$$

The Neveu-Schwarz superalgebra is the unique central extension of $\mathfrak{switt}$.

Proposition ([11] M, cf. [10] Manin 1988)

The Witt superalgebra acts on the moduli space $\mathfrak{M}_{g,1^\infty_{\mathrm{NS}}}$ by vector fields.

$$\Lambda : \; \mathfrak{switt} \to \Gamma(\mathfrak{M}_{g,1^\infty_{\mathrm{NS}}}, \mathscr{T}_{\mathfrak{M}})$$

The formal super Laurent series

Define the vector superspace $H = \mathbb{C}((z))[\zeta] = \mathbb{C}((z))\zeta + \mathbb{C}((z))$ where $\zeta^2 = 0$. In the below we use the notation $\widehat{V}_S := V \widehat{\otimes}_{\mathbb{C}} \mathcal{O}_S$.

Definition

Define a super subspace of H to be compact if it is commensurable with $\mathbb{C}[[z]][\zeta] = H^+$.

For a superscheme S with structure sheaf $\mathcal{O}_S$, a super submodule $L \subset \widehat{H}_S$ is discrete if for every $s \in S$ there exists a neighborhood U of s and a compact K such that the natural map $L_U \oplus \widehat{K}_U \to \widehat{H}_U$ is an isomorphism.

Explicitly, subspaces K and H^+ are commensurable when $K/(K \cap H^+)$ and $H^+/(K \cap H^+)$ both have finite even dimension and finite odd dimension.

The super Sato Grassmannian

The definition below is adapted to the super setting from the defintion in [1] Álvarez Vázquez, Muñoz Porras, Plaza Martín 1998.

Definition ([11] M)

The super Sato Grassmannian $\mathrm{Gr}(\mathbb{C}((z))[\zeta])$ is the as the complex superscheme representing the functor of the discrete super subspaces $D \subset H$:

$$S \mapsto \{\text{discrete sub-}\mathcal{O}_S\text{-modules of } H\widehat{\otimes}_{\mathbb{C}}\mathcal{O}_S\}$$

Berezinian line bundle

Generalizing the determinant line bundle for the supervector spaces gives a Berezinian line bundle $\mathscr{B}er$ on the super Sato Grassmannian.

$$\mathscr{B}er := \frac{\mathrm{Ber}(I \cap \widehat{H}^{+}_{\mathrm{Gr}})}{\mathrm{Ber}(\widehat{H}_{\mathrm{Gr}}/(I + \widehat{H}^{+}_{\mathrm{Gr}}))},$$

where $I \in \widehat{H}_{\mathrm{Gr}} = H \widehat{\otimes}_{\mathbb{C}} \mathscr{O}_{\mathrm{Gr}(H)}$ is the tautological bundle, a discrete super submodule.

I have shown the Lie algebra $\widetilde{\mathfrak{gl}}(H)$ acts by first order differential operators on $\mathscr{B}er$ by finding the explicit formula in local coordinates on $\mathrm{Gr}(\mathbb{C}((z))[\zeta])$. (on the next slide)

The Lie superalgebra action on $\mathscr{B}er$

Definition ([16] Ueno and Yamada 1988)

The super Japanese cocycle on $\mathfrak{gl}(H)$ is

$$\eta(F, G) := \mathrm{str}(F^{-+}G^{+-} - (-1)^{|F||G|}G^{-+}F^{+-})$$

where $F^{-+}: H^+ \to H^-$, etc. This cocycle defines the central extension $\widetilde{\mathfrak{gl}}(H)$.

Proposition ([11] M)

The Lie superalgebra $\widetilde{\mathfrak{gl}}(H)$ acts by first order differential operators on $\mathscr{B}er$. In the chart $U_{D,K} \cong \underline{\mathrm{Hom}}_{\mathbb{C}}(D, K)$, this action is given by the formula

$$\widetilde{L}_F(A) = F^{KD} + F^{KK}A - AF^{DD} - AF^{DK}A + \mathrm{str}(F^{DK}A) + \alpha(F)$$

where $\alpha \in C^1(\mathfrak{gl})$ is such that $d\alpha(F, G) = \alpha([F, G]) = \eta_{D,K}(F, G) - \eta(F, G)$.

The super Krichever map

Definition ([12] Mulase and Rabin 1991)

The super Krichever map $(\mathfrak{M}_{g,1^\infty_{NS}})_{\mathrm{red}} \to (\mathrm{Gr}(\mathbb{C}((z))[\zeta]))_{\mathrm{red}}$ is

$$\kappa_j(\Sigma, p, z|\zeta) = \Gamma(\Sigma \setminus p, \omega_\Sigma^{\otimes j}) \subset \mathbb{C}((z))[\zeta].$$

Consider a family of super Riemann surfaces $\pi \colon X \to S$ with NS puncture P. The sheaf $\pi_* \omega_{(X \setminus P)/S}^{\otimes j}$ is a discrete $\mathcal{O}_S$-submodule.

Definition ([11] M)

The super Krichever map $\mathfrak{M}_{g,1^\infty_{NS}} \to \mathrm{Gr}(\mathbb{C}((z))[\zeta])$ is

$$\kappa_j(X/S, p, z|\zeta) = \pi_* \omega_{(X \setminus p)/S}^{\otimes j} \subset \mathcal{O}_S((z))[\zeta].$$

Pullback along the super Krichever map

We pullback $\widetilde{\mathscr{G}} = \widetilde{\mathfrak{gl}} \ltimes \mathrm{Gr}$ along κ_j.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O}_{\mathfrak{M}} & \hookrightarrow & {\kappa_j}^! \widetilde{\mathscr{G}} & \twoheadrightarrow & {\kappa_j}^! \mathscr{G} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \widetilde{L}} & & \downarrow{\scriptstyle L} & & \\
0 & \longrightarrow & \mathscr{O}_{\mathfrak{M}} & \hookrightarrow & {\kappa_j}^! \mathscr{A}_{\mathscr{B}er} & \twoheadrightarrow & \mathscr{T}_{\mathfrak{M}} & \longrightarrow & 0
\end{array}
$$

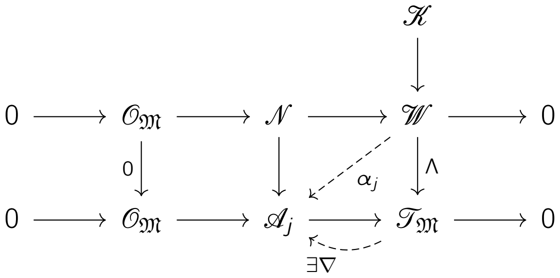
The action of the Neveu-Schwarz Lie superalgebroid $\mathscr{N} = \mathfrak{ns} \ltimes \mathfrak{M}$ results:

$$
\begin{array}{ccccc}
\mathscr{O}_{\mathfrak{M}} & \longrightarrow & \mathscr{N} & \longrightarrow & \mathscr{W} \\
\downarrow{\scriptstyle \cdot c_j} & & \downarrow & & \downarrow{\scriptstyle \varrho_j} \\
\mathscr{O}_{\mathfrak{M}} & \longrightarrow & {\kappa_j}^! \widetilde{\mathscr{G}} & \twoheadrightarrow & {\kappa_j}^! \mathscr{G} \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \widetilde{L}} & & \downarrow{\scriptstyle L} \\
\mathscr{O}_{\mathfrak{M}} & \longrightarrow & \mathscr{A}_{{\kappa_j}^* \mathscr{B}er} & \twoheadrightarrow & \mathscr{T}_{\mathfrak{M}}
\end{array}
\qquad \Lambda : \mathscr{W} \to \mathscr{T}_{\mathfrak{M}}
$$

A flat holomorphic connection

We notice that $\lambda_{j/2} \cong \kappa_j^* \mathscr{B}er$. Denote $\mathscr{A}_j := \mathscr{A}_{\lambda_{j/2} \otimes \lambda_{1/2}^{-c_j}}$.

Analyzing the diagram of super Lie superalgebroids below gives the main result.



Theorem ([10] Manin 1988, [11] M)

There exists a flat holomorphic connection ∇ *on the line bundle* $\lambda_{j/2} \otimes \lambda_{1/2}^{-c_j}$.

Lemma on the perfectness of a Lie superalgebroid

Proposition ([11] M)

Let $\pi : Y \to S$ be a family of open super Riemann surfaces. Denote the Lie superalgebroid of global superconformal vector fields as $\mathscr{K} := \pi_*(\mathscr{T}^s_{Y/S})$. Then the Lie superalgebroid $\mathscr{K}$ is perfect, that is to say, $\mathscr{K}/[\mathscr{K}, \mathscr{K}] = 0$.

We prove this using a strategy analogous to the proof for classical Riemann surfaces in [2]. In local coordinates, we can verify the proposition by noticing:

$$3[hD_\zeta, D_\zeta] = \Big[[hD_\zeta, D_\zeta], [zD_\zeta, D_\zeta]\Big] + \Big[[D_\zeta, D_\zeta], [hzD_\zeta, D_\zeta]\Big] + \Big[[h\zeta D_\zeta, D_\zeta], [\zeta D_\zeta, D_\zeta]\Big]$$

Then, we use a superconformal Nother normalization lemma over a family to show the proposition globally.

Main proof details

Proof of existence of a holomorphic connection.

Define $\mathscr{K}$ to be the kernel of the Witt superalgebroid action $\Lambda\colon \mathscr{W} \to \mathscr{T}_{\mathfrak{M}}$. By the Proposition on the last slide, we know that $\mathscr{K} = [\mathscr{K}, \mathscr{K}]$.

Now, let $k \in \Gamma(U, \mathscr{K})$ for some open U in S. We have shown that $k = [k_1, k_2]$ for some $k_1, k_2 \in \Gamma(U, \mathscr{K})$. Then notice that since $\Lambda(k_i) = 0 \in \Gamma(U, \mathscr{T}_{\mathfrak{M}})$, then $\alpha_j(k_i) \in \ker(\Gamma(U, \mathscr{A}_j) \to \Gamma(U, \mathscr{T}_{\mathfrak{M}})) = \Gamma(U, \mathscr{O}_{\mathfrak{M}})$. Thus $\alpha_j(k) = [\alpha_j(k_1), \alpha_j(k_2)] = 0$ as an element of $\Gamma(U, \mathscr{A}_j)$.

Note that $\mathscr{T}_{\mathfrak{M}} = \operatorname{coker}(\mathscr{K} \to \mathscr{W})$. Since α_j maps $\mathscr{K}$ to zero, then by the universal property of the cokernel, then α_j must factor through $\mathscr{T}_{\mathfrak{M}}$ uniquely. That is, we have a unique morphism of Lie superalgebroids $\nabla\colon \mathscr{T}_{\mathfrak{M}} \to \mathscr{A}_j$. □

References

[1] A. Álvarez Vázquez, J. M. Muñoz Porras, and F. J. Plaza Martín. The algebraic formalism of soliton equations over arbitrary base fields. In *Workshop on Abelian Varieties and Theta Functions (Spanish) (Morelia, 1996)*, volume 13 of *Aportaciones Mat. Investig.*, pages 3–40. Soc. Mat. Mexicana, México, 1998.

[2] E. Arbarello, C. De Concini, V. G. Kac, and C. Procesi. Moduli spaces of curves and representation theory. *Communications in Mathematical Physics*, 117(1):1–36, 1988. doi:10.1007/bf01228409.

[3] A. A. Belavin and V. G. Knizhnik. Complex geometry and the theory of quantum strings. *Akademiya Nauk SSSR. Zhurnal Èksperimental′ noĭ i Teoreticheskoĭ Fiziki*, 91(2):364–390, 1986.

[4] P. Deligne. Lettre à Manin, 1988.

[5] E. D'Hoker and D. H. Phong. Two-loop superstrings. I. Main formulas. *Physics Letters. B. Particle Physics, Nuclear Physics and Cosmology*, 529(3-4):241–255, 2002. doi:10.1016/S0370-2693(02)01255-8.

References (cont.)

[6] R. Donagi and E. Witten. Supermoduli space is not projected. In *String-Math 2012*, volume 90 of *Proc. Sympos. Pure Math.*, pages 19–71. Amer. Math. Soc., Providence, RI, 2015. doi:10.1090/pspum/090/01525.

[7] M. L. Kontsevich. The Virasoro algebra and Teichmüller spaces. *Akademija Nauk SSSR. Funkcional'nyi Analiz i ego Priloženija*, 21(2):78–79, 1987.

[8] Y. Kosmann-Schwarzbach and K. C. H. Mackenzie. Differential operators and actions of Lie algebroids. In *Quantization, Poisson brackets and beyond (Manchester, 2001)*, volume 315 of *Contemp. Math.*, pages 213–233. Amer. Math. Soc., Providence, RI, 2002. doi:10.1090/conm/315/05482.

[9] Y. I. Manin. Critical dimensions of the string theories and the dualizing sheaf on the moduli space of (super) curves. *Functional Analysis and Its Applications*, 20(3):244–246, July 1986. doi:10.1007/BF01078480.

[10] Y. I. Manin. Neveu-Schwarz sheaves and differential equations for Mumford superforms. *Journal of Geometry and Physics*, 5(2):161–181, 1988. doi:10.1016/0393-0440(88)90002-2.

References (cont.)

[11] K. A. Maxwell. The super Mumford form and Sato Grassmannian. Feb. 2020, arXiv:2002.06625.

[12] M. Mulase and J. M. Rabin. Super Krichever functor. *International Journal of Mathematics*, 2(6):741–760, 1991. doi:10.1142/S0129167X91000417.

[13] D. Mumford. Stability of projective varieties. *L'Enseignement Mathématique. Revue Internationale. 2e Série*, 23(1-2):39–110, 1977. doi:10.1007/978-1-4757-4265-7_7.

[14] M. Sato. Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold. In *Nonlinear partial differential equations in applied science (Tokyo, 1982)*, volume 81 of *North-Holland Math. Stud.*, pages 259–271. North-Holland, Amsterdam, 1983. doi:10.1016/s0304-0208(08)72096-6.

[15] G. Segal and G. Wilson. Loop groups and equations of KdV type. In *Surveys in differential geometry: integral systems [integrable systems]*, volume 4 of *Surv. Differ. Geom.*, pages 403–466. Int. Press, Boston, MA, 1998. doi:10.4310/SDG.1998.v4.n1.a8.

References (cont.)

[16] K. Ueno and H. Yamada. Some observations on geometric representations of the superconformal algebras and a super analogue of the Mumford sheaves. In *Algebraic analysis, Vol. II*, pages 893–900. Academic Press, Boston, MA, 1988. doi:10.1016/b978-0-12-400466-5.50032-9.

[17] A. A. Voronov. A formula for the Mumford measure in superstring theory. *Functional Analysis and Its Applications*, 22(2):139–140, 1988. doi:10.1007/bf01077608.