From Six Functors Formalisms to Derived Motivic Measures

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Overview

1. K-Theory of Varieties
2. The Gillet-Soulé Motivic Measure
3. (∞-)Categorical and K-Theoretic Preliminaries
4. Six Functors Formalisms
5. From Six Functors to Derived Motivic Measures
6. A Derived Lift of the Gillet-Soulé Motivic Measure
7. Conclusion
8. References

Preamble

Conventions:

- k will always refer to a perfect field
- Our model for ∞-categories will mostly be Kan-enriched categories (just because it gives us a strict horizontal composition and a nicely concrete model for K-theory)
- $\mathbf{Var}_S$ will be finite type separated schemes over S.

In this talk I will:

- Offer motivation for following work (Zakharevich et. al, Gillet-Soulé, Bondarko)
- Discuss several types of (∞-)categories from which K-theory can be extracted (Waldhausen, Campbell, Blumberg-Gepner-Tabuada)
- Describe abstract six functors formalisms/motivic categories (Khan, Drew-Gallauer, Cisinski-Déglise, Hoyois)
- Show how to extract maps of K-theory spectra from motivic categories (L.)
- Show that this applies to the Gillet-Soulé motivic measure (L.)

K-Theory of Varieties

The Grothendieck Ring of Varieties

Let S be an arbitrary scheme

- The *Grothendieck ring of S-varieties* $K_0(\mathbf{Var}_S)$ is the abelian group on isomorphism classes of varieties obtained by imposing the relation

$$[X] = [Z] + [X \setminus Z]$$

  for any closed subvariety $Z \subseteq X$.
- The ring structure arises naturally from $[X][Y] = [X \times_S Y]$ for all X and Y.
- If $S = \mathrm{Spec}(k)$ for k satisfying resolution of singularities and weak factorization, then $K_0(\mathbf{Var}_k)$ has an alternative presentation in terms of smooth projective varieties:
  - $[\emptyset] = 0$
  - $[X] - [Z] = [\mathrm{Bl}_Z(X)] - [E]$ for any $Z \subseteq X$ a closed subvariety and E the exceptional divisor of the blowup

Motivic measures

- Given k as in the previous slide, and any Weil Cohomology theory $H^\bullet$ with coefficients in K of characteristic 0, the assignment $X \mapsto \sum_i [H^i(X)] \in K_0(K)$ of the corresponding Euler characteristic factors through $K_0(\mathbf{Var}_k)$
- In other words, we have a ring homomorphism

$$K_0(\mathbf{Var}_k) \to K_0(K)$$

  induced by the Euler characteristic.
- over a general base, we call any ring homomorphism

$$K_0(\mathbf{Var}_S) \to R$$

  a *motivic measure*
- Motivic measures are useful for probing the structure of the Grothendieck ring of varieties and of varieties more generally.

Motivation for the K-Theory Spectrum of Varieties

- There is a spectral upgrade $K(\mathbf{Var}_S)$ of the Grothendieck ring of varieties (originally due to I. Zakharevich, with equivalent models by J. Campbell and provisionally considered by T. Ekedahl)
- This was used by I. Zakharevich to demonstrate that the kernel of multiplication by $\mathbb{L} := [\mathbb{A}^1]$ in $K_0(\mathbf{Var}_k)$ (for a "convenient" k) is given by classes of the form $[X] - [Y]$ where X and Y are such that $[X \times \mathbb{A}^1] = [Y \times \mathbb{A}^1]$, but $[X] \neq [Y]$ and $X \times \mathbb{A}^1$ and $Y \times \mathbb{A}^1$ are not piecewise equivalent.
- This shows that the K-theory spectrum of varieties is useful, but it is also quite a mysterious object
- One might ask in particular if it carries any higher homotopical data

Additional Properties

- The inclusion $\textbf{FinSet} \hookrightarrow \textbf{Var}_k$ induces a map $K(\textbf{FinSet}) \simeq \mathbb{S} \to K(\textbf{Var}_k)$, the cofiber of which is denoted $\tilde{K}(\textbf{Var}_k)$.
- If k is realizable as a subfield of $\mathbb{C}$, then the resulting cofiber sequence gives us infinitely many nonzero homotopy groups
- If k is finite, the assignment $X \to X(k)$ yields a splitting

$$K(\textbf{Var}_k) \simeq \mathbb{S} \vee \tilde{K}(\textbf{Var}_k)$$

- In this case, we can begin asking subtler questions about $\tilde{K}(\textbf{Var}_k)$
- It is shown by Campbell, Wolfson, and Zakharevich that by lifting the Hasse-Weil zeta function to a map of K-theory spectra (*a derived motivic measure*)

$$K(\textbf{Var}_k) \to K(\mathrm{Aut}\mathbb{Q}_l),$$

one can prove that $\tilde{K}(\textbf{Var}_k)$ contains higher homotopical data (with some further restrictions on the characteristic of k)

The Gillet-Soulé Motivic Measure

The Gillet-Soulé Motivic Measure

- The original purpose of this work was to lift the Gillet-Soulé motivic measure
- Note that one has a natural functor

$$h : \mathbf{SmProj}_k \to \mathrm{Chow}(k, \mathbb{Q})$$

given by $h(X) = (X, \Delta_X, 0)$ for any smooth projective variety over k.
- Supposing again that k satisfies resolution of singularities and weak factorization, one can show that for any closed subvariety $Z \subseteq X$, one has that

$$[h(X)] - [h(Z)] = [h(\mathrm{Bl}_Z(X))] - [h(E)]$$

- Consequently, one can define the *Gillet-Soulé motivic measure*

$$\chi^{gs} : K_0(\mathbf{Var}_k) \to K_0(\mathrm{Chow}(k, \mathbb{Q}))$$

via $\chi^{gs}([X]) = [h(X)]$, where $K_0(\mathcal{C})$ for an additive category is generated via isomorphism classes under direct sums, and the product is tensor.

The Motivic Weight Complex of Gillet-Soulé

- Given any pseudoabelian category $\mathfrak{A}$, there is an isomorphism $K_0(\mathrm{Hot}^\flat \mathfrak{A}) \cong K_0(\mathfrak{A})$ via $[c^\bullet] \mapsto \sum_i [c^i]$
- For any $X \in \mathbf{Var}_k$, one can prove the existence up to isomorphism of its *weight complex* $W(X) \in \mathrm{Hot}^\flat \mathrm{Chow}(k, \mathbb{Q})$
- $W(X)$ satisfies $[W(X)] = [h(X)]$ for $X \in \mathbf{SmProj}_k$
- The assignement $X \mapsto W(X)$ is functorial in the following sense:
  - W defines a contravariant functor
$$W^* : (\mathbf{Var}_k^{\text{closed}})^{op} \to \mathrm{Hot}^\flat \mathrm{Chow}(k, \mathbb{Q})$$
    (from varieties with closed immersions) where $W^*(f)$ is denoted f^*
  - W defines a covariant functor
$$W_* : \mathbf{Var}_k^{\text{open}} \to \mathrm{Hot}^\flat \mathrm{Chow}(k, \mathbb{Q})$$
    (from varieties with open immersions) where $W_*(f)$ is denoted f_*
  - $W(X \times_k Y) \cong W(X) \otimes W(Y)$
  - Given a closed/open decomposition $Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$, one has the distinguished triangle $W(U) \overset{j_*}{\to} W(X) \overset{i^*}{\to} W(Z) \to W(U)[1]$

The Weight Functor of Bondarko

- If k satisfies the same conditions as before and we replace $\mathrm{Chow}(k, \mathbb{Q})$ with its homological version (the opposite category), one can define a *motivic weight complex functor*

$$t_{\mathbb{Q}} : \mathrm{DM}_{gm}(k, \mathbb{Q}) \to \mathrm{Hot}^{\flat}\mathrm{Chow}(k, \mathbb{Q})$$

- $t_{\mathbb{Q}}$ descends to an isomorphism $K_0(\mathrm{DM}_{gm}(k, \mathbb{Q})) \cong K_0(\mathrm{Hot}^{\flat}\mathrm{Chow}(k, \mathbb{Q}))$
- Under this isomorphism, one has that $[M^c(X)] \mapsto [W(X)]$
- consequently, our attempts to lift the Gillet-Soulé motivic measure can focus on lifting the assignment $X \mapsto M^c(X)$ to the level of K-theory spectra

(∞-)Categorical and K-Theoretic Preliminaries

Some Motivations for K-Theory I

- K-theory is an invariant that informs you about the structure of certain categories
- The most classical setting (general) for algebraic K-theory is that of exact categories, which are
  - additive
  - are equipped with a well-behaved notion of exact sequence
- The K-groups of an exact category $\mathcal{A}$ are defined as the homotopy groups of an infinite loop space known as $K(\mathcal{A})$
- In particular, we have $\mathbf{Proj}_R$ for a commutative ring R (finitely generated projective modules), and define $K(R) := K(\mathbf{Proj}_R)$
- One has that $K_0(R)$ is the abelian group on isomorphism classes of projective R-modules modulo the relations $[B] = [A] + [C]$ whenever

$$0 \to A \to B \to C \to 0$$

is exact

Some Motivations for K-Theory II

- While $K_0(R)$ for a ring R can be used to study the structure of projective modules, higher K theory can be used to understand the structure of automorphisms
- $K_1(R)$ may be presented by classes $[f : P \to P]$ for finitely generated projective P, as there exists a unique map of groups $\mathrm{Aut}(P) \to K_1(R)$
- $K_2(R)$ may generally be thought of as probing the structure of pairs of commuting automorphisms
- From there, the situation gets much more mysterious
- Even just looking at the K-theory of rings and other exact categories, the structure of the K-groups is related to many deep conjectures
- Whenever we extend K-theory to a new setting or probe the structure of existing K-theory, we end up gaining a lot of insights into other areas of math

Motivation for Waldhausen Categories

- There are many situations where you have a category which is "like an exact category," but is not additive
- For example, **FinSet**$_*$, with "exact sequences" being cofiber sequences of the form

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ & & \ulcorner \\ * & \hookrightarrow & Z \end{array}$$

  defined by pushing out along a monomorphism
- This structure is analogous in exact categories, where a sequence is exact if and only if it may be fit into a pushout square

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ & & \ulcorner \\ * & \hookrightarrow & C \end{array}$$

Motivation for Waldhausen Categories II

- The commonality between the previous examples is the existence of pushouts/cofiber sequences along a special class of morphisms
- To define K-theory of these "nonlinear exact categories" (Waldhausen categories), we need a different technique from the one used in the exact world.
- This is Waldhausen's $S_\bullet$-construction (not described here)
- This can be iterated to obtain a spectrum
- For an exact category $\mathcal{A}$, Waldhausen's K-theory $K(\mathcal{A})$ is equivalent to Quillen's K-theory
- In our nonlinear example from before, we get

$$K(\mathbf{FinSet}_*) \simeq \mathbb{S}$$

by the Barratt-Priddy-Quillen theorem (this result is sometimes quoted as $K(\mathbb{F}_1) \simeq \mathbb{S}$).

Waldhausen Categories

- Waldhausen categories are at this point the "standard" categorical setting for K-theory
- They consist of a category $\mathcal{C}$ with two distinguished classes of morphisms: *cofibrations* **cof** (elements of which are denoted $\hookrightarrow$) and *weak equivalences* $\mathcal{W}$ (elements of which are denoted $\xrightarrow{\sim}$) which are required to satisfy the following axioms:
  - All isomorphisms are cofibrations
  - $\mathcal{C}$ has a zero object, and $0 \to X$ is a cofibration
  - Cofibrations are stable under pushout
  - All isomorphisms are weak equivalences
  - Weak equivalences are closed under composition and hence form a subcategory
  - $$\begin{array}{ccccc} Z & \longleftarrow & X & \hookrightarrow & Y \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ Z' & \longleftarrow & X' & \hookrightarrow & Y' \end{array}$$ commutative implies that the induced map $Y \cup_X Z \xrightarrow{\sim} Y' \cup_{X'} Z'$ is an equivalence

Motivation for SW-Categories

- Not every category we wish to obtain K-theory from is Waldhausen
- For example, $\mathbf{Var}_S$ does not admit a Waldhausen structure
- We want a framework for which the basic structures are not cofiber sequences, but rather subtraction sequences (abstract scissors congruences)
- SW-categories are like Waldhausen categories, but are centered around subtraction, as opposed to cofiber sequences
- This allows us to extract K-theory from many categories for which the natural notion is subtraction as opposed to cofiber sequences, such as $\mathbf{Var}_S$

SW-Categories

- SW-categories were axiomatized to categorify "cutting and pasting"
- The basic data consists of:
  - An essentially small category $\mathcal{C}$
  - A wide subcategory $\mathbf{cof}(\mathcal{C})$ whose morphisms are called *cofibrations* and denoted $\hookrightarrow$
  - A wide subcategory $\mathbf{comp}(\mathcal{C})$ whose morphisms are called *complements* and denoted $\overset{\circ}{\rightarrow}$
  - A wide subcategory containing all isomorphisms $w\mathcal{C}$ whose morphisms are called *weak equivalences* and denoted $\overset{\sim}{\rightarrow}$
  - A class of diagrams $\mathbf{sub}(\mathcal{C})$ of the form $Z \hookrightarrow X \overset{\circ}{\leftarrow} U$
- The axioms that this data is mandated to satisfy are quite complex, and we only give the most important ones here
  - Cofibrations and complements are stable under pullback
  - $X \hookrightarrow X \coprod Y \overset{\circ}{\leftarrow} Y$
  - Every cofibration extends uniquely to a subtraction sequence up to unique isomorphism. The same is true for all complements
  - Subtraction is stable under pullback
  - The pushout of subtraction sequences along cofibrations are subtraction sequences

Examples of SW-Categories

- The category $\mathbf{Var}_S$ of varieties over S is an SW-category with
  - Cofibrations closed immersions
  - Complements open immersions
  - Weak equivalences isomorphisms
  - Subtraction sequences open/closed decompositions

  This was, in many ways, THE motivating example
- The category $\mathbf{Sch}_S$ of schemes over S is an SW-category with
  - Cofibrations closed immersions
  - Complements open immersions
  - Weak equivalences isomorphisms
  - Subtraction sequences open/closed decompositions

  Furthermore, $\mathbf{Var}_S \to \mathbf{Sch}_S$ is an "exact functor of SW-categories"
- The category **FinSet** of finite sets is also an SW-category

Motivation for Stable ∞-categories

- Triangulated categories are central to algebraic geometry in the form of derived categories (of schemes, rings, etc.)
- They have many drawbacks
- For example, mapping cones are not functorial, and their notion of K-theory is often poorly behaved
- Stable ∞-categories provide a nice alternative, as many constructions which are not functorial for triangulated categories become functorial for stable ∞-categories
- Stable ∞-categories have a nice notion of K-theory that is in many ways the most general notion for "linear" categories

Stable ∞-categories

- A composition $A \to B \to C$ in an ∞-category is called a *(co)fiber sequence* if the diagram

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ * & \hookrightarrow & Z \end{array}$$

  is commutative and homotopy (co)cartesian
- An ∞-category $\mathcal{A}$ is *stable* if
  - $\mathcal{A}$ has a (homotopy) zero object
  - Every morphism has a kernel and cokernel (may be extended to a fiber and cofiber sequence)
  - Every fiber sequence is a cofiber sequence and vice versa
- stable categories admit all (homotopy) pushouts and pullbacks, and these squares coincide
- If $\mathcal{A}$ is stable, then $\mathbf{Ho}(\mathcal{A})$ is a triangulated category

K-Theory of Kan-Enriched Categories

- Let $\mathcal{A}$ be a finitely homotopy cocomplete, homotopy pointed ∞-category
- Define $\mathcal{P}(\mathcal{A})$ to be pointed simplicial presheaves on $\mathcal{A}$ with the projective model structure, and let $\mathcal{P}_{ex}(\mathcal{A})$ be the left Bousfeld localization to the model category of pointed simplicial presheaves which preserve homotopy colimits
- We then obtain a Waldhausen category $\mathcal{M}(\mathcal{A}) \subset \mathcal{P}_{ex}(\mathcal{A})$ as the cofibrant objects which are weakly equivalent to representable presheaves
- This assignment is functorial in weakly exact functors $F : \mathcal{A} \to \mathcal{B}$ (via restriction of left Kan extension)
- We define the K-theory $K(\mathcal{A})$ of $\mathcal{A}$ to be the Waldhausen K-theory

$$K(\mathcal{A}) := K(\mathcal{M}(\mathcal{A}))$$

- We obtain a K-theory functor from the model category of pointed, finitely homotopy cocomplete ∞-categories to spectra, which sends weak equivalences to weak equivalences

Weakly W-Exact Maps

Given a SW-category $\mathcal{C}$ and a pointed, finitely homotopy cocomplete ∞-category $\mathcal{A}$, a weakly W-exact functor $F := (F_!, F^!, F_w) : \mathcal{C} \to \mathcal{A}$ is a triple such that

- $F_!$ is a functor $F_! : \mathbf{cof}(\mathcal{C}) \to \mathcal{A}$. We abbreviate $F_!(i)$ to $i_!$
- $F^!$ is a functor $F^! : \mathbf{comp}(\mathcal{C})^{op} \to \mathcal{A}$. We abbreviate $F^!(j)$ to $j^!$
- F_w is a functor $F_w : w\mathcal{C} \to \iota(\mathcal{A})$. We abbreviate $F_w(f)$ to f_w
- For all objects $X \in \mathcal{C}$, one has $F_!(X) = F^!(X) = F_w(X) =: F(X)$
- $$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & Z \\ {\scriptstyle\circ}\downarrow{\scriptstyle i} & & {\scriptstyle\circ}\downarrow{\scriptstyle i'} \\ Y & \underset{j'}{\hookrightarrow} & W \end{array} \text{ cartesian } \Rightarrow \begin{array}{ccc} F(X) & \xrightarrow{j_!} & F(Z) \\ {\scriptstyle i^!}\uparrow & & \uparrow{\scriptstyle i'^!} \\ F(Y) & \xrightarrow{j'_!} & F(W) \end{array} \text{ commutes}$$
- $$Z \overset{i}{\hookrightarrow} X \overset{j}{\underset{\circ}{\leftarrow}} U \in \mathbf{sub}(\mathcal{C}) \Rightarrow \begin{array}{ccc} F(Z) & \xrightarrow{i_!} & F(X) \\ \downarrow & & \downarrow{\scriptstyle j^!} \\ 0 & \longrightarrow & F(U) \end{array} \text{ weakly cocartesian}$$

Weakly W-Exact Functors II

- For all commutative squares

$$
\begin{array}{ccc}
X & \overset{f}{\hookrightarrow} & Z \\
\sim\big\downarrow g & & \sim\big\downarrow g' \\
Y & \overset{f'}{\hookrightarrow} & W
\end{array}
\Rightarrow
\begin{array}{ccc}
F(X) & \xrightarrow{f_!} & F(Z) \\
\big\downarrow g_w & & \big\downarrow g'_w \\
F(Y) & \xrightarrow{f'_!} & F(W)
\end{array}
$$

  commutes in $\mathcal{A}$. One gets an analogous diagram if one replaces cofibrations with complements

This definition is central because it induces a map on K-theory

$$K(F) : K(\mathcal{C}) \to K(\mathcal{A})$$

which will be used to define our lift of the Gillet-Soulé motivic measure later.

This definition is directly analogous to that of Campbell with image a Waldhausen category.

Six Functors Formalisms

Extra Conditions on ∞-Categorical Presheaves on Schemes

Let $\mathcal{S}$ be a nice subcategory of schemes (we will not define this) in which all elements are Noetherian (such as schemes of finite-type or varieties over a Noetherian base)

- Given a presheaf $\mathbb{D}^*$ of ∞-categories on $\mathcal{S}$, we set $\mathbb{D}(S) := \mathbb{D}^*(S)$ for all $S \in \mathcal{S}$ and set $f^* : \mathbb{D}(Y) \to \mathbb{D}(X)$ for all morphisms $f : X \to Y$
- If $\mathbb{D}$ takes values in presentable ∞-categories and colimit-preserving functors, we say that it is a *presheaf of presentable ∞-categories*
- Then each f^* admits a right adjoint $f_* : \mathbb{D}(X) \to \mathbb{D}(Y)$
- If $\mathbb{D}^*$ factors throught symmetric monoidal ∞-categories for which the tensor product commutes with colimits, we say that it is a *presheaf of presentable symmetric monoidal ∞-categories*
- Over $\mathbb{D}(S)$ for any S, we denote the tensor product by $\otimes$ and the unit by $\mathbf{1}_S$
- We have an internal Hom by definition
- From now on, we omit the $*$ and simply use the notation $\mathbb{D} := \mathbb{D}^*$

$(*, \#, \otimes)$-Formalisms

A *premotivic ∞-category* or $(*, \#, \otimes)$*-formalism* $\mathcal{S}$ is a presheaf of symmetric monoidal presentable ∞-categories $\mathbb{D}$ such that:

- For every smooth $f : T \to S$ in $\mathcal{S}$, f^* admits a left-adjoint $f_\# : \mathbb{D}(T) \to \mathbb{D}(S)$
- $f_\#$ is a morphism of $\mathbb{D}(S)$-modules
- Given

  $$\begin{array}{ccc} T' & \xrightarrow{g} & S' \\ \downarrow{\scriptstyle q} & \lrcorner & \downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & S \end{array}$$

  cartesian with p and q smooth, then there is an equivalence

  $$\mathrm{Ex}^*_\# : q_\# g^* \xrightarrow{\sim} f^* p_\#$$

- Given any finite family S_α in $\mathcal{S}$, the induced functor

  $$\mathbb{D}(\amalg_\alpha S_\alpha) \to \Pi_\alpha \mathbb{D}(S_\alpha)$$

  is an equivalence.

Voevodsky Criteria and Motivic ∞-Categories

A premotivic ∞-category on $\mathcal{S}$ *satisfies the Voevodsky conditions* if:

- For $S \in \mathcal{S}$ and $p : E \to S$ a vector bundle, the unit map

$$\mathrm{id} \to p_* p^*$$

  is an equivalence
- For every closed/open decomposition

$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$$

  in $\mathcal{S}$, i_* is fully faithful with essential image spanned by objects in $\ker j^*$ (this will imply that $i_! i^! \to \mathrm{id} \to j_* j^*$ is a cofiber sequence)
- For $S \in \mathcal{S}$ and $\mathcal{E}$ a locally free sheaf on S with associated vector bundle $p : E \to S$, the Thom twist endofunctor

$$\mathcal{F} \mapsto \mathcal{F}\langle\mathcal{E}\rangle := p_\# s_*(\mathcal{F}).$$

  is an equivalence

Any such premotivic ∞-category $\mathbb{D}$ is called a motivic ∞-category. By the properties above $\mathbb{D}(S)$ is stable for all $S \in \mathcal{S}$.

The Exceptional Functors

Consider a motivic ∞-category $\mathbb{D}$. For $f : X \to Y$ of finite type, there exists an adjunction

$$(f_! \dashv f^!) : \mathbb{D}(X) \rightleftarrows \mathbb{D}(Y)$$

and a natural transformation $\alpha_f : f_! \to f_*$ such that:

- If f is an open immersion, then $f_! \simeq f_\#$ and $f^! \simeq f^*$
- α_f is an equivalence if f is a proper morphism
- $\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow q & \lrcorner & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$ cartesian implies that

  $\mathrm{Ex}^*_! : v^* f_! \to g_! u^*$ and $\mathrm{Ex}^!_* : u_* g^! \to f^! v_*$ are equivalences
- The functor $f_!$ is a morphism of $\mathbb{D}(Y)$-modules. Furthermore, the canonical morphisms

  $$\mathcal{F} \otimes f_!(\mathcal{G}) \to f_!(f^*(\mathcal{F}) \otimes \mathcal{G}),\ \underline{\mathrm{Hom}}(f^*(\mathcal{F}), f^!(\mathcal{F}')) \to f^!(\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}')),$$

  $$\mathrm{f}_*(\underline{\mathrm{Hom}}(\mathcal{F}, f^!(\mathcal{G}))) \to \underline{\mathrm{Hom}}(f_!(\mathcal{F}), \mathcal{G})$$

  are equivalences

Various Forms of Base Change

If $\mathbb{D}$ is a motivic category over $(\mathcal{S}, \mathcal{A})$ and

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow q & \lrcorner & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

is cartesian, then:

- *Proper base change:* If f is proper, then $\mathrm{Ex}^*_* : p^* f_* \xrightarrow{\sim} g_* q^*$ is an equivalence
- *Smooth-proper base change:* If f is proper and p and q are smooth, then $\mathrm{Ex}_{\#*} : p_\# g_* \xrightarrow{\sim} f_* q_\#$ is an equivalence
- *Finite type-smooth base change:* If f is finite type and p and q are smooth, then $\mathrm{Ex}^{*!} : q^* f^! \xrightarrow{\sim} g^! p^*$ is an equivalence
- *Finite type-proper base change:* If f is finite type and p is proper, then $\mathrm{Ex}_{!*} : f_! q_* \xrightarrow{\sim} p_* g_!$ is an equivalence

Constructible Objects and Generation

- An object in $\mathbb{D}(S)$ is *constructible* if it lies in the thick subcategory generated by $f_\# f^*(\mathbf{1}_S)\langle -n\rangle \simeq f_! f^!(\mathbf{1}_S)\langle -n\rangle$ with $f : X \to S$ smooth of finite presentation and $n \in \mathbb{Z}_{\geq 0}$
- $\mathbb{D}$ is *compactly generated* if
  - For every $S \in \mathcal{S}$, the ∞-category $\mathbb{D}(S)$ is compactly generated
  - For every morphism $f : T \to S$ in $\mathcal{S}$, the inverse image functor $f^* : \mathbb{D}(S) \to \mathbb{D}(T)$ is a compact functor (preserves compact objects)
- $\mathbb{D}$ is *constructibly generated* if it is compactly generated and every constructible object is compact. In this case, compactness coincides with constructibility
- Constructibility is preserved by:
  - $(-) \otimes \mathcal{F}$ with constructible $\mathcal{F}$
  - f^* for any $f : X \to Y$
  - $f_\#$ for $f : X \to Y$ finitely presented smooth
  - $(-)\langle \mathcal{E}\rangle$ for any vector bundle $\mathcal{E}$
  - $f_!$ for any $f : X \to Y$ finite type

Examples of Motivic ∞-categories

- The *stable motivic homotopy category* **SH** is a motivic ∞-category, where $\mathbf{SH}(S)$ is defined in steps:
  - $\mathbf{H}(S)$ is itself the (Bousfeld) localization of presheaves on $\mathbf{Sm}_S$ valued in spaces by Nisnevich descent and $\mathbb{A}^1$-homotopy invariance
  - $\mathbf{SH}(S)$ is the stabilization of $\mathbf{H}_\bullet(S)$, pointed objects in $\mathbf{H}(S)$, under the operation of suspension relative to the Thom sphere $\mathbb{T}_S := \mathbb{A}^1_S/(\mathbb{A}1_S - S)$
- The rational motivic stable homotopy category $\mathbf{SH}_\mathbb{Q}$ is a motivic ∞-category defined as $\mathbf{SH}_\mathbb{Q}(S) = \mathbf{SH}(S) \otimes \mathbf{D}(\mathbb{Q}) \simeq \mathbf{D}_{\mathbb{A}^1}(S, \mathbb{Q})$, where the latter is the ∞-category of complexes of Tate spectra over rational sheaves on S satisfying Nisnevich descent and $\mathbb{A}^1$-homotopy invariance and stabilized relative to tate twist as before
- The derived stable ∞-category $\mathbf{D}_{\acute{e}t}(-, \mathbb{Z}/l\mathbb{Z})$ of étale sheaves is a motivic ∞-category
- The derived stable ∞-category $\mathbf{D}(-, \mathbb{Q}_l)$ of l-adic sheaves is a motivic ∞-category

From Six Functors to Derived Motivic Measures

Outline

- We notice that in the triangulated category of cdh-motives $f_* f^!(\mathbf{1}_S) \cong M^c(X)$
- Inspired by this, we want to show that for any $\mathbb{D}$ motivic with certain niceness properties, $(f : X \to S) \mapsto f_* f^!(\mathbf{1}_S)$ defines a weakly W-exact functor
- We note that for any $\mathbb{D}$ motivic there is a weakly W-exact functor $\mathbf{Var}_S \to \mathrm{End}(\mathbb{D}(S))$ defined by $(f : X \to S) \mapsto f_* f^!$ (this is the purpose of our "Essential Lemmas" below)
- We then determine some niceness conditions on $\mathbb{D}$ that ensure that evaluation at $\mathbf{1}_S$ defines a weakly W-exact functor $\mathbf{Var}_S \to \mathbb{D}_{\mathrm{cons}}(S)$
- The above conditions are needed to avoid swindles
- This then allows us to obtain a map on K-theory $K(\mathbf{Var}_S) \to K(\mathbb{D}_{\mathrm{cons}}(S))$

Essential Lemmas

- $$\begin{array}{ccccc} W & \xhookrightarrow{i} & Z & \xhookrightarrow{j} & X \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle h} & \\ & & S & & \end{array} \quad \text{commutes} \Rightarrow \quad \begin{array}{ccc} g_* g^! & \longrightarrow & f_* f^! \\ & \searrow & \downarrow \\ & & h_* h^! \end{array} \quad \text{commutes}$$

  In other words, the assignment $(f : X \to S) \mapsto f_* f^!$ is covariantly functorial on closed immersions

- $$\begin{array}{ccccc} U & \xrightarrow[\circ]{k} & V & \xrightarrow[\circ]{l} & Y \\ & {\scriptstyle u}\searrow & \downarrow{\scriptstyle s} & \swarrow{\scriptstyle t} & \\ & & S & & \end{array} \quad \text{commutes} \Rightarrow \quad \begin{array}{ccc} t_* t^! & \longrightarrow & s_* s^! \\ & \searrow & \downarrow \\ & & u_* u^! \end{array} \quad \text{commutes}$$

  In other words, the assignment $(s : V \to S) \mapsto s_* s^!$ is contravariantly functorial on open immersions

Essential Lemmas II

- $$\begin{array}{ccc} X & \xrightarrow[\circ]{j} & Y \\ {\scriptstyle i}\big\downarrow & \searrow^{f_X} \; S \; \swarrow_{f_Y} & \big\downarrow{\scriptstyle i'} \\ Z & \xrightarrow[\circ]{j'} & W \end{array} \quad (f_Z : Z \to S,\ f_W : W \to S)$$ cartesian $\Rightarrow$ $$\begin{array}{ccc} f_{X*}f_X^! & \xleftarrow{\eta_j^*} & f_{Y*}f_Y^! \\ {\scriptstyle \epsilon_i^!}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_{i'}^!} \\ f_{Z*}f_Z^! & \xleftarrow[\eta_{j'}^*]{} & f_{W*}f_W^! \end{array}$$ commutes
- $$\begin{array}{ccccc} Z & \xhookrightarrow{i} & X & \xleftarrow[\circ]{j} & U \\ & {\scriptstyle g}\searrow & \big\downarrow{\scriptstyle f} & \swarrow{\scriptstyle h} & \\ & & S & & \end{array}$$ a closed/open decomposition implies that

  $g_*g^! \to f_*f^! \to h_*h^!$ is a cofiber sequence
- $$\begin{array}{ccc} X & \xrightarrow[\sim]{j} & Y \\ {\scriptstyle i}\big\downarrow & \searrow^{f_X} \; S \; \swarrow_{f_Y} & \big\downarrow{\scriptstyle i'} \\ Z & \xrightarrow[\sim]{j'} & W \end{array} \quad (f_Z : Z \to S,\ f_W : W \to S)$$ cartesian $\Rightarrow$ $$\begin{array}{ccc} f_{X*}f_X^! & \xrightarrow{\epsilon_{j'}^!} & f_{Y*}f_Y^! \\ {\scriptstyle \epsilon_i^!}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_{i'}^!} \\ f_{Z*}f_Z^! & \xrightarrow[\epsilon_j^!]{} & f_{W*}f_W^! \end{array}$$ commutes
- if i and i' are open immersions, the corresponding diagram commutes

An Example Proof Sketch: Localization

- Start with $$\begin{array}{ccccc} Z & \xhookrightarrow{i} & X & \xleftarrow[\circ]{j} & U \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle h} & \\ & & S & & \end{array}$$ a closed/open decomposition
- Note that since $Z \overset{i}{\hookrightarrow} X \underset{\circ}{\overset{j}{\leftarrow}} U$, one has the cofiber sequence $i_! i^! \to \mathrm{id} \to j_* j^*$
- Since i is proper, $i_! \simeq i_*$
- Since j is étale, $j^* \simeq j^!$
- Consequently, $i_* i^! \to \mathrm{id} \to j_* j^!$ is a cofiber sequence
- Precomposing with $f^!$, we obtain a cofiber sequence $i_* i^! f^! \to f^! \to j_* j^! f^!$
- Since any fiber sequence is a cofiber sequence and vice-versa, composing with f_* yields a cofiber sequence $f_* i_* i^! f^! \to f_* f^! \to f_* j_* j^! f^!$
- This yields the cofiber sequence $g_* g^! \to f_* f^! \to h_* h^!$

An Example Proof Sketch: Base Change

- Start with $\begin{array}{ccccc} X & & \xrightarrow[\circ]{j} & & Y \\ & \searrow^{f_X} & & \swarrow_{f_Y} & \\ {\scriptstyle i}\big\downarrow & & S & & \big\downarrow{\scriptstyle i'} \\ & \nearrow_{f_Z} & & \nwarrow^{f_W} & \\ Z & & \xrightarrow[\circ]{j'} & & W \end{array}$ a cartesian square of S-varieties

- This implies that $\begin{array}{ccc} X & \xrightarrow[\circ]{j} & Y \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i'} \\ Z & \xrightarrow[\circ]{j'} & W \end{array}$ is cartesian

- The heart of this proof lies in showing from here that

$$\begin{array}{ccc} j'_* i_! i^! j'^* \simeq i'_! j_* j^* i'^! & \xleftarrow[\eta_j^*]{} & i'_! i'^! \\ \big\downarrow{\scriptstyle \epsilon_i^!} & & \big\downarrow{\scriptstyle \epsilon_{i'}^!} \\ j'_* j'^* & \xleftarrow[\eta_{j'}^*]{} & \mathrm{id} \end{array}$$

commutes

- From here, precomposing with $f_W^!$ and postcomposing with f_{W*} yields our desired commutative square

An Example Proof Sketch: Base Change II

- Given an adjunction of cospans of ∞-categories

$$\begin{array}{ccccc} C & \xrightarrow{g} & A & \xleftarrow{f} & B \\ u \uparrow \dashv \downarrow r & & s \uparrow \dashv \downarrow p & & t \uparrow \dashv \downarrow q \\ F & \xrightarrow{i} & D & \xleftarrow{h} & E \end{array}$$

Induces an adjunction $\mathrm{Hom}_D(h,i) \perp \mathrm{Hom}_A(f,g)$

- Applying this to

$$\begin{array}{ccccc} \mathbb{D}(X) & \xrightarrow{i_! \simeq i_*} & \mathbb{D}(Z) & \xleftarrow{j'^*\mathcal{F}} & \mathbf{1} \\ j^* \uparrow \dashv \downarrow j_* & & j'^* \uparrow \dashv \downarrow j'_* & & \Vert \\ \mathbb{D}(Y) & \xrightarrow{i'_! \simeq i'_*} & \mathbb{D}(W) & \xleftarrow{j'_* j'^*\mathcal{F}} & \mathbf{1} \end{array}$$

for any $\mathcal{F} \in \mathbb{D}(W)$

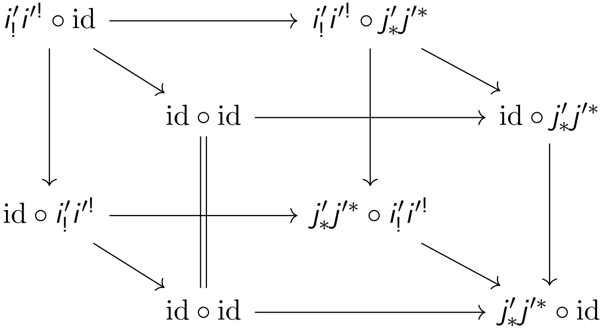
yields an adjunction $\mathrm{Hom}_{\mathbb{D}(W)}(i'_!, j'_* j'^* \mathcal{F}) \perp \mathrm{Hom}_{\mathbb{D}(Z)}(i_!, j'^*\mathcal{F})$

An Example Proof Sketch: Base Change III

- Thus, $\mathrm{Hom}_{\mathbb{D}(Z)}(i_!, j'^*\mathcal{F}) \to \mathrm{Hom}_{\mathbb{D}(W)}(i'_!, j'_*j'^*\mathcal{F})$ is a right adjoint, and preserves terminal objects
- The terminal objects of these comma categories are the components of the counits $(\epsilon^!_i)_{j'^*\mathcal{F}} : i_! i^! j'^*\mathcal{F} \to j'^*\mathcal{F}$ and $(\epsilon^!_{i'})_{j'_*j'^*\mathcal{F}} : i'_! i'^! j'_* j'^*\mathcal{F} \to j'_* j'^*\mathcal{F}$ (via the universal property of counits)
- This lets us prove that
$$\begin{array}{ccc} i'_! i'^! j'_* j'^* & \overset{\sim}{\text{———}} & j'_* i_! i^! j'^* \\ & {\scriptstyle \epsilon^!_{i'}} \searrow \quad \swarrow {\scriptstyle \epsilon^!_i} & \\ & j'_* j'^* & \end{array}$$
commutes
- Commutativity of
$$\begin{array}{ccc} & i'_! i'^! & \\ & {\scriptstyle \eta^*_j} \swarrow \quad \searrow {\scriptstyle \eta^*_{j'}} & \\ i'_! j_* j^* i'^! & \overset{\sim}{\text{———}} & j'_* j'^* i'_! i'^! \end{array}$$
follows dually

An Example Proof Sketch: Base Change IV

- Finally, the commutative triangles on the previous slide reduce the proof to the commutativity of the cube



The Central Theorem

Suppose that the constructibly-generated motivic ∞-category $\mathbb{D}$ is such that each of the six functors preserve constructible objects over Notherian quasi-excellent schemes of finite dimension

- Defining $M^c_{\mathbb{D}(S)}(X) := f_* f^!(\mathbf{1}_S)$ for any S-variety $f : X \to S$ and taking S to be Noetherian quasi-excellent of finite dimension, one has by the above central lemmas that

$$M^c_{\mathbb{D}(S)} : \mathbf{Var}_S \to \mathbb{D}_{\text{cons}}(S)$$

  is a weakly W-exact functor.
- This result is proven by evaluating each of the commuting diagrams in the essential lemmas at the tensor unit $\mathbf{1}_S$ and noting that these correspond to $M^c_{\mathbb{D}(S)}$ being weakly W-exact
- This, in turn, yields a map on K-theory

$$K(M^c_{\mathbb{D}(S)}) : K(\mathbf{Var}_S) \to K(\mathbb{D}_{\text{cons}}(S))$$

Interpretations of $M^c_{\mathbb{D}(k)}(X)$ in Different Contexts

- For $k \subseteq \mathbb{C}$, $\mathbb{D} = \mathbf{D}^{top}((-)(\mathbb{C}), \mathbb{Q})$, and $X \in \mathbf{Var}_k$,

$$M^c_{\mathbb{D}(k)}(X) \simeq H^{\mathrm{BM}}_*(X(\mathbb{C}), \mathbb{Q}) = H^*_c(X(\mathbb{C}), \mathbb{Q})^\vee,$$

  which categorifies the compactly-supported Euler characteristic
- For $k = \mathbb{F}_q$, $\mathbb{D} = \mathbf{D}_{\text{ét}}(-, \mathbb{Q}_l)$, and $X \in \mathbf{Var}_k$,

$$M^c_{\mathbb{D}(k)}(X) \simeq \mathrm{Gal}(\overline{k}/k) \circlearrowright H^{\mathrm{BM}}_*(X \times_k \overline{k}, \mathbb{Q}_l),$$

  which categorifies the l-adic Hasse-Weil zeta function (upon remembering only Frobenius action, not full action)
- For k perfect, $\mathbb{D} = \mathbf{DM}_B$ (defined next section), and $X \in \mathbf{Var}_k$,

$$M^c_{\mathbb{D}(k)}(X) \simeq M^c(X),$$

  which categorifies the Gillet-Soulé motivic measure

A Derived Lift of the Gillet-Soulé Motivic Measure

Beilinson Motives

- Given any $S \in \mathcal{S}$, one can define a representative of algebraic K-theory KGL_S so that for all $f : X \to Y$, one has $f^* KGL_Y \simeq KGL_X$
- Over the rationalization $\mathbf{SH}_{\mathbb{Q}}(S)$, one has that $KGL_{S,\mathbb{Q}}$ decomposes as the sum
$$KGL_{S,\mathbb{Q}} \simeq \bigoplus_{i \in \mathbb{Z}} KGL_S^{(i)}$$
compatibly with base change
- We define the *Beilinson motivic cohomology* to be $H_{B,S} := KGL_S^{(0)}$ for all S
- The category of *Beilinson Motives* is defined to be $\mathbf{DM}_B(S) := \mathrm{Mod}_{H_{B,S}}$
- $\mathbf{DM}_B$ defines a motivic ∞-category

Properties of Beilinson Motives

In addition to admitting a six functors formalism, the motivic ∞-category of Beilinson motives $\mathbf{DM}_B$ satisfies several other good properties. In particular:

- *Finiteness:* over quasi-excellent schemes, the six functors preserve constructibility
- *Absolute Purity:* for any smooth $f : X \to S$ of Noetherian schemes, one obtains an equivalence

$$\mathbf{1}_X \langle \mathrm{rank}(T_{X/S}) \rangle \simeq f^!(\mathbf{1}_S) \text{ in } \mathbf{DM}_B(X)$$

- *Duality:* for $f : X \to S$ separated of finite type and S quasi-excellent and regular, one has $f^!(\mathbf{1}_S)$ is a dualizing object in $\mathbf{DM}_B(X)$

Finally, via the weakly W-exact functor $M^c_S := M^c_{\mathbf{DM}_B(S)} : \mathbf{Var}_S \to \mathbf{DM}^c_B(S)$, we obtain our desired derived motivic measure

$$K(M^c_S) : K(\mathbf{Var}_S) \to K(\mathbf{DM}^c_B(S))$$

Showing that our Map of Spectra Lifts the Gillet-Soulé Motivic Measure

We specialize to the case $S = \mathrm{Spec} k$

- In the derived category of cdh-motives with rational coefficients $\mathrm{DM}_{cdh}(k, \mathbb{Q})$, we have for $f : X \to \mathrm{Spec} k$ that

$$f_* f^! (\mathbf{1}_k) \simeq M^c(X)$$

- Furthermore, one has a string of equivalences

$$\mathrm{DM}_B(k) \xrightarrow{\sim} \mathrm{DM}(k, \mathbb{Q}) \xrightarrow{\sim} \mathrm{DM}_{cdh}(k, \mathbb{Q})$$

  such that the composition commutes with six functors
- Therefore, $f_* f^! (\mathbf{1}_k)$ must map to the compactly supported motive $M^c(X)$ of X, given that each map is fully faithful, and the two coincide in the image.
- The above equivalence descends to compact objects

$$\mathrm{DM}^c_B(k) \xrightarrow{\sim} \mathrm{DM}_{gm}(k, \mathbb{Q}) \xrightarrow{\sim} \mathrm{DM}^c_{cdh}(k, \mathbb{Q})$$

Showing that our Map of Spectra Lifts the Gillet-Soulé Motivic Measure II

- Note that for any stable ∞-category $\mathcal{A}$, $K_0(\mathcal{A}) \simeq K_0(\mathbf{Ho}(\mathcal{A}))$, where the latter is the Grothendieck group of a triangulated category.
- If k satisfies resolution of singularities and weak factorization, on triangulated categories, one has for any $f : X \to \mathrm{Spec} k$ that

$$f_* f^!(\mathbf{1}_k) \mapsto M^c(X) \mapsto W(X)$$

under $\mathrm{DM}^c_B(k) \xrightarrow{\sim} \mathrm{DM}_{gm}(k, \mathbb{Q}) \xrightarrow{t_{\mathbb{Q}}} \mathrm{Hot}^\flat \mathrm{Chow}(k, \mathbb{Q})$

- Consequently, we can factorize Gillet-Soulé as

$$\chi^{gs} : K_0(\mathbf{Var}_k) \xrightarrow{K_0(M^c_k)} K_0(\mathbf{DM}_B(k)) \xrightarrow{\cong} K_0(\mathrm{DM}_B(k)) \to K_0(\mathrm{DM}_{gm}(k, \mathbb{Q})) \underset{\cong}{\xrightarrow{t_{\mathbb{Q}}}} K_0(\mathrm{Hot}^\flat \mathrm{Chow}(k, \mathbb{Q})) \xrightarrow{\cong} K_0(\mathrm{Chow}(k, \mathbb{Q}))$$

- Thus, $K(M^c_S) : K(\mathbf{Var}_S) \to K(\mathbf{DM}_B(S))$ lifts the Gillet-Soulé motivic measure when $S = \mathrm{Spec} k$

If Time Permits: An Alternative Approach to Lifting the Gillet-Soulé Motivic Measure

- Let R be a commutative ring
- From the work of Beilinson and Vologodsky, there exists a natural pretriangulated DG upgrade $\mathbf{DM}_{gm}(k, R)$ of Voevodsky's category of geometric motives (i.e. $\mathrm{DM}_{gm}(k, R)$ is an algebraic triangulated category)
- From the work of Schwede, any algebraic triangulated category $\mathbf{Ho}(\mathcal{A})$ (where $\mathcal{A}$ is our pretriangulated DG model) is naturally a topological triangulated category (one that arises from a stable cofibration category)
- In other words, there is a Waldhausen category attached to $\mathcal{A}$ known as the *cycle category* $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$ such that $\mathbf{Ho}(\mathcal{Z}(\mathcal{A})) \cong \mathbf{Ho}(\mathcal{A})$

If Time Permits: An Alternative Approach to Lifting the Gillet-Soulé Motivic Measure II

- In particular, $\mathcal{Z}(\mathcal{A})$ is defined so that
  - $\mathrm{Ob}(\mathcal{Z}(\mathcal{A})) = \mathrm{Ob}(\mathcal{A})$
  - $\mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(X, Y) := \mathrm{Ker}(\mathrm{Hom}_{\mathcal{A}}(X, Y)^0 \xrightarrow{d} \mathrm{Hom}_{\mathcal{A}}(X, Y)^1)$ for all $X, Y \in \mathcal{Z}(\mathcal{A})$
  - $f : X \to Y$ in $\mathcal{Z}(\mathcal{A})$ is a weak equivalence if its image in $\mathbf{Ho}(\mathcal{A})$ is an isomorphism
  - $f : X \to Y$ in $\mathcal{Z}(\mathcal{A})$ is a cofibration if for every $Z \in \mathcal{A}$, the induced map $\mathrm{Hom}_{\mathcal{A}}(f, Z) : \mathrm{Hom}_{\mathcal{A}}(Y, Z) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$ is surjective
- In particular, our derived motivic measure will come from a weakly W-exact map $M^c : \mathbf{Var}_k \to \mathcal{Z}(\mathbf{DM}_{gm}(k, R))$
- Recall that for any k-variety, one has the assignment $X \mapsto R^c_{tr}[X]$, where $R^c_{tr}[X]$ is the presheaf with transfers such that for every $Y \in \mathbf{Var}_k$, $R^c_{tr}[X](Y)$ is the free R-module generated by cycles in $X \times Y$ whose projection to Y is quasi-finite and dominant (as opposed to finite)

If Time Permits: An Alternative Approach to Lifting the Gillet-Soulé Motivic Measure III

- By work of Suslin and Voevodsky, the assignment $X \mapsto R^c_{tr}[X]$ is covariant in closed immersions and contravariant in open immersions
- Letting $M^c(X)$ be the image of $R^c_{tr}[X]$ (i.e., $0 \to R^c_{tr}[X] \to 0$) in $\mathbf{DM}_{gm}(k, R)$, we note that the above functorialities allow us to define our weakly W-exact map $M^c : \mathbf{Var}_k \to \mathcal{Z}(\mathbf{DM}_{gm}(k, R))$
- Commutativity of the appropriate diagrams is due originally to Suslin-Voevodsky and Beilinson-Vologodsky
- Given the weakly W-exact functor above, we get a map on K-theory $K(M^c) : K(\mathbf{Var}_k) \to K(\mathcal{Z}(\mathbf{DM}_{gm}(k, R)))$ which specializes to the a lift of the Gillet-Soulé motivic measure when $R = \mathbb{Q}$ and k satisfies resolution of singularities

If Time Permits: Hypothetical K-Theory of the Abelian Category of Motives

- Suppose that one has a *motivic t-structure* μ on the triangulated category $\mathrm{DM}_{gm}(k, \mathbb{Q})$ of geometric mixed motives. Namely, μ satisfies:
  - the cohomology functor is conservative, or f in $\mathrm{DM}_{gm}(k, \mathbb{Q})$ is an isomorphism if and only if ${}^{\mu}H^{a}(f)$ is an isomorphism for all a
  - $\otimes$ is t-exact
  - all realization functors are t-exact
- Shown by Sasha Beilinson that μ is bounded (if it exists)
- Can lift μ to a t-structure on $\mathbf{DM}^{c}_{B}(k)$ via $\mathrm{DM}_{gm}(k, \mathbb{Q}) \simeq \mathrm{DM}^{c}_{B}(k)$
- Barwick's Theorem of the Heart: If $\mathcal{A}$ is a stable ∞-category equipped with a bounded t-structure τ, then $K(\mathcal{A}) \simeq K(\mathcal{A}^{\heartsuit})$, where the latter term is the K-theory of an exact ∞-category
- $\mathcal{A}^{\heartsuit}$ is is equivalent to the nerve of an abelian category, and $K(\mathcal{A}^{\heartsuit})$ is its classical abelian K-theory
- $K(\mathbf{DM}^{c}_{B}(k))$ models the K-theory of the hypothetical abelian category of mixed motives

Conclusion

Further Directions

- Upgrade the adjunctions used to demonstrate our lift of Gillet-Soulé to the ∞-categorical level (already done over perfect base field)
- Apply the results of this enquiry to other classical motivic measures, especially those arising from sheaf theory
- Using the extension by Khan of the six functors formalism to (derived) algebraic stacks, analyze the equivalent setup for equivariant motives
- Demonstrate equivalence of the derived l-adic zeta function resulting from this work with that of Campbell-Wolfson-Zakharevich
- Extend the recently obtained structure results of Braunling-Groechenig to (slightly) more general base
- Generalize the results of L.-Manin-Marcolli using the six functors formalism as our categorical framework of choice (instead of Nori motives)

Thank You!

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