Topological expansions, Random matrices and operator algebras

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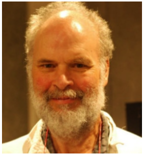
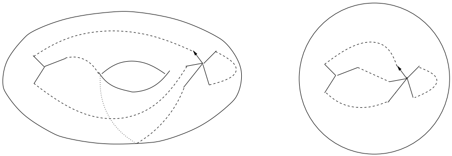
Algebra, Geometry and Physics Bonn/Berlin seminar



Joint work with V. Jones and D. Shlyakhtenko.

What is in common between



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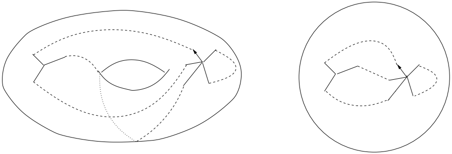

And

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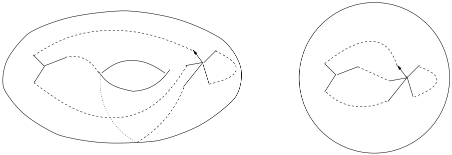


And

And

Outline

Topological expansions, Random matrices and operator algebras

What is a map ?

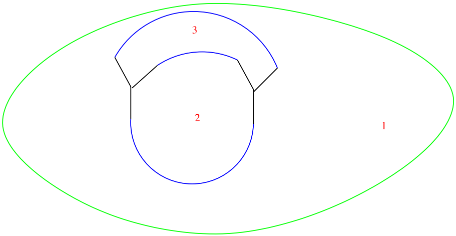
A map is a connected graph which is properly embedded into a surface, that is embedded in such a way that its edges do not cross and the faces (obtained by cutting the surface along the edges of the graph) are homeomorphic to disks. The genus of a map is the genus of such a surface.

By Euler formula,

$$2-2g = \#\{vertices\}$$

$$+\#\{faces\} - \#\{edges\}\,.$$

$$= 2+3-3$$



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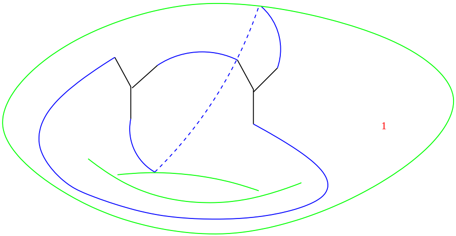

Enumeration of maps

Being given vertices with given valence, how many maps with genus g can we build ?

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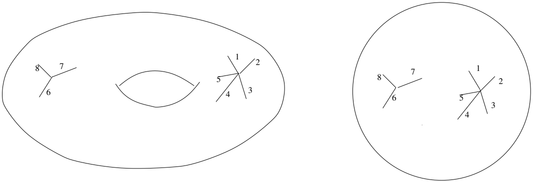
Recipe :

• Draw labeled vertices with labeled half-edges on a surface of genus g,

• Match the end points of these half-edges,

• Check the resulting map is properly embedded and could not be properly embedded on a surface with genus smaller than g,

• Count such matchings (which are the same only if matched labelled half-edges are the same).



Enumeration of maps

Being given vertices with given valence, how many maps with genus g can we build ?

Recipe :

- Draw vertices with labeled half-edges on a surface of genus g,
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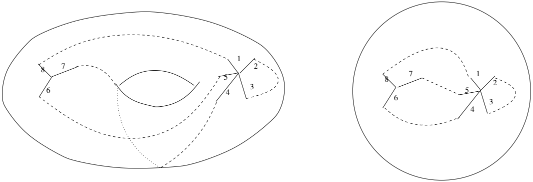

Topological expansions, Random matrices and operator algebras

The law of the GUE and the enumeration of maps

Let X^N be a matrix following the Gaussian Unitary Ensemble, that is a $N \times N$ Hermitian matrix with i.i.d centered complex Gaussian entries with covariance N^{-1}, that is

$$d\mathbb{P}(X^N) = \frac{1}{Z^N} \exp\{-\frac{N}{2} \mathrm{Tr}((X^N)^2)\} dX^N$$

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Theorem (Harer-Zagier 86)

For all $p \in \mathbb{N}$

$$\int \frac{1}{N}\mathrm{Tr}((X^N)^{2p}) d\mathbb{P}(X^N) = \sum_{g \geq 0} N^{-2g} M(2p; g) .$$

equals $\sum_{n=1}^{N} \left(\begin{array}{c} N \\ n \end{array} \right) (2p-1)!!2^{n-1} \left(\begin{array}{c} p \\ n-1 \end{array} \right)$. $M(2p; g)$ *denotes the number of maps with genus* g *build over a vertex of valence* $2p$.

Proof " Feynman diagrams"

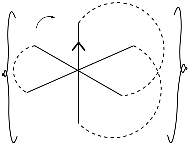
$$\mathbb{E}[\frac{1}{N}\mathrm{Tr}((X^N)^p)] = \frac{1}{N}\sum_{i(1),\ldots,i(p)=1}^{N}\mathbb{E}[X^N_{i(1)i(2)}X^N_{i(2)i(3)}\cdots X^N_{i(p)i(1)}]$$

Wick formula : If $(G_1,\cdots,G_{2n})$ is a centered Gaussian vector,

$$\mathbb{E}[G_1 G_2\cdots G_{2n}] = \sum_{\substack{1\le s_1<s_2..<s_n\le 2n\\ r_i>s_i}}\prod_{j=1}^{n}\mathbb{E}[G_{s_j}G_{r_j}].$$

Example : If $G_i = G$ follows the standard Gaussian distribution

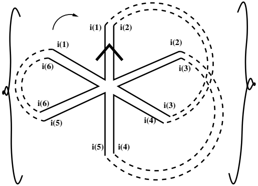
$E[G^p] = \#\{$ pair partitions of p points$\}$ $= \#$



Proof " Feynman diagrams"

$$\mathbb{E}[\mathrm{Tr}(X^N)^p] = \sum_{i(1),\ldots,i(p)=1}^{N} \mathbb{E}[X^N_{i(1)i(2)} X^N_{i(2)i(3)} \cdots X^N_{i(p)i(1)}]$$

$$\mathbb{E}[X^N_{i(1)i(2)} \cdots X^N_{i(p)i(1)}] = \sum$$



As $\mathbb{E}[X^N_{ij} X^N_{k\ell}] = N^{-1} 1_{ij=\ell k}$, only matchings so that indices are constant along the boundary of the faces contribute.

$$\mathbb{E}[\mathrm{Tr}((X^N)^p)] = \sum_{\substack{\text{graph 1 vertex} \\ \text{degree p}}} N^{\#\text{faces}-p/2}$$

$$= \sum N^{-2g+1} M((x^p,1);g) \text{ by Euler formula}$$

Random matrices and the enumeration of maps

't Hooft 74' and Brézin-Itzykson-Parisi-Zuber 78'

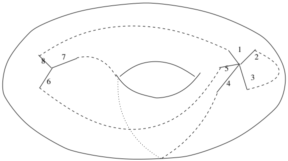
Let $\mathbf{t} = (t_i)_{1 \le i \le n} \in \mathbb{R}^n$ and set $V_{\mathbf{t}} = \sum_{i=1}^n t_i x^i$. Formally,

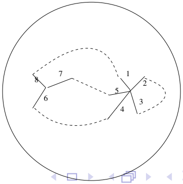
$$\frac{1}{N^2} \log \int e^{N \mathrm{tr}(V_{\mathbf{t}}(X^N))} d\mathbb{P}(X^N)$$

$$= \sum_{k_1,..,k_n \in \mathbb{N}} \sum_{g \ge 0} N^{-2g} \prod_{j=1}^n \frac{(t_j)^{k_j}}{k_j!} M((k_i)_{1 \le i \le n}; g)$$

with

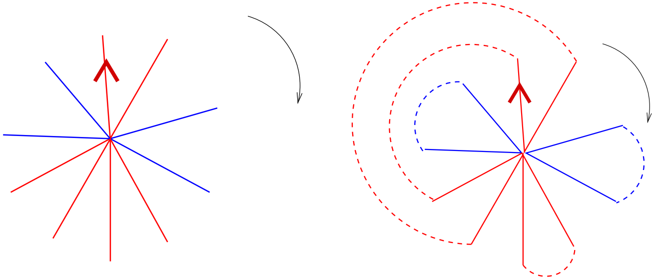
$M((k_i)_{1 \le i \le n}; g) = \sharp\{$maps of genus g with k_i vertices of degree $i\}$





Enumeration of colored maps

Consider vertices with colored half-edges and enumerate maps build by matching half-edges of the same color.



Such vertices are in bijection with monomials :
to $q(X_1,\ldots,X_d) = X_{i_1}X_{i_2}\cdots X_{i_p}$ associate a "star of type q" given by the vertex with p drawn on the plan so that the first half-edge has color i_1, the second color i_2 etc until the last which has color i_p.
$M((q_i,k_i)_{1\le i\le m},g)$ denotes the number of maps with genus g build on k_i stars of type q_i, $1\le i\le m$.

Random matrices and the enumeration of maps

't Hooft (1974) and Brézin-Itzykson-Parisi-Zuber (1978)

Let $(q_1, \ldots, q_n)$ be monomials. Let $\mathbf{t} = (t_i)_{1 \le i \le n} \in \mathbb{R}^n$ and set $V_{\mathbf{t}}(X_1, \ldots, X_m) = \sum_{i=1}^n t_i q_i(X_1, \ldots, X_m)$. Formally,

$$F_{V_{\mathbf{t}}}^N = \frac{1}{N^2} \log \int e^{N \mathrm{tr}(V_{\mathbf{t}}}(A_1, \cdots, A_m)) d\mathbb{P}^N(A_1) \cdots d\mathbb{P}^N(A_m)$$

$$= \sum_{k_1,..,k_n \in \mathbb{N}} \sum_{g \ge 0} N^{-2g} \prod_{j=1}^n \frac{(t_j)^{k_j}}{k_j!} M((q_i, k_i)_{1 \le i \le n}, g)$$

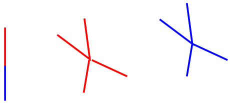
with

$$M((q_i, k_i)_{1 \le i \le n}, g) = \sharp\{\text{maps of genus } g \text{ with } k_i \text{ vertices of type } q_i\}$$

where maps are constructing by matching half-edges of the same color.

Example : The Ising model on random graphs

Take $q_1(X_1, X_2) = X_1 X_2$, $q_2(X_1, X_2) = X_1^4$, $q_3(X_1, X_2) = X_2^4$ represented by

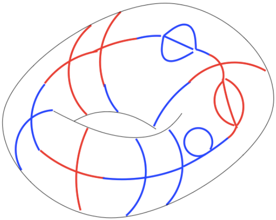


Then,

$$\frac{1}{N^2} \log \int e^{N \mathrm{Tr}(\sum_{i=1}^{3} t_i q_i(X_1^N, X_2^N))} d\mathbb{P}(X_1^N) d\mathbb{P}(X_2^N)$$

is a generating function for the enumeration of the

the Ising model on random graphs. Solved by Mehta (1986).



Random matrices, maps and tracial states

't Hooft 74' and Brézin-Itzykson-Parisi-Zuber 78'

Let $(q_1, \cdots, q_n)$ be monomials, $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$ and put

$$d\mathbb{P}_{V_{\mathbf{t}}}(X_1^N, \cdots, X_m^N) = e^{-N^2 F_{V_{\mathbf{t}}}^N + N \mathrm{Tr}\left(V_{\mathbf{t}}(X_1^N, \cdots, X_m^N)\right)} d\mathbb{P}(X_1^N) \cdots d\mathbb{P}(X_m^N)$$

Formally, for any monomial P

$$\begin{aligned} \tau_{\mathbf{t}}^N(P) &:= \int \frac{1}{N} \mathrm{Tr}\left(P(X_1^N, \ldots, X_m^N)\right) d\mathbb{P}_{V_{\mathbf{t}}}(X_1^N, \ldots, X_m^N) \\ &= \partial_s F^N_{V_{\mathbf{t}} + sP/N^2}\big|_{s=0} \\ &= \sum_{g \geq 0} N^{-2g} \sum_{k_1, \ldots, k_n \in \mathbb{N}} \prod_{j=1}^n \frac{(t_j)^{k_j}}{k_j!} M((P,1), (q_i, k_i)_{1 \leq i \leq n}; g) \end{aligned}$$

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$\tau_{\mathbf{t}}^N$ is a tracial state :

$$\tau_{\mathbf{t}}^N(PP^*) \geq 0, \tau_{\mathbf{t}}^N(1) = 1, \tau_{\mathbf{t}}^N(PQ) = \tau_{\mathbf{t}}^N(QP).$$

What is a non-commutative law ?

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What is a classical law on $\mathbb{R}^d$?

It is a non-negative linear map

$$Q : f \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R}) \to Q(f) = \int f(x) dQ(x) \in \mathbb{R}, \quad Q(1) = 1$$

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A non-commutative law τ of n self-adjoint variables is a linear map

$$\tau : P \in \mathbb{C}\langle X_1, \cdots, X_d \rangle \to \tau(P) \in \mathbb{C}$$

It should satisfy

- $\tau(PP^*) \geq 0$ for all P, $(zX_{i_1} \cdots X_{i_k})^* = \bar{z} X_{i_k} \cdots X_{i_1}$.
- $\tau(1) = 1$
- $\tau(PQ) = \tau(QP)$ for all $P, Q \in \mathbb{C}\langle X_1, \cdots, X_d \rangle$.

The law of free semicircle variables

Take $X_1^N, \cdots, X_d^N$ be independent GUE matrices, that is

$$\mathbb{P}\left(dX_1^N, \cdots, dX_d^N\right) = \frac{1}{(Z^N)^d} \exp\{-\frac{N}{2}\mathrm{Tr}(\sum_{i=1}^d (X_i^N)^2)\} \prod dX_i^N .$$

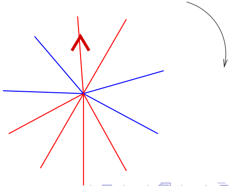
Theorem (Voiculescu(91))

For any polynomial $P \in \mathbb{C}\langle X_1, \cdots, X_d\rangle$

$$\lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\mathrm{Tr}(P(X_1^N, \cdots, X_d^N))] = \sigma(P)$$

σ is the law of d free semicircle variables.

If $P = X_{i_1}X_{i_2}\cdots X_{i_k}$, $\sigma(P)$ is the number of planar maps build over a star of type P.



From formal to asymptotic topological expansions

For $m \in \mathbb{N}$ and $(q_1, \cdots, q_n)$ monomials, $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$, $M > 2$

$$d\mathbb{P}^M_{V_{\mathbf{t}}}(X_1^N, \cdots, X_m^N) = \frac{1_{\|X_i^N\| \le M}}{Z_V^{N,M}} e^{N \mathrm{Tr}\left(V_{\mathbf{t}}(X_1^N, \ldots, X_m^N)\right)} d\mathbb{P}(X_1^N) \cdots d\mathbb{P}(X_m^N)$$

For $M > 2$, all $K \in \mathbb{N}$, t_i small enough so that $V_{\mathbf{t}} = V^*_{\mathbf{t}}$, for any monomial P

$$\tau^N_{\mathbf{t}}(P) = \int \frac{1}{N} \mathrm{Tr}\left(P(X_1^N, \ldots, X_m^N)\right) d\mathbb{P}^M_{V_{\mathbf{t}}}(X_1^N, \cdots, X_m^N)$$

$$= \sum_{g=0}^{K} N^{-2g} \sum_{k_1, .., k_n \in \mathbb{N}} \prod_{j=1}^{n} \frac{(t_j)^{k_j}}{k_j!} M((P, 1), (q_i, k_i)_{1 \le i \le n}; g) + o(N^{-2K})$$

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For $m \in \mathbb{N}$ and $(q_1, \cdots, q_n)$ monomials, $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$, $M > 2$

$$d\mathbb{P}_{V_{\mathbf{t}}}^M(X_1^N, \cdots, X_m^N) = \frac{1_{\|X_i^N\| \le M}}{Z_V^{N,M}} e^{N \mathrm{Tr}\left(V_{\mathbf{t}}(X_1^N, \dots, X_m^N)\right)} d\mathbb{P}(X_1^N) \cdots d\mathbb{P}(X_m^N)$$

For $M > 2$, all $K \in \mathbb{N}$, t_i small enough so that $V_{\mathbf{t}} = V_{\mathbf{t}}^*$, for any monomial P

$$\tau_{\mathbf{t}}^N(P) = \int \frac{1}{N} \mathrm{Tr}\left(P(X_1^N, \dots, X_m^N)\right) d\mathbb{P}_{V_{\mathbf{t}}}^M(X_1^N, \cdots, X_m^N)$$

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-$m = 1$: Ambjórn et al. 95', Albeverio, Pastur, Scherbina 01', Ercolani-McLaughlin 03'

-$m \ge 2$: G-Maurel-Segala 06', G-Shlyakhtenko 09', Dabrowski 18' Jekel 19'

Topological expansions, Random matrices and operator algebras

Maps

Random Matrices and the enumeration of maps

SD equations

Loop models

Subfactors theory

Transport

Schwinger-Dyson equations

Both matrix integrals and map enumerations are related with a third mathematical objects : The Schwinger-Dyson equations.

- They describe relations between moments, obtained thanks to integration by parts, for matrix integrals,
- They describe the induction relations for the enumeration of maps.

First loop equation

Let V be a polynomial and set

$$d\mathbb{P}_V(X_1^N,\ldots,X_m^N) = (Z_V^N)^{-1} e^{N\mathrm{Tr}\left(V(X_1^N,\ldots,X_m^N)\right)} d\mathbb{P}(X_1^N)\cdots d\mathbb{P}(X_m^N)$$

Then, for any polynomial P, any $i \in \{1,\ldots,m\}$

$$\int \frac{1}{N}\mathrm{Tr}\otimes\frac{1}{N}\mathrm{Tr}(\partial_i P(X_1^N,\ldots,X_m^N))d\mathbb{P}_V(X_1^N,\ldots,X_m^N)$$

$$= \int \frac{1}{N}\mathrm{Tr}((X_i - D_i V)P(X_1^N,\ldots,X_m^N))d\mathbb{P}_V(X_1^N,\ldots,X_m^N)$$

where for any monomial q

$$\partial_i q = \sum_{q=q_1X_iq_2} q_1\otimes q_2 \qquad D_i q = \sum_{q=q_1X_iq_2} q_2 q_1$$

Proof : Based on $\int f'(x)e^{-V(x)}dx = \int f(x)V'(x)e^{-V(x)}dx$.

First order asymptotics

Let V be a polynomial and set

$$d\mathbb{P}_V(X_1^N,\ldots,X_m^N) = (Z_V^N)^{-1} e^{N\mathrm{Tr}\left(V(X_1^N,\ldots,X_m^N)\right)} d\mathbb{P}(X_1^N)\cdots d\mathbb{P}(X_m^N)$$

Assume V small (and add a cutoff if needed). The limit points τ_V of

$$\tau_{X^N}(P) := \frac{1}{N}\mathrm{Tr}(P(X_1^N,\ldots,X_d^N))$$

satisfy

$$(A) \quad \tau_V(X_i P) = \tau_V \otimes \tau_V(\partial_i P) + \tau_V(D_i V P)$$

with $\partial_i q = \sum_{q=q_1 X_i q_2} q_1 \otimes q_2, \quad D_i q = \sum_{q=q_1 X_i q_2} q_2 q_1,$

$$(B) \quad |\tau_V(X_{i_1}\cdots X_{i_k})| \le 4^k .$$

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with $\partial_i q = \sum_{q=q_1 X_i q_2} q_1 \otimes q_2, \quad D_i q = \sum_{q=q_1 X_i q_2} q_2 q_1,$

$$(B) \quad |\tau_V(X_{i_1}\cdots X_{i_k})| \le 4^k\,.$$

Proof : as $\mathbb{P}_V$ is log-concave, τ_{X^N} self-averages and satisfies (B) for $k \le \sqrt{N}$. Hence (A) comes from the loop equation

$$\int \tau_{X^N} \otimes \tau_{X^N}(\partial_i P) d\mathbb{P}_V = \int \tau_{X^N}((X_i - D_i V)P) d\mathbb{P}_V$$

First order asymptotics

If V is small enough, there exists a unique solution to

$$(A) \quad \tau_V(X_i P) = \tau_V \otimes \tau_V(\partial_i P) + \tau_V(D_i V P)$$

$$\Leftrightarrow \tau_V(X_i q) = \sum_{q=q_1 X_i q_2} \tau_V(q_1)\tau_V(q_2) + \sum_j t_j \sum_{q_j = q_1^j X_i q_2^j} \tau_V(q_2^j q_1^j q)$$

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$$(B) \quad |\tau_V(X_{i_1} \cdots X_{i_k})| \leq 4^k \,,$$

Hence τ_{X^N} converges to this solution.

First order asymptotics

If V is small enough, there exists a unique solution to

$$(A) \quad \tau_V(X_i P) = \tau_V \otimes \tau_V(\partial_i P) + \tau_V(D_i V P)$$

$$\Leftrightarrow \tau_V(X_i q) = \sum_{q=q_1 X_i q_2} \tau_V(q_1)\tau_V(q_2) + \sum_j t_j \sum_{q_j = q_1^j X_i q_2^j} \tau_V(q_2^j q_1^j q)$$

$$(B) \quad |\tau_V(X_{i_1} \cdots X_{i_k})| \leq 4^k ,$$

Hence τ_{X^N} converges to this solution.

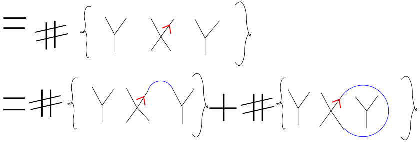
It is the generating function of planar maps

$$\tau_V(P) = \sum \prod \frac{t_i^{k_i}}{k_i!} M((P,1),(q_i,k_i);0) .$$

Induction relations and non-commutative derivatives

Tutte's surgery =Induction relations on maps.
Let $M(p, n)$ be the number of planar maps with p vertices of degree 3 and one of degree n.

$M(p, n)$



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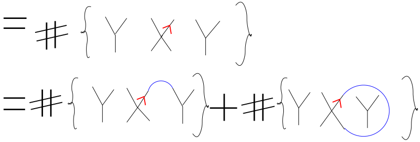

$$= 3pM(p-1, n+1) + \sum_{k=0}^{n-2} \sum_{\ell=0}^{p} C_p^{\ell} M(\ell, k) M(p-\ell, n-k-2)$$

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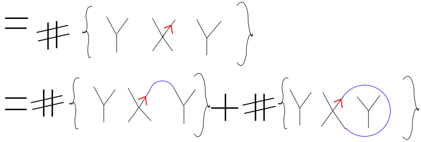

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$M_t(x^n) = \sum_{p\geq 0} \frac{t^p}{p!} M(p,n)$ satisfies the loop equation with $V = x^3$

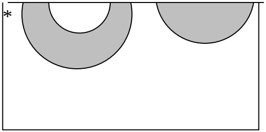
$$(A) \quad M_t(x^n) = tM_t(x^{n-1}3x^2) + M_t \otimes M_t(\partial x^{p-1})$$

$$(B) \quad |M_t(x^n)| \leq 4^n .$$

Topological expansions, Random matrices and operator algebras

Loop models

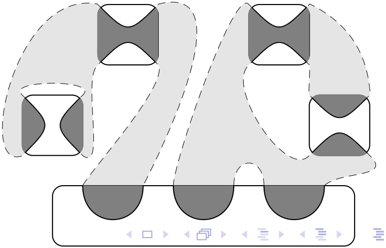
The Temperley-Lieb elements (TLE) are boxes with boundary points connected by non-intersecting strings, a shading and a marked boundary point.



Let $S_1, \ldots, S_n$ be (TLE) and $\beta_1, \cdots, \beta_n$ be small real numbers. The loop model is given, for any Temperley-Lieb element S,by

$$\mathrm{Tr}_{\beta,\delta}(S) = \sum_{n_i \geq 0} \sum \prod_{1 \leq i \leq n} \frac{\beta_i^{n_i}}{n_i!} \delta^{\sharp \mathrm{loops}}$$

where we sum over all planar maps with n_i elements S_i and one element S.



Main results

Theorem (G-Jones-Shlyakhtenko-Zinn Justin 10')

Let $S_1, \dots, S_n$ be Temperley-Lieb elements, $\beta_1, \dots, \beta_n \in \mathbb{R}^n$ and consider the loop model

$$\mathrm{Tr}_{\beta,\delta}(S) = \sum_{n_i \geq 0} \sum \prod_{1 \leq i \leq n} \frac{\beta_i^{n_i}}{n_i!} \delta^{\sharp \mathrm{loops}}$$

Then, for $\delta \in I := \{2\cos(\frac{\pi}{n})\}_{n \geq 3} \cup [2, \infty[$ and β_i small enough $Tr_{\beta,\delta}$ is a limit of matrix models.

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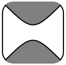
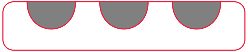
For the Potts model, i.e $S_1 =$  , $S_2 =$ 

Theorem (G-Jones-Shlyakhtenko-Zinn Justin 10')

For $\delta \in I$ and a Temperley-Lieb element S of the form

 *there exists an explicit formula for $\mathrm{Tr}_{\beta,\delta}(S)$.*

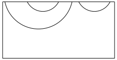
Cf Bousquet-Melou–Bernardi, Borot, Duplantier, Eynard, Kostov, Staudacher ...

Random matrices and loop enumeration ; $\beta = 0$

Let $\delta = m \in \mathbb{N}$. For a (TLE) B, we denote $p \overset{B}{\sim} \ell$ if a string joins the pth boundary point with the ℓth boundary point in B, then we associate to B with k strings the polynomial

$$q_B(X) = \sum_{\substack{i_j = i_p \text{ if } j \overset{B}{\sim} p \\ 1 \le i_\ell \le m}} X_{i_1} \cdots X_{i_{2k}}.$$

$$q_B(X) = \sum_{i,j,k=1}^{n} X_i X_j X_j X_i X_k X_k \Leftrightarrow$$



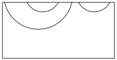
Theorem

If ν^N denotes the law of m independent GUE matrices,

$$\lim_{N \to \infty} \int \frac{1}{N} tr(q_B(X))\, \nu^N(dX) = \sum m^{\sharp \text{loops}} = Tr_0(B)$$

where we sum over all planar maps that can be built on B.

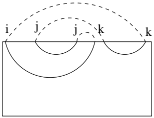
Proof

By Voiculescu's theorem, if $B =$ ,

$$\lim_{N\to\infty} \int \frac{1}{N}\mathrm{tr}\left(q_B(X)\right)\nu^N(dX)$$

$$= \sum_{i,j,k=1}^{n} \lim_{N\to\infty}\int \frac{1}{N}\mathrm{tr}\left(X_iX_jX_jX_iX_kX_k\right)\nu^N(dX)$$

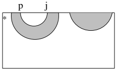
$$= \sum_{i,j,k}\sum$$



$$= \sum n^{\sharp \mathrm{loops}}$$

because the indices have to be constant along loops.

Non integer fugacities, $\beta = 0$

Based on the construction of the planar algebra of a bipartite graph, Jones 99'. Recall $p \overset{B}{\sim} j$ if a string joins the pth dot with the jth do in the TL element B 

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$$q_B(X) = \sum_{i_j = i_p \text{ if } j \overset{B}{\sim} p} X_{i_1} \cdots X_{i_{2k}} \Rightarrow q_B^v(X) = \sum_{e_j = e_p^o \text{ if } j \overset{B}{\sim} p} \sigma_B(w) X_{e_1} \cdots X_{e_{2k}}$$

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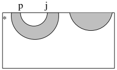
Based on the construction of the planar algebra of a bipartite graph, Jones 99'. Recall $p \overset{B}{\sim} j$ if a string joins the pth dot with the jth do in the TL element B 

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- e_i edges of a bipartite graph $\Gamma = (V = V_+ \cup V_-, E)$ so that the adjacency matrix of Γ has eigenvalue δ with eigenvector $(\mu_v)_{v \in V}$ with $\mu_v \geq 0$ ($\exists$ for any $\delta \in \{2\cos(\frac{\pi}{n})\}_{n \geq 3} \cup [2, \infty[$)

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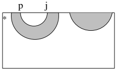
Based on the construction of the planar algebra of a bipartite graph, Jones 99'. Recall $p \overset{B}{\sim} j$ if a string joins the pth dot with the jth do in the TL element B 

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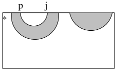
Based on the construction of the planar algebra of a bipartite graph, Jones 99'. Recall $p \overset{B}{\sim} j$ if a string joins the pth dot with the jth do in the TL element B 

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- The sum runs over loops $w = e_1 \cdots e_{2k}$ in Γ which starts at v. $v \in V_+$ iff $*$ is in a white region.
- $\sigma_B(w)$ is a well chosen weight.

Non integer fugacities,the matrix model, $\beta = 0$

For $e \in E$, $e = (s(e), t(e))$, X_e^M are independent (except $X_{e^o} = X_e^*$) $[M\mu_{s(e)}] \times [M\mu_{t(e)}]$ matrices with i.i.d centered Gaussian entries with variance $1/(M\sqrt{\mu_{s(e)}\mu_{t(e)}})$.

$$\text{Recall} \qquad q_B^v(X^M) = \sum_{\substack{w=e_1\cdots e_{2k}\in L_B \\ s(e_1)=v}} \sigma_B(w) X_{e_1}^M \cdots X_{e_{2k}}^M$$

Theorem (G-Jones-Shlyakhtenko 07')

Let Γ be a bipartite graph as before. Let B be Temperley-Lieb element. For all $v \in V$

$$\lim_{M\to\infty} E[\frac{1}{M\mu_v} tr(q_B^v(X^M))] = \mathrm{Tr}_{0,\delta}(B) = \sum \delta^{\sharp \mathrm{loops}}$$

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Based on $\sum_{e\in E:s(e)=v} \mu_{t(e)} = \delta\mu_v$.

Non integer fugacities, $\beta \neq 0$

Let B_i be Temperley Lieb elements with $*$ with color $\sigma_i \in \{+,-\}$, $1 \le i \le p$. Let Γ be a bipartite graph whose adjacency matrix has eigenvalue δ as before. Let ν^M be the law of the previous independent rectangular Gaussian matrices and set

$$d\nu^M_{(B_i)_i}(X_e) = \frac{1_{\|X_e\|_\infty \le L}}{Z_B^N} e^{M \mathrm{tr}(\sum_{i=1}^p \beta_i \sum_{v \in V_{\sigma_i}} \mu_v q^v_{B_i}(X))} d\nu^M(X_e).$$

Theorem (G-Jones-Shlyakhtenko-Zinn Justin 10')

For any $L > 2$, for β_i small enough real numbers, for any Temperley-Lieb element B with color σ, any $v \in V_\sigma$,

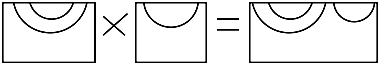
$$\lim_{M\to\infty} \int \frac{1}{M\mu_v} tr(q^v_B(X)) d\nu^N_{(B_i)_i}(X) = \sum_{n_i \ge 0} \sum \delta^{\sharp \mathrm{loops}} \prod_{i=1}^p \frac{\beta_i^{n_i}}{n_i!}$$

where we sum over the planar maps build on n_i TL elements B_i and one B. This is $\mathrm{Tr}_{\beta,\delta}(B)$.

Topological expansions, Random matrices and operator algebras

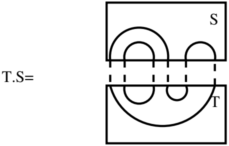
Application to subfactors theory

Temperley-Lieb elements are boxes containing non-intersecting strings. We can endow this set with the multiplication :



and the trace given by

$$\tau(S) = \sum_{R \in \mathrm{TL}} \delta^{\sharp \text{loops in } S.R}$$



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-τ *is a tracial state, as a limit of matrix (or free var.) models.*

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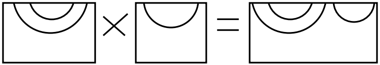

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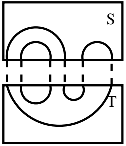

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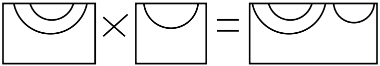

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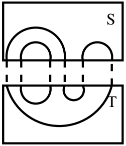

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-*A tower of factors with index* δ^2 *can be built* .

Topological expansions, Random matrices and operator algebras

Convergence of the empirical distribution of matrices

Let $X^N = (X_1^N, \dots, X_d^N)$ be a sequence of $N \times N$ (random) Hermitian matrices and let $\hat{\mu}_N$ be its empirical distribution

$$\hat{\mu}_N(P) = \frac{1}{N}\mathrm{Tr}(P(X^N))$$

Assume that for any polynomial P

$$\lim_{N\to\infty} \hat{\mu}_N(P) = \lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}(P(X^N)) = \tau(P)\,.(*)$$

Then τ is a tracial state :

$$\tau(PP^*) \geq 0, \quad \tau(PQ) = \tau(QP), \tau(I) = 1\,.$$

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Z. Ji, A. Natarajan,T. Vidick, J. Wright and H. Yuen (2020) : Answer is no (MIP*=RE). But a mystake in the proof was found and a patch posted.

The classification problem

Let τ, μ be two non-commutative laws of d (resp. m) variables. Can we find "transport maps" $T = (T_1, \ldots, T_m)$ and $T' = (T_1, \ldots, T'_d)$ of d (resp. m) variables so that for all polynomials P, Q

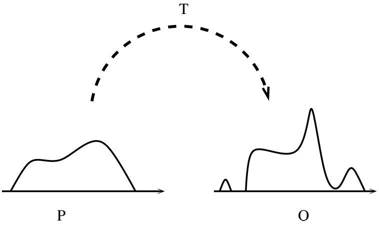
$$\begin{aligned} \tau(P(X_1, \ldots, X_d)) &= \mu(P(T_1(Y_1, \ldots, Y_m), \ldots, T_d(Y_1, \ldots, Y_m))) \\ \mu(Q(Y_1, \ldots, Y_m)) &= \tau(Q(T'_1(X_1, \ldots, X_d), \ldots, T'_m(X_1, \ldots, X_d))) \end{aligned}$$

The free group isomorphism problem : Does there exists transport maps from σ_d to σ_m, the law of d (resp. m) free variables with $d \neq m$?

Classical transport

Let P, Q be two probability measures on $\mathbb{R}^d$ and $\mathbb{R}^m$ respectively.
A transport map from P to Q is a measurable function $T : \mathbb{R}^d \to \mathbb{R}^m$ so that for all bounded continuous function f

$$\int f(T(x))dP(x) = \int f(x)dQ(x)\,.$$



We denote $T\#P = Q$.

Fact (von Neumann [1932]) : If $P, Q \ll dx$, T exists.
According to Ozawa [2004], transport map can not "always" exists as in the classical case, i.e there is no "universal" von Neumann algebras such as dx in the non-commutative case.

Free transport

Recall that

$$\tau_W(P) = \lim_{N\to\infty} \int \frac{1}{N}\mathrm{Tr}(P(X_1^N,\ldots,X_d^N))d\mathbb{P}_V^N(X_1^N,\ldots,X_d^N)$$

with

$$V = \frac{1}{2}\sum X_i^2 + W \quad \text{with} \quad W \text{ self-adjoint}$$

Theorem (G-Shlyakhtenko 12', Dabrowski -G-S 16', Jekel 19')

Assume W small or V strictly convex.
There exists F^W, T^W smooth transport maps between $\tau_W, \sigma^d = \tau_0$ so that for all polynomial P

$$\tau_W = T^W \sharp \tau_0 \quad \tau_0 = F^W \sharp \tau_W$$

In particular the related C^* algebras and von Neumann algebras are isomorphic.

Rmk : applies to q-Gaussian algebras. Extends to loop models.

What about general potentials ?

$$\mathbb{P}_N^V(dX_1^N,\ldots,dX_d^N) = \frac{1}{Z_N^V}\exp\{-N\mathrm{Tr}(V(X_1^N,\ldots,X_d^N))\}dX^N$$

Theorem (WIP G–Maurel Segala)

Let $\mathcal{D}_i V$ be the cyclic derivative $\mathcal{D}_i(X_{i_1}\cdots X_{i_k}) = \sum_{i_j=i} X_{i_{j+1}}\cdots X_{i_k}X_{i_1}\cdots X_{i_{j-1}}$ and assume that V is (η, A) trapping in the sense that $\forall k\in\mathbb{N}$

$$\mathrm{Tr}(\sum X_i^{2k}X_i.\mathcal{D}_i V) \geq \mathrm{Tr}(\eta\sum X_i^{2k+2} - A\sum X_i^{2k})$$

for some $\eta > 0$. Then there exists $L(\eta, A) < \infty$ such that

$$\limsup_{N\to\infty}\|X_i^N\|_\infty \leq L(\eta, A)$$

Moreover, any limit point of $\hat{\mu}^N(P) = \frac{1}{N}\mathrm{Tr}P(X^N)$ satisfy Dyson-Schwinger equations.

What about general potentials ?

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What kind of limit/transition can we expect ?

Low temperature expansion (WIP G–Maurel Segala)

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- If $V(X) = \beta \sum V_i(X_i) + W$ with $V_i'' \geq c$ minimum at x_i. Then for $\beta > \beta(c)$ $\hat{\mu}_N$ converges to the distribution of

$$X_i = x_i I + \frac{1}{\sqrt{V''(x_i)\beta}} S_i + \frac{1}{\sqrt{\beta}} F_i^\beta(S)$$

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$$X_i = x_i I + \frac{1}{\sqrt{V''(x_i)\beta}}S_i + \frac{1}{\sqrt{\beta}}F_i^\beta(S)$$
- $V(X) = \beta\sum V_i(X_i) + W$ with V_i minimum at $(x_j^i)_{1\leq j\leq m_i}$ where $V_i''(x_j^i) = c_j^i > 0$, $W = \sum V_i(X_i)Z_i(X)$. If β large enough, $\hat{\mu}_N$ converges towards the distribution of
$$X_i = U\begin{pmatrix} x_1^i + \frac{S_1^i}{\sqrt{\beta}} & 0 & \cdots & 0 \\ 0 & x_2^i + \frac{S_2^i}{\sqrt{\beta}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & x_i^{m_i} + \frac{S_{m_i}^i}{\sqrt{\beta}} \end{pmatrix} U^* + \frac{1}{\beta}F_i^\beta(S,(P_j^i))$$
P_j^i are projections st $\sum P_j^i = 1$, $\tau_V(P_j^i) = 1/m_i + o(\beta)$.

Thanks for listening