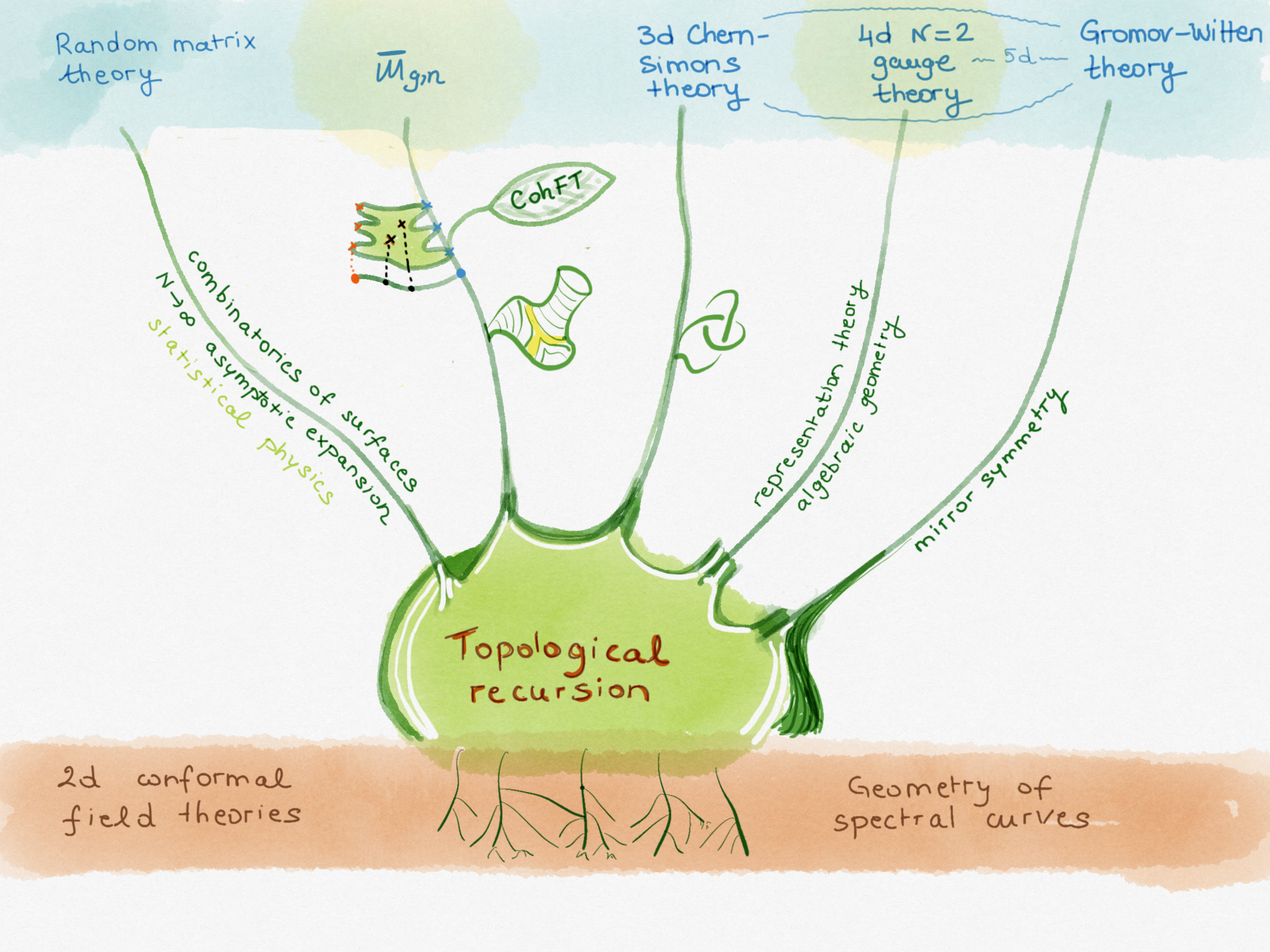
Topological recursion, geometry and QFT

@Sheffield University, 13th May 2021

(c) A. Giacchetto

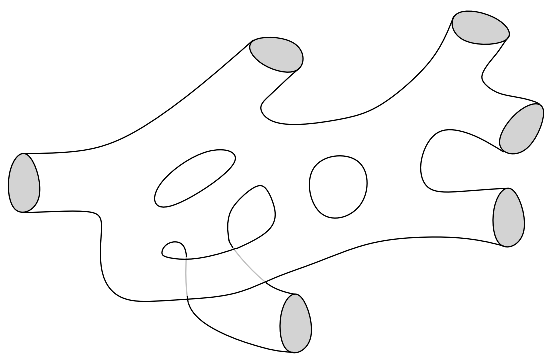
Gaëtan Borot



Random matrix theory
$\overline{\mathcal{M}}_{g,n}$
3d Chern-Simons theory
4d $\mathcal{N}=2$ gauge theory
5d
Gromov-Witten theory
CohFT
combinatorics of surfaces
$N \to \infty$ asymptotic expansion
statistical physics
representation theory
algebraic geometry
mirror symmetry
Topological recursion
2d conformal field theories
Geometry of spectral curves

In many problems I am studying …

Compact oriented surface
genus g, n punctures/boundaries



→ **quantities $\mathbf{F}_{g,n}$ or $\mathbf{\Omega}_{g,n}$**

numeric	*geometric*
$\mathcal{V}^{\otimes n}$	$H^{\bullet}(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{V}^{\otimes n}$ $\mathrm{Fun}(\mathcal{M}_{g,n}, \mathcal{V}^{\otimes n})$

⟵ integration

We want to compute $F_{g,n}$ and better understand the *algebraic structures* governing these computations, and their *ubiquity*. E.g.

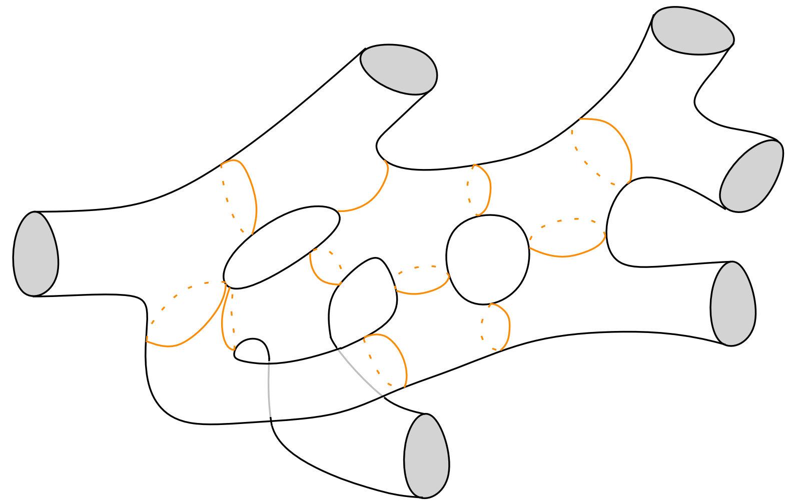
mirror symmetry $\quad F_{g,n} =$ periods on X^n, $\quad X =$ algebraic variety

non-linear integrable PDEs
linear PDEs

$$\text{for} \quad Z_{\hbar} = \exp\Big(\sum_{g,n} \frac{\hbar^{g-1}}{n!} F_{g,n}\Big) \in \mathrm{Fun}_{\hbar}(\mathcal{V})$$

In many problems I am studying …

$|\chi_{g,n}| = 2g - 2 + n$



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mirror symmetry $\quad F_{g,n} = \text{periods on } X^n, \quad X = \text{algebraic variety}$

non-linear integrable PDEs

linear PDEs $\longleftrightarrow$ recursion on $|\chi_{g,n}|$

$$\text{for} \quad Z_{\hbar} = \exp\left(\sum_{g,n} \frac{\hbar^{g-1}}{n!} F_{g,n}\right) \in \mathrm{Fun}_{\hbar}(\mathcal{V})$$

implementing the idea of cutting surfaces into smaller pieces

1 What is topological recursion ?

- Input and output
- … and where to get input

  2d TQFT, spectral curves, 2d CFT

1. Initial data

$\mathscr{D}^{\hbar}_{\mathcal{V}} = \mathbb{C}[[\hbar]]\langle x_i, \hbar\partial_{x_i} \quad i \in I\rangle \qquad \deg x_i = 1 \qquad \deg \hbar = 2$

graded algebra of differential operators on $\mathcal{V}$

basis $(e_i)_{i\in I}$

(dual) linear coordinates $(x_i)_{i\in I}$

A **quantum Airy structure** is a linear map $\mathcal{L} : \mathcal{V} \to \mathscr{D}^{\hbar}_{\mathcal{V}}$ such that

- $\mathcal{L}(e_i) = \hbar\partial_{x_i} + O(2)$
- $[\mathcal{L}, \mathcal{L}] \subseteq \hbar\mathscr{D}^{\hbar}_{\mathcal{V}} \cdot \mathcal{L}$

Theorem Kontsevich Soibelman 17

For a given quantum Airy structure, there exists a unique

$$F = \sum_{\substack{2g-2+n>0 \\ n>0}} \frac{\hbar^g}{n!} F_{g,n} \qquad F_{g,n} \in \mathrm{Sym}^n\mathcal{V}^*$$

satisfying $\quad \forall v \in \mathcal{V} \qquad \mathcal{L}(v)\cdot e^{F/\hbar} = 0$

2. The recursion

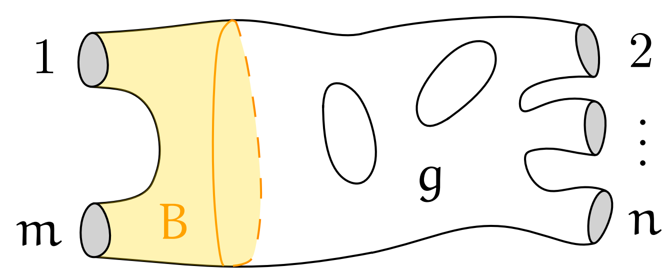
Kontsevich Soibelman 17
Andersen B Chekhov Orantin 17

initial data (A, B, C, D) —— **Topological recursion (TR)** ——→ $F_{g,n} \in \mathrm{Sym}^n \mathcal{V}^*$

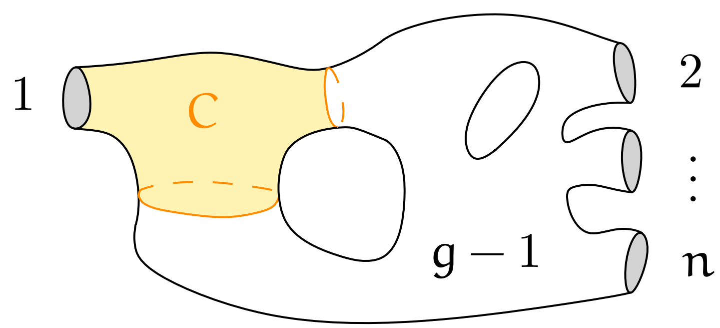
Degree 2 case $\mathcal{L}(e_i) = \hbar\partial_{x_i} - \sum_{a,b} \left(\frac{1}{2}A^i_{a,b}x_a x_b + B^i_{a,b}x_a\,\hbar\partial_{x_b} + \frac{1}{2}C^i_{a,b}\,\hbar^2\partial_{x_a}\partial_{x_b}\right) - \hbar D^i$

$|\chi| = 1 \qquad F_{0,3} = A \qquad F_{1,1} = D$

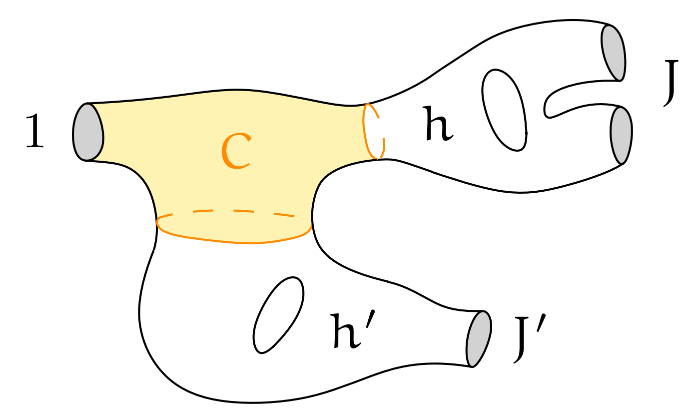
$|\chi| \geqslant 2 \qquad F_{g,n} = \sum$



$+$



$+\sum$

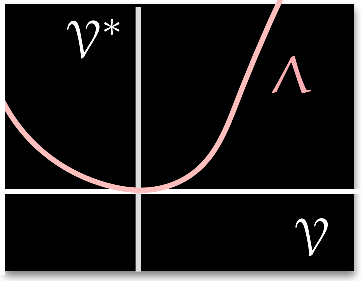


Terms $\longleftrightarrow$ $\left\{\begin{matrix}\text{embeddings of pairs of pants } P \hookrightarrow \Sigma_{g,n} \\ \text{such that } \partial_1 P = \partial_1\Sigma_{g,n} \text{ and } \Sigma_{g,n} - P \text{ is stable}\end{matrix}\right\} \Big/ \mathrm{Diff}^{\partial}(\Sigma_{g,n})$

3. Meaning and symmetries

- This fits in a quantization scheme

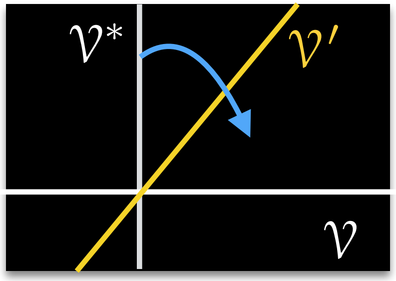
	quantum $\quad \hbar\partial_{x_i} \longrightarrow y_i$	*classical*
	$[x_i, \hbar\partial_{x_j}] = \hbar\,\delta_{i,j}$ asso. algebra $\mathscr{D}_{\mathcal{V}}^{\hbar}$	$\{x_i, y_j\} = \delta_{i,j}$ Poisson algebra $\mathbb{C}[T^*\mathcal{V}]$
	$\mathscr{D}$-module $\mathscr{D}_{\mathcal{V}}^{\hbar}/\mathscr{D}_{\mathcal{V}}^{\hbar}\cdot\mathcal{L}$	$\Lambda = \cap_i\{\mathcal{L}(e_i)_{\mathrm{cl}} = 0\} \subset T^*\mathcal{V}$
	Quantum Airy structure	Lagrangian tangent to 0-section at 0
TR	solution $e^{F/\hbar}$	classical solution $F_{g=0}$



- (Quantum) Airy structures are not easy to find !

- Change of polarisation operates on them

$$x_i \to x_i + u_{i,a}\,\hbar\partial_{x_a}$$



$$A[u]^i_{j,k} = A^i_{j,k}$$

$$B[u]^i_{j,k} = B^i_{j,k} + u_{j,a}A^i_{a,k}$$

$$C[u]^i_{j,k} = C^i_{j,k} + u_{j,a}B^i_{a,k} + u_{k,a}B^i_{a,j} + u_{j,a}u_{k,b}A^i_{a,b}$$

$$D[u]^i = D^i + \tfrac{1}{2}\,u_{a,b}A^i_{a,b}$$

Vertex operator algebra (2d chiral CFT)
+ twist by an automorphism σ $\rightsquigarrow$ quantum Airy structures
+ representation in $\mathcal{D}^{\hbar}_{\mathcal{V}}$

Fundamental example $\mathcal{W}(\mathfrak{gl}_r)_{\mathfrak{c}}$ (= Virasoro for $r = 2$)

generators H^{α}_{k} $k \in \mathbb{Z}$ $\alpha \in \{1, \ldots, r\}$

automorphisms $\mathfrak{c} = r$ $\mathfrak{S}_r \ltimes \mathbb{Z}_2$

$\mathfrak{c} \neq r$ $\mathbb{Z}_2$

admit representations in $\mathcal{D}^{\hbar}_{\mathbb{C}^r[[z]]}$

(quantum Miura transform)

Lemma 1

For $\mathfrak{c} = r$, twist by $\sigma = (1 \cdots r)$, $s \in \{1, \ldots, r+1\}$ such that $r = \pm 1 \bmod s$

$\mathcal{H} = \mathrm{span}\big(H^{\alpha}_{k} \ : \ k \geqslant \alpha - 1 - \lfloor (\alpha - 1)\frac{s}{r} \rfloor\big)$ satisfies $[\mathcal{H}, \mathcal{H}] \subseteq \hbar \mathcal{W}(\mathfrak{gl}_r) \cdot \mathcal{H}$

Uses representation theory.

4. Construction from VOAs

Fundamental example $\mathcal{W}(\mathfrak{gl}_r)_{\mathfrak{c}}$

generators H_k^{α} $k \in \mathbb{Z}$ $\alpha \in \{1, \ldots, r\}$ have representation in $\mathcal{D}^{\hbar}_{\mathbb{C}^r[[z]]}$

Lemma 1 B Bouchard Chidambaram Creutzig Noshchenko 18

For $\mathfrak{c} = r$, twist by $\sigma = (1 \cdots r)$, $s \in \{1, \ldots, r+1\}$ such that $r = \pm 1 \bmod s$

$\mathcal{H} = \mathrm{span}\big(H_k^{\alpha} \;:\; k \geqslant \alpha - 1 - \lfloor (\alpha - 1)\tfrac{s}{r} \rfloor\big)$ satisfies $[\mathcal{H}, \mathcal{H}] \subseteq \hbar \mathcal{W}(\mathfrak{gl}_r) \cdot \mathcal{H}$

and supports a quantum Airy structure

$\longrightarrow$ partition functions $Z^{(r,s)}$ computed by TR

Lemma 2 B Bouchard Chidambaram Creutzig Noshchenko 18

Likewise with $\sigma = (1\,2 \cdots r-1) \in \mathfrak{S}_r$ and $s | r$

$\longrightarrow$ partition functions $\widetilde{Z}^{(r,s)}$ ...

Theorem 3 B Kramer Schüler 20

For $\mathfrak{c} = r$, general twist $\sigma \in \mathfrak{S}_r$: many more (almost classified)

5. Spectral curve description

A **spectral curve** is a branched cover $\tilde{C} \xrightarrow{x} \mathbb{C}$ together with

$\mathfrak{a} =$ zeroes of dx

$y : \tilde{C} \to \mathbb{C}$ meromorphic, and $\omega_{0,2} \in H^0(K_{\tilde{C}}^{\boxtimes 2}(2\Delta))^{\mathfrak{S}_2}$

assuming $y - y(\alpha) \sim (x - x(\alpha))^{s_\alpha/r_\alpha - 1}$ with $s_\alpha \in \{1, \ldots, r_\alpha + 1\}$ and $r_\alpha = \pm 1 \bmod s_\alpha$

Eynard-Orantin TR

by computations of periods on $\tilde{C}$

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{i_1, \ldots, i_n} F_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n}) \prod_{a=1}^{n} d\xi_{i_a}(z_i)$$

generating series (meromorphic differential on $\tilde{C}^n$)

Kontsevich Soibelman 17

B et al. 17, 18, 20

quantum Airy structure on $\mathcal{V} = \oplus_{\alpha \in \mathfrak{a}} \oplus_{k \geqslant 0} \mathbb{C}.e_{\alpha,k}$ based on $\oplus_\alpha \mathcal{W}(\mathfrak{gl}_{r_\alpha})$

- The theory of Airy structures proves it is well-defined

→ Definition of B-model for 1d Landau-Ginzburg model

- Extended to certain singular curves (*Theorem 3* + B Kramer Schüler 20)

2 Geometry of $\overline{\mathcal{M}}_{g,n}$

- cohomological field theories
- Potentially new classes
- open Gromov-Witten theory

1. Cohomological field theories (CohFT)

$\mathcal{M}_{g,n}$ moduli space of genus g Riemann surfaces C, with punctures $p_1, \ldots, p_n$

$\overline{\mathcal{M}}_{g,n}$ compactification by allowing nodal curves with $\chi < 0$ for each component

= complex orbifold of dimension $3g - 3 + n$.

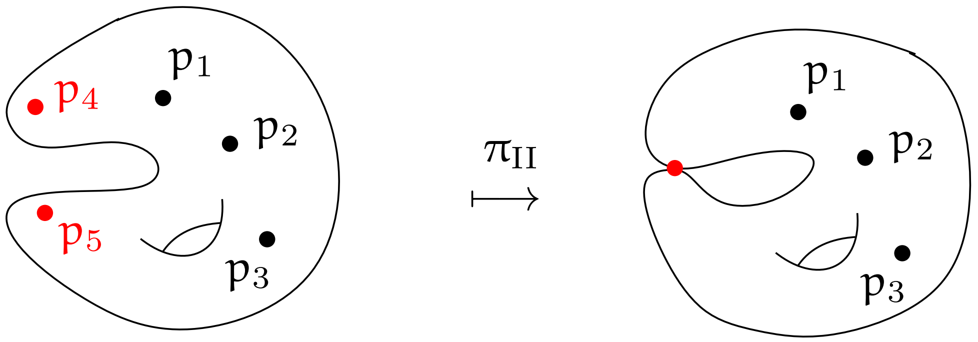
- $\psi_i = c_1\big(T^*_{p_i}C \to \overline{\mathcal{M}}_{g,n}\big)$
- A **CohFT on** $(\mathcal{A}, \langle\cdot,\cdot\rangle)$ is a sequence $\Omega_{g,n} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{A}^{*\otimes n}$ such that

$\Omega_{0,3}$ represents an associative product

$\Omega_{g,n}$ is permutation invariant

is compatible with natural morphisms

$$\pi_I : \overline{\mathcal{M}}_{g_1,1+n_1} \times \overline{\mathcal{M}}_{g_2,1+n_2} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$
$$\pi_{II} : \overline{\mathcal{M}}_{g-1,2+n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$
$$\tilde{\pi} : \overline{\mathcal{M}}_{g,1+n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$



$$\pi_{II}^*\Omega_{g,n}(-) = \sum_i \Omega_{g-1,2+n}(e_i \otimes e_i^* \otimes -)$$

1. Cohomological field theories (CohFT)

- $\psi_i = c_1\big(T^*_{p_i}C \to \overline{\mathcal{M}}_{g,n}\big)$

- A **CohFT on** $(\mathcal{A}, \langle\cdot,\cdot\rangle)$ is a sequence $\Omega_{g,n} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{A}^{*\otimes n}$ such that

$\Omega_{0,3}$ represents the product in $\mathcal{A}$

$\Omega_{g,n}$ is $\mathfrak{S}_n$-invariant

is compatible with natural morphisms

$$\pi_I : \overline{\mathcal{M}}_{g_1,1+n_1} \times \overline{\mathcal{M}}_{g_2,1+n_2} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$
$$\pi_{II} : \overline{\mathcal{M}}_{g-1,2+n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$
$$\tilde{\pi} : \overline{\mathcal{M}}_{g,1+n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

? Compute the correlation functions $F_{g,n} \in \big(\mathcal{A}^*[[t]]\big)^{\otimes n}$

$$F_{g,n}(v_1 t^{d_1} \otimes \cdots \otimes v_n t^{d_n}) = \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) \prod_{i=1}^{n} \psi_i^{d_i}$$

2. Examples of CohFTs

- Fundamental class of $\overline{\mathcal{M}}_{g,n}$
  $\mathcal{A} = \mathbb{C}$

$$F_{g,n}(t^{d_1} \otimes \cdots \otimes t^{d_n}) = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i}$$

- **Gromov-Witten theory/topological string theory**
  for Kähler varieties X
  $\mathcal{A} = QH^\bullet(X)$

$$\overline{\mathcal{M}}_{g,n}(X;\beta) \xrightarrow{p} \overline{\mathcal{M}}_{g,n}, \qquad \overline{\mathcal{M}}_{g,n}(X;\beta) \xrightarrow{ev_i} X^n$$

$$\Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) = \sum_{\beta \in H^2_{\mathrm{eff}}(X,\mathbb{Z})} t^\beta p_*\big([\overline{\mathcal{M}}_{g,n}(X;\beta)]_{\mathrm{vir}} \cap ev_1^* v_1 \cup \cdots \cup ev_n^* v_n\big)$$

Kontsevich Manin 94
Tian | Behrend Fantechi 90s

- Chern character of bundles of conformal blocks obtained from **modular functors/2d rational CFTs**

  $\mathcal{A} =$ fusion ring

Marian et al. 13 for $SU(N)_k$
Andersen B Orantin 15 in general

2. Examples of CohFTs

$\overline{\mathcal{M}}^{s/r}_{g,n}$ proper moduli stack of $(L \to C, p_1, \ldots, p_n, \phi)$

[Jarvis 98] such that $L^{\otimes r} \overset{\phi}{\simeq} K_C^{\otimes s}\left(-\sum_i (a_i - 1)p_i\right)$

$1 \leqslant s \leqslant r$
$1 \leqslant a_1, \ldots, a_n \leqslant r$
$s(2g-2+n) - \sum_{i=1}^n a_i \in r\mathbb{Z}$

We have $\mathcal{L} \xrightarrow{\pi} \mathcal{C} \longrightarrow \overline{\mathcal{M}}^{s/r}_{g,n}(\mathbf{a}) \xrightarrow{p} \overline{\mathcal{M}}_{g,n}$

$\mathcal{L}$ universal line bundle
$\mathcal{C}$ universal curve

- **Witten r -spin class** $\mathscr{W}(\frac{1}{r}; \mathbf{a})$ Polishchuk Vaintrob 01, Chiodo 02

  plays the role of p_*(virtual fundamental class of $\overline{\mathcal{M}}^{1/r}_{g,n}(\mathbf{a})$)

  defines a CohFT on $\mathcal{A} = \oplus_{a=2}^r \mathbb{C}$

  In genus 0 : $\mathscr{W}(\frac{1}{r}; \mathbf{a}) = r^{-g}\, p_* c_{top}(R^1\pi_*\mathcal{L})$ Witten 91

- **Chiodo class** $\mathscr{C}(\frac{s}{r}; \mathbf{a}) = p_*\mathrm{Ch}(-R^\bullet\pi_*\mathcal{L})$ Chiodo 06

  explicitly computed (GRR), defines a CohFT on $\mathcal{A} = \oplus_{a=1}^r \mathbb{C}$

  $\mathscr{C}(\frac{1}{1}; \mathbf{1}) = \Lambda =$ Chern character of $H^0(K_C)^\vee$ (dual of Hodge bundle)

3. Relation with TR

CohFTs governed by TR on spectral curve		*Semisimple*
$[1]_{\overline{\mathcal{M}}_{g,n}}$	$x = y^2$	yes
Witten r-spin	$x = y^r$	no

Conjecture: Witten 91 | Theorem $r = 2$: Dijkgraaf, Verlinde, Verlinde 91 + Kontsevich 92
Theorem r general: Faber, Shadrin, Zvonkine 07 + Milanov 16

- Givental group acts transitively on semisimple CohFTs   Teleman 07

⟶ Correlators of semisimple CohFTs are governed by TR

Dunin-Barkowski, Orantin, Shadrin, Spitz 12

⟶ In particular, GW invariants of toric CY3 are governed by TR

spectral curve = mirror curve $H(e^x, e^y) = 0$

Bouchard, Klemm, Mariño, Pasquetti conjecture 07
Eynard, Orantin 13 | Fang, Liu, Zong 16

3. Relation with TR

$\Omega_{g,n}$	*TR on spectral curve*	*Partition function*	
$[1]_{\overline{\mathcal{M}}_{g,n}}$	$x = y^2$	$Z^{(3,2)}$	… Kontsevich 92
Witten r-spin	$x = y^r$	$Z^{(r,r+1)}$	… Milanov 16
$p_* c_{top}(-R^\bullet \pi_* \mathcal{L})$ from $\overline{\mathcal{M}}_{g,n}^{-1/r}$	$x = y^{-r}$	$Z^{(r,r-1)}$	$r = 2$: Norbury 17 Conjecture B et al. 20
$\Omega_{g,n}^{(r,s)}$?	$x^{r-s} y^r = 1$	$Z^{(r,s)}$	BBCCN 18 B Kramer Schüler 20
open r-spin GW(pt) Pandharipande Solomon, Tessler 15- …	$y(x - y^r) = 0$	$\widetilde{Z}^{(r,r)}$	$r = 2$: Alexandrov 16 all r : conjecture BBCCN 18

Expectation To each Airy structure based on a VOA computes intersection theory of a class $\Omega_{g,n}$ on moduli of curves

Fact Using symmetries, enough to know it for basic Airy structure (i.e each type of local behavior in spectral curves)

3. Relation with TR

Theorem 4 Eynard 12 | Chekhov Norbury 17 { B Kramer Schüler 20 modulo existence of $\Omega^{(r,s)}$

For any admissible spectral curve, one can construct $\Omega_{g,n} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{A}^{*\otimes n}$

with $\mathcal{A} = \mathrm{span}_{\mathbb{C}}(\gamma_\alpha = \text{thimble for } x : \tilde{C} \to \mathbb{C}\}$ such that

$$\int_{z_i \in \gamma_{\alpha_i}} \omega_{g,n}(z_1,\ldots,z_n) \prod_{i=1} e^{-\mu_i x(z_i)} = \left(\prod_{i=1}^{n} C(\mu_i)\right) \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega_{g,n}\left(\otimes_{i=1}^{n} e_{\alpha_i}\right)}{1 - \mu_i \psi_i}$$

- Givental group $\subseteq$ {symmetries of Airy structures}

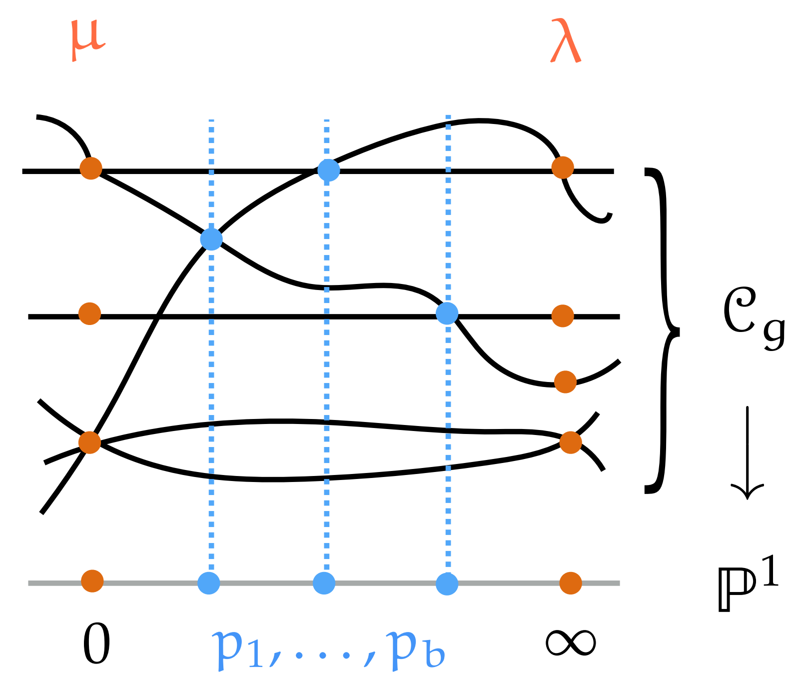
  = {deformation of spectral curves respecting local behavior}

→ retrieves TR for semisimple CohFTs of Dunin-Barkowski, Orantin, Shadrin, Spitz 12
for smooth spectral curves with simple ramification

3 Enumeration of branched covers

- Hurwitz theory
- ELSV-like formulas and TR
- consequences in algebraic geometry

1. Enumerative geometry of branched covers



Double Hurwitz numbers

$$\#\left\{ \begin{array}{l} \Sigma_g \xrightarrow{d:1} \mathbb{S}^2 \text{ ramified with profile} \\ \mu \;\text{ above } 0,\ \lambda \text{ above } \infty,\ b \text{ simple branch pts.} \end{array} \right\}$$

$$= h_{g,\mu,\lambda} = \frac{1}{d!}[C_{\text{id}}]\left(C_\lambda C_\mu \frac{(C^b_{(12)})}{b!}\right) \quad \text{in } Z\mathbb{C}[\mathfrak{S}_d]$$

Riemann-Hurwitz $\Rightarrow b = 2g - 2 + \ell(\mu) + \ell(\lambda)$

? How to compute them (before 07)

Integrability Generating series is a 2d Toda tau function

Okounkov, Pandharipande 01

Intersection theory

$$\frac{h_{g,\mu,1^d}}{b!} = \left(\prod_{i=1}^n \frac{\mu_i^{\mu_i}}{\mu_i!}\right) \int_{\overline{\mathcal{M}}_{g,n}} \Lambda \prod_{i=1}^n \frac{1}{1 - \mu_i \psi_i}$$

Ekedahl, Lando, Shapiro, Vainshtein 99

In general: vague conjecture of Goulden, Jackson, Vakil 03

1. Enumerative geometry of branched covers

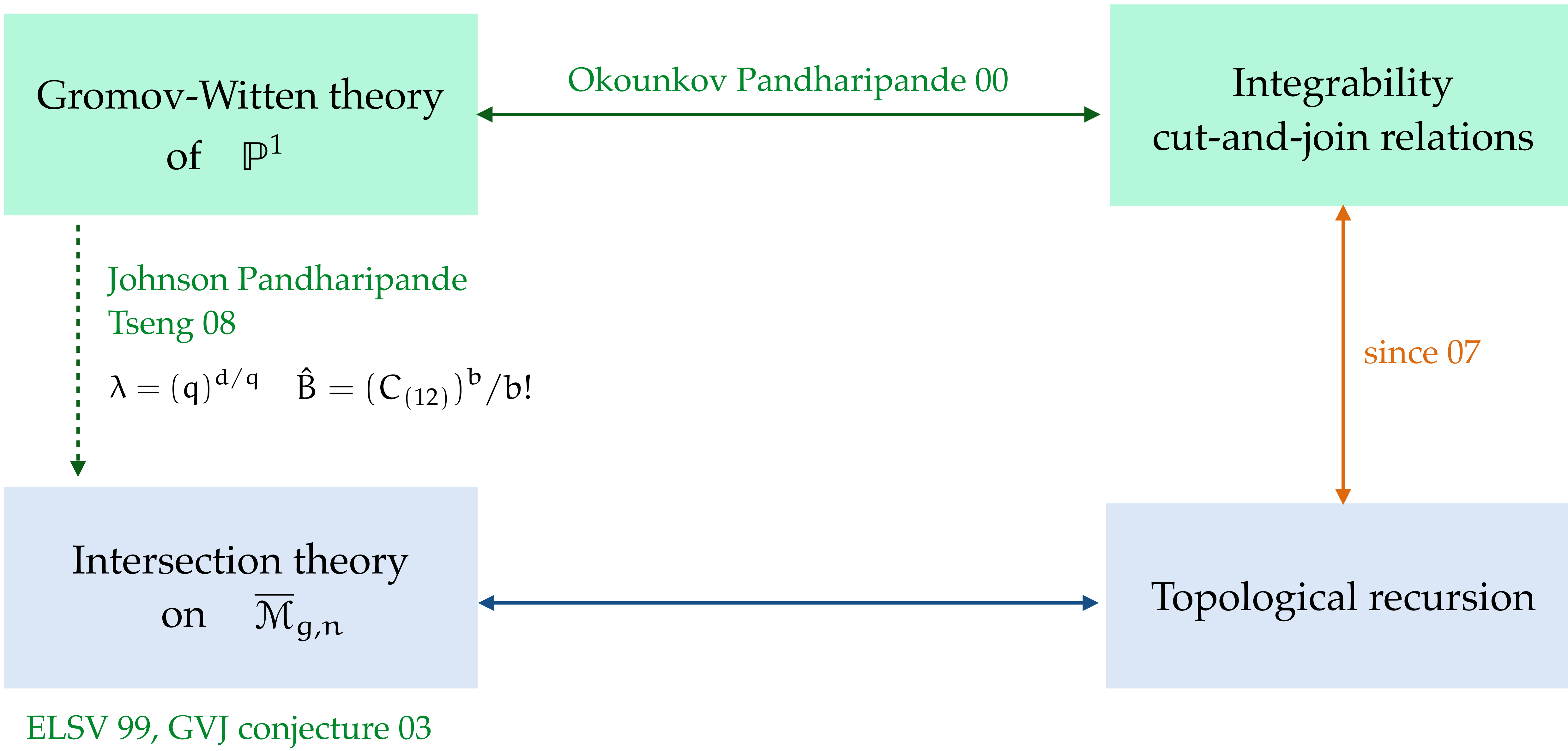
More general Hurwitz numbers $\hat{B} \in Z\mathbb{C}[\mathfrak{S}_d]$

$$h^{\hat{B}}_{g,\mu,\lambda} = \frac{1}{d!}[C_{\mathrm{id}}]\big(C_\mu C_\lambda \hat{B}\big)$$

Example : r-spin

$$\hat{B} = \hat{C}_{(1\,2\,\cdots\,r+1)} + \cdots$$

completed (r+1)-cycle



We would like to have this picture in full generality

2. TR and ELSV-like formulas for Hurwitz numbers

Proofs: polynomiality by semi-infinite wedge techniques + analysis of cut-and-join

Type	*Spectral curve/TR*	$\Omega_{g,n}$	*Contributions*
$\lambda = (1^d)$ $\hat{B}$ = simple	$ye^{-y} = e^{-x}$	Hodge class	Conjecture : Bouchard-Mariño 07 Heuristic : B Eynard Mulase Safnuk 09 EMS 09
$\lambda = (q)^{d/q}$ $\hat{B}$ = simple	$y^{1/q} e^{-y^{1/q}} = e^{-x}$	JPT 08	Do Leigh Norbury 13 Bouchard Hernandez, Liu, Mulase 13
$\lambda = (q)^{d/q}$ r-spin	$y^{1/q} e^{-y^{r/q}} = e^{-x}$	Chiodo class $\mathcal{C}(\frac{q}{qr}; \mathbf{a})$	Conjecture : Zvonkine 07 B Kramer Lewanski Popolitov Shadrin 17 Kramer, Popolitov, Shadrin 19
$\hat{B}$ = simple	$\begin{cases} x(z) = -\ln z + P(z) \\ y(z) = P(z) \end{cases}$	*	B Do Karev Lewanski Moskowsky 20

and more : Shadrin et al. 15-21, Alexandrov, Chapuy, Eynard, Harnad 17, B Garcia-Failde 17

3. Double Hurwitz numbers

- From algebraic geometry on $\overline{\mathcal{M}_{g,n}(\mathbb{P}^1[q])}$ JPT 08 + Kramer Lewanski Shadrin Popolitov 17

$h_{g,\mu,\lambda} =$ intersection numbers of $\mathcal{C}\left(\frac{q}{q}, -\boldsymbol{\mu}, -\boldsymbol{\lambda}\right)$

when $\max_{i \neq j}(\lambda_i + \lambda_j) \leqslant \max_i \mu_i \leqslant q$

and vanishing properties of Chiodo integrals when $|\lambda| > |\mu|$

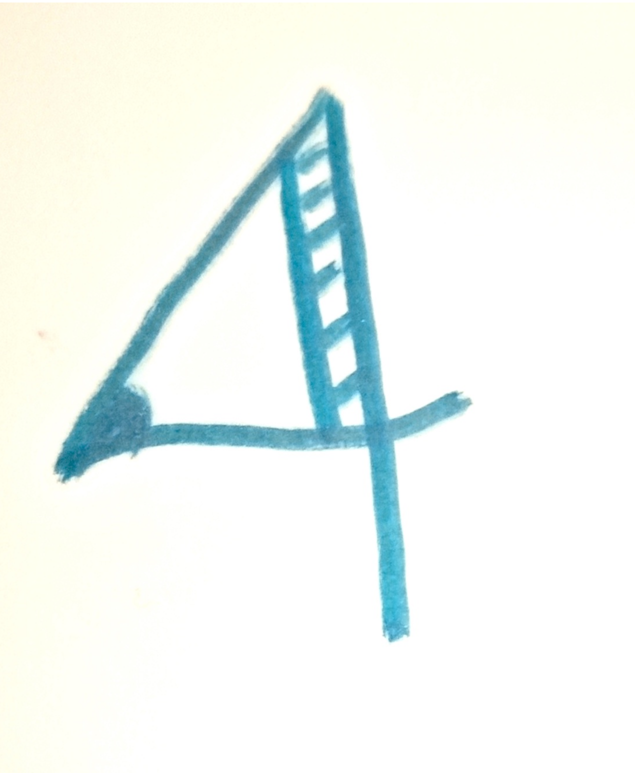
Theorem 5 (ELSV for double Hurwitz numbers) B Do Karev Lewanski Moskowsky 20

$h_{g,\mu,\lambda} =$ intersection numbers of $\mathcal{C}\left(\frac{q}{q}, -\boldsymbol{\mu}, -\boldsymbol{\lambda}\right)$

when $\max_i \mu_i \leqslant q$

and vanishing properties of Chiodo integrals when $|\lambda| > |\mu|$

- Proof by combining deformation of TR with JPT formula
- Generalises JPT formula unconditionally (more terms)
- An answer (not in expected form) to Goulden-Jackson-Vakil problem



Instanton counting in 4d gauge theories

- $N = 2$ SUSY gauge theory
- Whittaker vectors
- Nekrasov partition function

1. 4d N = 2 supersymmetric gauge theory

Donaldson 84

- $$\mathcal{M}^d_G = \left\{ \begin{array}{c} \text{Moduli space of anti-self-dual} \\ SU(r) \text{ instantons on } \mathbb{S}^4 \\ \text{framed at } \infty, \text{ instanton \# } = d \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{Algebraic } SL(r,\mathbb{C})\text{-bundles on } \mathbb{P}^2 \\ \text{with } c_2 = d \ + \text{ trivialisation on } l_\infty \end{array} \right\}$$

- In N = 2 supersymmetric pure gauge theory with equivariant parmeters $\epsilon_1, \epsilon_2, (Q_i)_{i=1}^{r-1}$ for the action of $\mathcal{G} = (\mathbb{C}^*)^2 \times$ Cartan torus, the partition function reduces to "integrals" over $\mathcal{M}^d_G$

$$Z_{\mathrm{Nek}}(\Lambda) = \exp\left(\sum_{g \geqslant 0} \hbar^{g-1} \mathfrak{F}_g(\Lambda, \alpha)\right)$$

$\hbar = -\epsilon_1 \epsilon_2 \to 0$

$\alpha = \epsilon_1 + \epsilon_2 \in \mathbb{C}$

Λ coupling / energy scale

- Mathematically:

$$|1^d\rangle \in IH^*_{\mathcal{G}}(\widetilde{\mathcal{M}}^d_G) \quad \rightsquigarrow \quad |\mathbf{1}\rangle = \sum_{d \geqslant 0} \Lambda^{dr} |1^d\rangle \quad \rightsquigarrow \quad Z_{\mathrm{Nek}}(\Lambda) = \langle \mathbf{1} | \mathbf{1} \rangle$$

Fundamental class of a (partial) compactif. Uhlenbeck, Donaldson, …

Gaiotto vector

Nekrasov partition function

2. Gaiotto vector = Whittaker vector

Alday-Gaiotto-Tachikawa conjectures a relation to $W(\mathfrak{sl}_r)_\mathfrak{c}$- conformal blocks

$\mathfrak{c} = r - 1 - r(r^2 - 1)\hbar^{-1}\alpha^2$

The mathematical theorem incarnating this is (here for $\mathfrak{gl}_r$)

Schiffmann, Vasserot 13
Braverman, Finkelberg, Nakajima 14

- $\mathcal{H} = \bigoplus_{d \geqslant 0} IH^*_{\mathcal{G}}(\widetilde{M}^d_G)$ is a Verma module for $W(\mathfrak{gl}_r)_\mathfrak{c}$ (highest weight vector $|0\rangle$)
- $W^i_k|\mathbf{1}\rangle = \delta_{i,r}\delta_{k,1}\Lambda^{rd}\,|\mathbf{1}\rangle$ for all $i \in \{1,\ldots,r\}$ and $k > 0$ (Whittaker vector)
- An explicit description of the intersection pairing in $\mathcal{H}$

→ W^i_k can be represented as differential operators $\in \mathcal{D}^\hbar_\mathcal{V}$ acting on $\mathcal{H} \cong \mathrm{Fun}_\hbar \mathcal{V}$, with $\mathcal{V} = \mathbb{C}^r[[z]]$

3. Whittaker vectors from TR

Theorem 6 B Bouchard Chidambaram Creutzig 21

$\Lambda = \hbar^{\frac{r}{2}}\widehat{\Lambda}$

$(W_k^i - \hbar^{\frac{r}{2}}\widehat{\Lambda}\delta_{i,r}\delta_{k,1})_{1\leqslant i\leqslant r}^{k>0}$ is a quantum Airy structure

and its partition function is $|\mathbf{1}\rangle = \exp\left(\sum_{g\in\frac{1}{2}\mathbb{N},\, n>0} \frac{\hbar^{g-1}}{n!} F_{g,n}^{(\widehat{\Lambda})}\right) \in \mathrm{Fun}_{\hbar}(\mathcal{V})$

It is computed by TR associated to

- the (unramified) spectral curve $\prod_{a=1}^{r}(y - \frac{Q_a}{x}) = 0$
  if $\epsilon_1 + \epsilon_2 = 0$

  periods of algebraic functions

- the non-commutative spectral curve $\prod_{a=1}^{r}(\alpha\partial_x - \frac{Q_a}{x})$
  if $\epsilon_1 + \epsilon_2 \neq 0$
  (refined)

  (regular D-module on $x \in \mathbb{P}^1$)

  periods from solutions of the D-module

  (new construction of TR)

  For $r = 2$: Eynard et al. 13-19

In progress

- For finite Λ (instead of $O(\hbar^{\frac{r}{2}})$), not directly quantum Airy structure

  but still solved by TR on spectral curve $\prod_{a=1}^{r}\left(y-\frac{Q_a}{x}\right)=\frac{\Lambda}{x^{r+1}}$

  or a non-commutative version of it

corollary → new proof of Z_{Nek} = generating series of weighted Hurwitz numbers
(Witten, Nekrasov 90-00s)

→ proof (?) that TR on SW curve should compute Z_{Nek} to all order

- Whittaker vectors for smaller subalgebra of modes

  correspond to Argyres-Douglas theories

→ approach to show existence of their partition function (and computation by TR)

- If we could do the same in 5d: new proof of remodeling the B-model
  and extension to refined topological strings

5 Geometric recursion

- Recursion for mapping class group invariants
- Mirzakhani-McShane identities
- Masur-Veech volumes

1. Glueing vs. mapping class group invariance

Surf — objects : topological, compact, bordered, oriented, stable surfaces

Surf — morphisms : isotopy class of diffeomorphisms

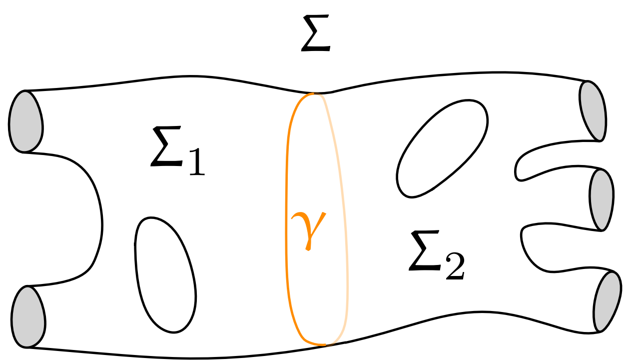
$\Gamma_\Sigma = \mathrm{Diff}(\Sigma, \partial\Sigma)/\mathrm{Diff}_0(\Sigma, \partial\Sigma)$ = mapping class group

Given a functor $\mathcal{E}$: **Surf** $\longrightarrow$ **V** we would like to construct functorial assignments $\Sigma \longmapsto \Omega_\Sigma \in \mathcal{E}(\Sigma)$ by induction on χ_Σ

- Naive glueing cannot work

  $\Omega_{\Sigma_1} \underset{\gamma}{*} \Omega_{\Sigma_2}$ can only have $\mathrm{Stab}(\gamma)$-invarance

  $\longrightarrow$ sums over all ways of cutting to restore Γ_Σ-invariance



- Idea from TR : sum over excisions of isotopy classes of pairs of pants

  $\longrightarrow$ use **Surf**$_1$: surfaces with a *first* boundary

- Countable sums : need to address convergence

  $\longrightarrow$ use a category of topological vector spaces as **V**

2. Geometric recursion valued in Fun($\mathcal{T}_\Sigma$)

Given a functor $\mathcal{E} : \mathbf{Surf}_1 \longrightarrow \mathbf{V}$ + some functorial data satisfying axioms
the geometric recursion developed in Andersen B Orantin 17 achieves this goal

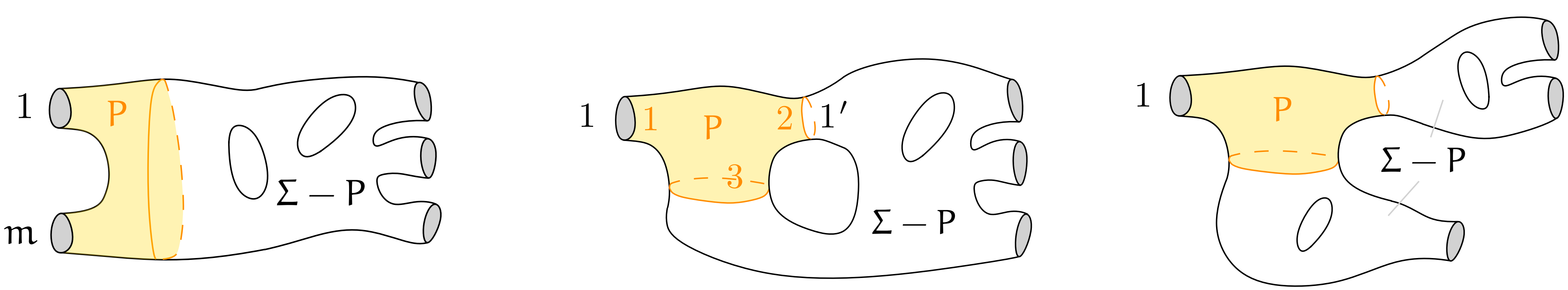
I present it for $\mathcal{E}(\Sigma) = C^0(\mathcal{T}_\Sigma)$

- $\mathcal{T}_\Sigma$ = Teichmüller space = $\left\{ \begin{array}{l} \text{hyperbolic metrics on } \Sigma \\ \text{with geodesic boundaries} \end{array} \right\} \Big/ \mathrm{Diff}_0(\Sigma, \partial\Sigma)$

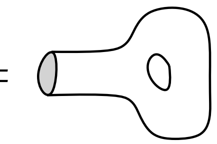
- $M_\Sigma(L) = \mathcal{T}_\Sigma(L)/\Gamma_\Sigma^\partial$ = moduli space of bordered Riemann surfaces with fixed boundary lengths $L \in \mathbb{R}_+^n$

  equipped with μ_{WP} = Weil-Petersson volume form

- $\mathcal{P}_\Sigma = \left\{ \begin{array}{l} \text{isotopy class of } P \hookrightarrow \Sigma \text{ with labeled boundaries} \\ \text{such that } \partial_1 P = \partial_1 \Sigma \text{ and } \Sigma - P \text{ is stable} \end{array} \right\} = \left(\bigsqcup_{m=2}^n \mathcal{P}_\Sigma^m\right) \sqcup \mathcal{P}_\Sigma^\emptyset$



2. Geometric recursion valued in $\mathrm{Fun}(\mathcal{T}_\Sigma)$

- Initial data $A, B, C \in C^0(\mathcal{T}_P) \cong C^0(\mathbb{R}_+^3)$ and $D \in C^0(\mathcal{T}_T)$ $T =$ 

  $X(L_1, L_2, L_3) = X(L_1, L_3, L_2)$ for $X = A, C$

- $|\chi| = 1$ $\Omega_P = A$ $\Omega_T = D$

  union $\Omega_{\Sigma_1 \sqcup \Sigma_2}(\sigma_1, \sigma_2) = \Omega_{\Sigma_1}(\sigma_1)\Omega_{\Sigma_2}(\sigma_2)$

  $|\chi| \geqslant 2$ $\Omega_\Sigma(\sigma) = \sum_{b=2}^{n} \sum_{[P] \in \mathcal{P}_\Sigma^b} B(\vec{\ell}_\sigma(\partial P))\Omega_{\Sigma - P}(\sigma|_{\Sigma - P}) + \frac{1}{2} \sum_{[P] \in \mathcal{P}_\Sigma^\emptyset} C(\vec{\ell}_\sigma(\partial P))\Omega_{\Sigma - P}(\sigma|_{\Sigma - P})$

Theorem 7 Andersen B Orantin 17

If A,B,C,D satisfy some (explicit) bounds

$\Sigma \longmapsto \Omega_\Sigma \in C^0(\mathcal{T}_\Sigma)$ is a well-defined functorial assignment
(absolute convergence on any compact)

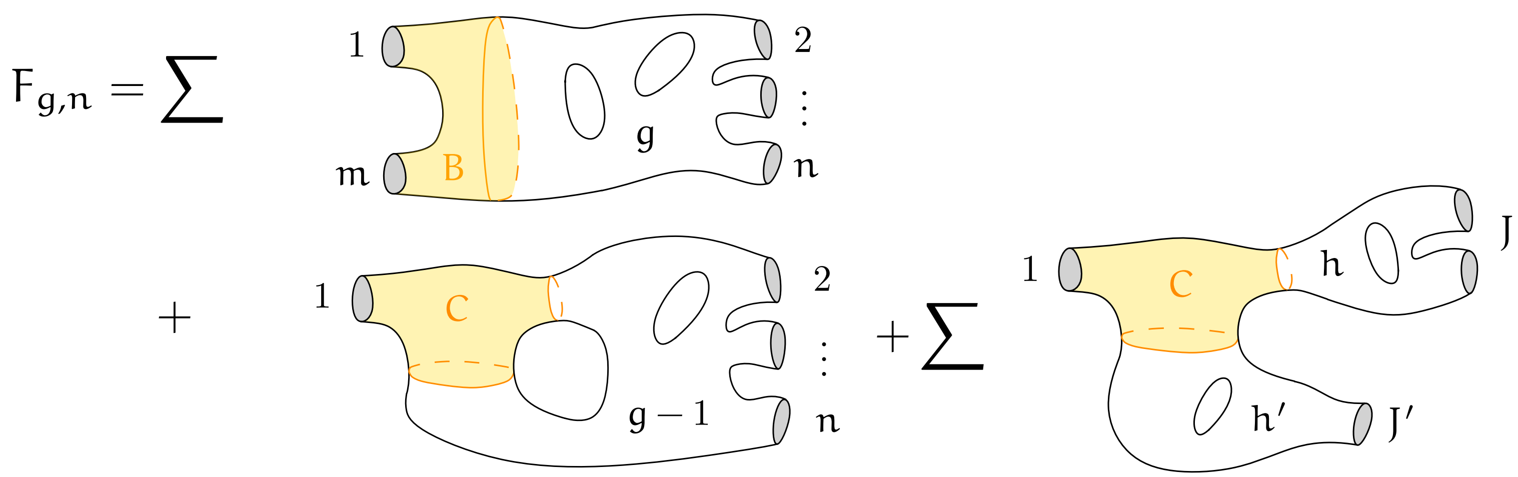
3. GR implies TR

GR formula $\Omega_\Sigma(\sigma) = \sum_{b=2}^{n} \sum_{[P]\in\mathcal{P}_\Sigma^b} B(\vec{\ell}_\sigma(\partial P))\Omega_{\Sigma-P}(\sigma|_{\Sigma-P}) + \frac{1}{2} \sum_{[P]\in\mathcal{P}_\Sigma^\emptyset} C(\vec{\ell}_\sigma(\partial P))\Omega_{\Sigma-P}(\sigma|_{\Sigma-P})$

Theorem 7 Andersen B Orantin 17

Under assumptions, $\Sigma \longmapsto \Omega_\Sigma \in C^0(\mathcal{T}_\Sigma)$ is a well-defined functorial assignment

and $F_{g,n}(L) = \int_{M_{\Sigma_{g,n}}(L)} \Omega_{\Sigma_{g,n}}(\sigma)\, d\mu_{WP}(\sigma)$ satisfies TR



Terms

$\updownarrow$

$\mathcal{P}_\Sigma/\mathrm{Diff}_\Sigma^\partial$

(finite)

Key for integration is that Fenchel-Nielsen coordinates are

- compatible with cutting/gluing
- Darboux for Weil-Petersson form

4. Examples

Mirzakhani generalisation of McShane identities (07)

$$A^M(L_1, L_2, L_3) = 1$$

$$B^M(L_1, L_2, \ell) = 1 - \frac{1}{L_1} \ln \left(\frac{\cosh\left(\frac{L_2}{2}\right) + \cosh\left(\frac{L_1+\ell}{2}\right)}{\cosh\left(\frac{L_2}{2}\right) + \cosh\left(\frac{L_1-\ell}{2}\right)} \right)$$

$$C^M(L_1, \ell, \ell') = \frac{2}{L_1} \ln \left(\frac{e^{\frac{L_1}{2}} + e^{\frac{\ell+\ell'}{2}}}{e^{-\frac{L_1}{2}} + e^{\frac{\ell+\ell'}{2}}} \right)$$

$$D^M(\sigma) = \sum_{\gamma \text{ simple}} C(L_1, \ell_\sigma(\gamma), \ell_\sigma(\gamma))$$

GR $\longrightarrow$ $\Omega_\Sigma \equiv 1$

$\downarrow$

TR $\longrightarrow$ $F_{g,n}(L) =$ WP volume of $M_{g,n}(L)$

$$= \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(2\pi^2 \kappa_1 + \sum_{i=1}^n \frac{L_i^2}{2} \psi_i\right)$$

associated with

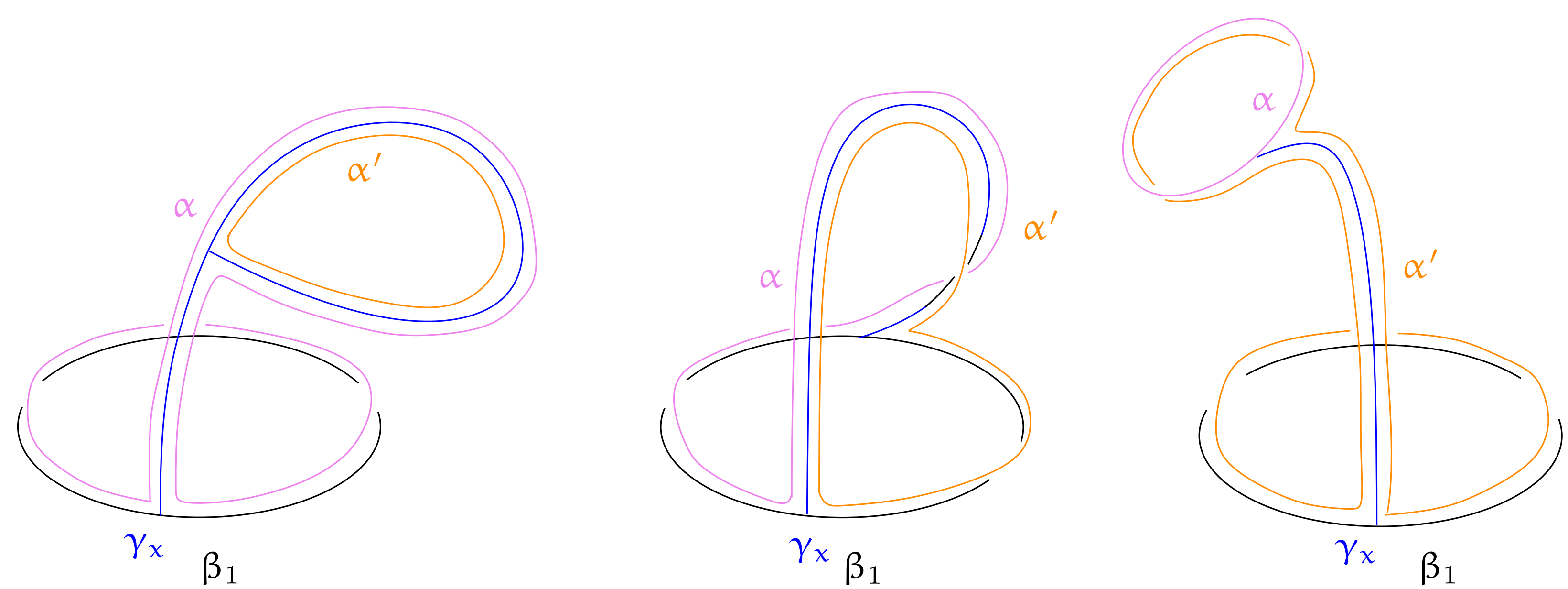
Eynard Orantin 07

$$\begin{cases} x(z) = \frac{z^2}{2} \\ y(z) = -\frac{\sin(2\pi z)}{2\pi} \\ \omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \end{cases}$$

The proof is geometric

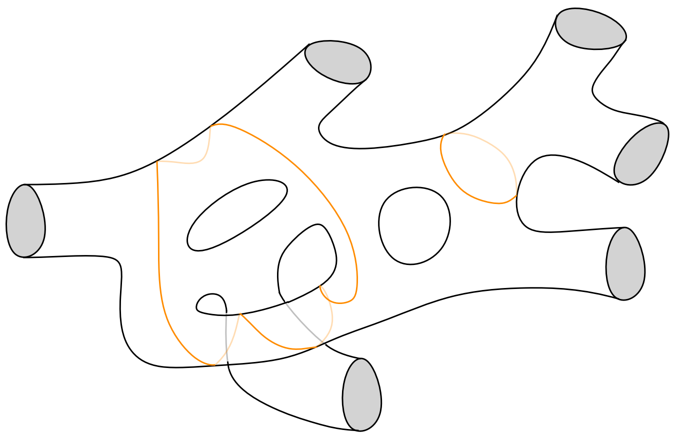
partition the 1st boundary using

$\perp$-shot geodesic $\longrightarrow \mathcal{P}_\Sigma$



4. Examples

Generalizations of Mirzakhani identities



$S_\Sigma = \{\text{primitive multicurves on } \Sigma\}$

Theorem 8 Andersen B Orantin 18

For any test function $f \in C^0(\mathbb{R}_+)$ with fast decay

$$\Omega_\Sigma^M[f](\sigma) = \sum_{c \in S_\Sigma} \prod_{\gamma \in \pi_0(c)} f(\ell_\sigma(\gamma))$$ is computed by GR for twisted initial data

$$A^M[f](L_1, L_2, L_3) = A^M(L_1, L_2, L_3)$$
$$B^M[f](L_1, L_2, \ell) = B^M(L_1, L_2, \ell) + f(\ell) A^M(L_1, L_2, \ell)$$
$$C^M[f](L_1, \ell, \ell') = C^M(L_1, \ell, \ell') + f(\ell) B^M(L_1, \ell, \ell') + f(\ell') B^M(L_1, \ell', \ell) + f(\ell) f(\ell') A^M(L_1, \ell, \ell')$$
$$D^M[f](\sigma) = \sum_{\gamma \text{ simple}} C^M(L_1, \ell_\sigma(\gamma), \ell_\sigma(\gamma)) + f(\ell_\sigma(\gamma)) A^M(L_1, \ell_\sigma(\gamma), \ell_\sigma(\gamma))$$

In the spectral curve description : amounts to changing $\omega_{0,2}$

= lift of (Givental) symmetries to hyperbolic geometry

5. Application to Masur-Veech volumes

- MF_Σ space of measured laminations on Σ = punctured surface

  {multicurves} = lattice in MF_Σ $\rightsquigarrow$ μ_{Th} = Thurston counting measure on MF_Σ

- $\mathcal{QT}_\Sigma \longrightarrow \mathcal{T}_\Sigma$ moduli space of quadratic meromorphic differentials with simple poles at punctures

$$\begin{array}{ccccc} \mu_{\mathrm{MV}} & = & \mu_{\mathrm{Th}} \otimes \mu_{\mathrm{Th}} & & \mu_{\mathrm{Th}} \otimes \mu_{\mathrm{WP}} \\ \mathcal{QT}_\Sigma & \overset{\sim}{\dashrightarrow} & \mathrm{MF}_\Sigma \times \mathrm{MF}_\Sigma & \overset{\sim}{\dashleftarrow} & \mathrm{MF}_\Sigma \times \mathcal{T}_\Sigma \end{array}$$

Masur-Veech volume $\quad \nu_{g,n} = \mu_{\mathrm{MV}}\big(\{\mathrm{Area}(q) \leqslant 1\}/\mathrm{Mod}_\Sigma\big)$

Bonahon
Mirzakhani 07

via integrating on $(\mathcal{M}_{g,n}, \mu_{\mathrm{WP}})$
an asymptotic count of curves

Andersen B Charbonnier Delecroix
Giacchetto Lewanski Wheeler 19

Theorem 8

Computed by TR(1)

via intersection theory of
$\mathrm{Segre}(\mathbb{P}\mathcal{Q}_{g,n} \to \mathcal{M}_{g,n}) \sim \mathcal{C}^{-1}\left(\tfrac{1}{2}\right)$

Chen, Möller
Sauvaget 19

B Giacchetto
Lewanski 20

Theorem 8'

Computed by TR(2)

6. Combinatorial Teichmüller space

$$\mathcal{T}_{\Sigma}^{\mathrm{comb}} = \left\{ \begin{array}{c} \text{embedded metric ribbon graphs } \mathbb{G} \hookrightarrow \Sigma \\ \Sigma \text{ retracts onto } \mathbb{G} \end{array} \right\} \quad \text{homeo. to} \quad \mathcal{T}_{\Sigma}$$

but different (symplectic) geometry : Kontsevich vs. Weil-Petersson

Kontsevich 91

$$\int_{\mathcal{M}_{g,n}^{\mathrm{comb}}(L)} d\mu_K = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\sum_{i=1}^{n} \frac{L_i^2}{2}\psi_i\right)$$

Theorem 9 Andersen B Charbonnier Giacchetto Lewanski Wheeler 21

There are combinatorial FN coordinates $\mathcal{T}_{\Sigma}^{\mathrm{comb}} \to (\mathbb{R}^+ \times \mathbb{R})^{3g-3+n} \times \mathbb{R}_+^n$

Image is open dense with zero measure complement

$\omega_K = \sum_{i=1}^{3g-3+n} d\ell_i \wedge d\tau_i$ on locus with fixed boundary lengths

One can set up geometric recursion to get $\Omega_\Sigma \in C^0(\mathcal{T}_{\Sigma}^{\mathrm{comb}})$

and $F_{g,n}(L) = \int_{\mathcal{M}_{\Sigma}^{\mathrm{comb}}(L)} \Omega_\Sigma \, d\mu_K$ satisfy TR

6. Combinatorial Teichmüller space

Andersen B Charbonnier Giacchetto Lewanski Wheeler 21

Most tools of hyperbolic geometry have an analogue in $\mathcal{T}_{\Sigma}^{\mathrm{comb}}$

Theorem 10 (combinatorial Mirzakhani-McShane identity)

$$A^{K}(L_1, L_2, L_3) = 1$$

$$B^{K}(L_1, L_2, \ell) = \frac{1}{2L_1}\left([L_1 - L_2 - \ell]_+ - [-L_1 + L_2 - \ell]_+ + [L_1 + L_2 - \ell]_+\right)$$

$$C^{K}(L_1, \ell, \ell') = \frac{1}{L_1}[L_1 - \ell - \ell']_+$$

$$D^{K}(\mathbb{G}) = \sum_{\gamma \text{ simple}} C^{K}\left(\vec{\ell}_{\mathbb{G}}(\partial(\Sigma - \gamma))\right)$$

GR ↓

$$\Omega_{\Sigma}^{K} \equiv 1$$

integration →

Kontsevich 92, Zvonkine 03

TR ↘

$$F_{g,n}(L) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\sum_{i=1}^{n} \frac{L_i^2}{2}\psi_i\right)$$

Spectral curve description

$$\begin{cases} x(z) = \frac{z^2}{2} \\ y(z) = -z \\ \omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \end{cases}$$

Fully geometric proof
(bypassing matrix models)

Thank you for your attention !