Donaldson–Thomas invariants of quivers with potentials from the flow tree flormula

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April 13, 2021

Plan of the talk

- (Refined) Donaldson–Thomas Invariants
  - Moduli space of quiver representations
  - The trace function
  - Refined DT invariants
- Attractor Invariants
- Wall crossing in the space of stability parameters
- The flow tree formula
  - The proof

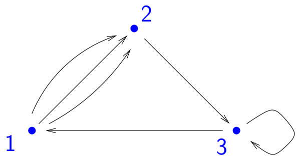
The flow tree formula computes refined DT invariants in terms of simpler attractor invariants. (A.–Bousseau, arXiv:2102.11200)

Quivers

Definition

A *quiver* is a finite oriented graph $Q = (Q_0, Q_1, s, t)$.

- Q_0: set of vertices.
- Q_1: set of arrows.
- $s : Q_1 \to Q_0$ maps an arrow to its *source*.
- $t : Q_1 \to Q_0$ maps an arrow to its *target*.



$$Q_0 = \{1, 2, 3\}$$

Definition

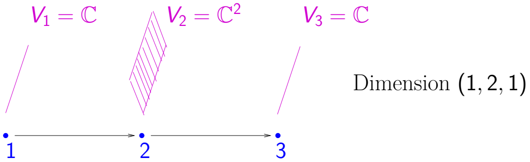
A *representation of a quiver* is an assignement of

- a vector space V_v, for each vertex $v \in Q_0$, and
- a linear transformation $\mathrm{Hom}_{\mathbb{C}}(V_{s(e)}, V_{t(e)})$ for each edge $e \in Q_1$.

- *Dimension* of a quiver representation is a vector

$$\gamma = (\gamma_i)_{i \in Q_0} \in N^+,$$

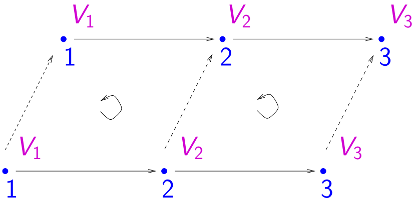
where $N := \mathbb{Z}^{Q_0}$ and $N^+ = \mathbb{N}^{Q_0} \setminus \{0\}$, encoding dimensions of the vector spaces assigned to vertices.



- $\{\text{Representations of } Q \text{ of dimension } \gamma\} = \bigoplus_{\alpha: i \to j} \mathrm{Hom}(\mathbb{C}^{\gamma_i}, \mathbb{C}^{\gamma_j})$

Representations of Quivers

- There is a natural notion of morphisms/isomorphisms between quiver representations.



- $\mathcal{M}_\gamma := \{\text{Representations of } Q \text{ of dimension } \gamma\}/\ \text{Isom} = \bigoplus_{\alpha: i \to j} \text{Hom}(\mathbb{C}^{\gamma_i}, \mathbb{C}^{\gamma_j}) / \prod_{i \in Q_0} GL(\gamma_i, \mathbb{C})$
  - "Stacky quotient" $\implies$ $\mathcal{M}_\gamma$ as an Artin stack
  - "GIT quotient" of the moduli space of *semi-stable representations* $\implies$ algebraic variety

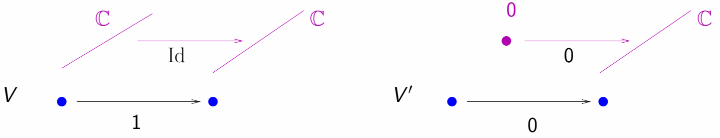
Representations of Quivers

Definition (King's notion of stability)

- V: quiver representation of dimension $\gamma \in N^+$.
- $M := \mathrm{Hom}(N, \mathbb{Z})$ and $M_\mathbb{R} = \mathrm{Hom}(N, \mathbb{R}) = M \otimes \mathbb{R}$
- $\theta \in \gamma^\perp := \{\theta \in M_\mathbb{R}, \theta(\gamma) = 0\} \subset M_\mathbb{R}$: **stability parameter**.
  - V: *θ-stable* if $\forall \{0\} \subsetneq V' \subsetneq V$ we have $\theta(\dim(V')) < 0$.
  - V: *θ-semi-stable* if $\forall V' \subsetneq V$ we have $\theta(\dim(V')) \leq 0$.

Example

- Q: A_2 quiver (1-Kronecker quiver),
- V: representation with $\gamma := \dim(V) = (1,1) \in N \cong \mathbb{Z}^2$,
- $\theta = (\theta_1, -\theta_1) \implies \theta \in \gamma^\perp \in M_\mathbb{R}$.
- $V' \subset V$ with $\dim(V') = (0,1) \implies \theta(\dim(V')) = -\theta_1$



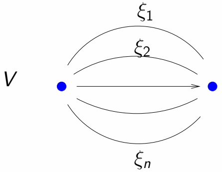
- $\theta_1 > 0 \implies V$ is **stable** and $\theta_1 < 0 \implies V$ is **unstable**.

The moduli space of semi-stable quiver representations

- $\mathcal{M}_\gamma^\theta$: Moduli space of θ semi-stable quiver representations of Q dimension γ.
  - Follows from the GIT construction that $\mathcal{M}_\gamma^\theta$ is a quasi-projective algebraic variety / $\mathbb{C}$

Example

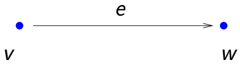
- Q: m-Kronecker quiver
- V: representation with $\gamma := \dim(V) = (1,1) \in N$
- $\theta = (\theta_1, -\theta_1) \in \gamma^\perp \subset M_\mathbb{R}$.



- $\theta_1 > 0$ and $(\xi_1, \ldots, \xi_n) \neq 0 \implies V$ is θ semi-stable, $\mathcal{M}_\gamma^\theta \cong \mathbb{CP}^{m-1}$
- $\theta_1 < 0 \implies \mathcal{M}_\gamma^\theta = \emptyset$.

Quivers with potentials

- Path algebra $\mathbb{C}Q$: $\mathbb{C}$-linear combinations of paths in Q with concatenation product.

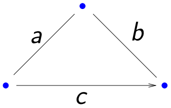


$$\mathbb{C}Q = \mathbb{C}v \oplus \mathbb{C}e \oplus \mathbb{C}w$$
$$v^2 = v, \; w^2 = w$$
$$ev = we = e$$

- Potential $W \in \mathbb{C}Q$: Formal linear combination of oriented cycles.

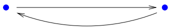
Acyclic Quiver

$W = 0$



$W = 2abc + 5(abc)^2$

Not allowed!



- We assume quivers do not have oriented two-cycles.

The trace function

- For $(Q, W = \sum \lambda_c c)$ define the **trace function** $\mathrm{Tr}(c)^\theta_\gamma : \mathcal{M}^\theta_\gamma \to \mathbb{C}$ by
$$V = ((V_i)_{i\in Q_0}, (f_\alpha)_{\alpha\in Q_1}) \longmapsto \mathrm{Tr}(f_{\alpha_n} \circ \ldots \circ f_{\alpha_1})$$

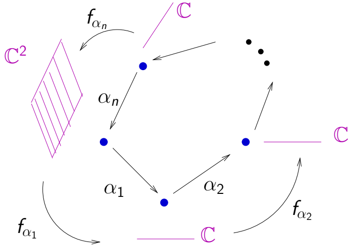


Figure: A cycle $c \in \mathbb{C}Q$

$$\mathrm{Tr}(W)^\theta_\gamma = \sum_c \lambda_c \mathrm{Tr}(c)^\theta_\gamma$$

DT invariants from the critical locus of the trace function

- C_γ^θ: Critical locus of $\mathrm{Tr}(W)_\gamma^\theta \subset \mathcal{M}_\gamma^\theta$.
- "In nice cases" ($\mathcal{M}_\gamma^\theta$: smooth, C_γ^θ: non-degenerate)
  $\implies$ DT invariants $\Omega_\gamma^\theta(y,t)$: (normalized) Hodge polynomial of C_γ^θ.

$$\Omega_\gamma^\theta(y,t) = (-y)^{-\dim C_\gamma^\theta} \sum_{p,q} h^{p,q}(C_\gamma^\theta) y^{p+q} t^{p-q}$$

Example

- Q: m-Kronecker quiver, V: reprs. with $\gamma := \dim(V) = (1,1) \in N$
- $\theta = (\theta_1, -\theta_1) \in \gamma^\perp \subset M_\mathbb{R}$.
- $\theta_1 > 0 \implies \mathcal{M}_\gamma^\theta \cong \mathbb{CP}^{m-1}$. Hence;

$$\Omega_\gamma^\theta(y,t) = (-y)^{-(m-1)}(1 + y^2 + \dots y^{2(m-1)})$$

- $W = 0 \implies \Omega_\gamma^\theta(y,t) \in \mathbb{Z}[y^{\pm 1}]$ (i.e. no Hodge numbers with $p \neq q$)

Refined DT invariants

Definition

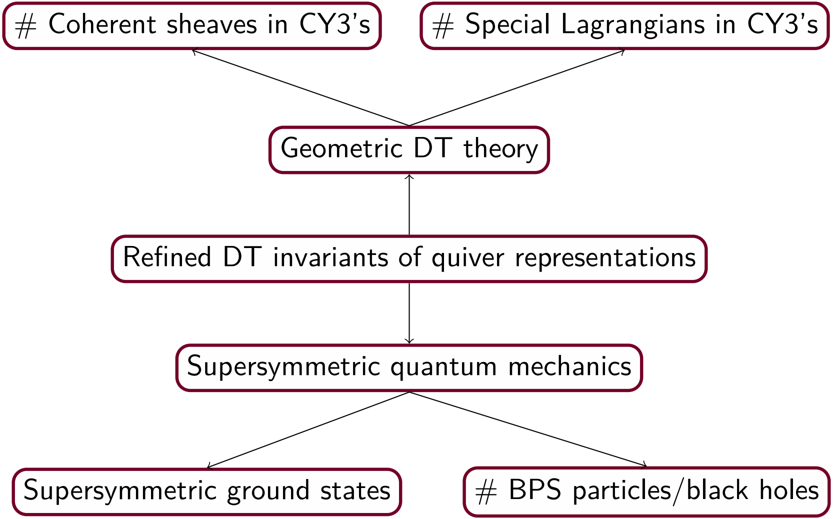
- (Q, W): quiver with potential
- $\gamma \in N^+$
- $\theta \in \gamma^\perp \subset M_\mathbb{R}$

The **refined Donaldson–Thomas (DT) invariant** $\Omega_\gamma^\theta(y,t) \in \mathbb{Z}[y^{\pm 1}, t^{\pm 1}]$ for $((Q,W),\gamma,\theta)$ is defined by

$$\Omega_\gamma^\theta(y,t) = (-y)^{-\dim C_\gamma^\theta} \sum_{p,q} h^{p,q}(H^*(C_\gamma^\theta, \phi_{\mathrm{Tr}(W)_\gamma^\theta} \mathcal{IC}_{M_\gamma^\theta})) y^{t+q} t^{p-q}$$

- $\mathcal{IC}_{M_\gamma^\theta}$: intersection cohomology sheaf on M_γ^θ
  - $\mathcal{IC}_{M_\gamma^\theta}$ is a perverse sheaf
  - M_γ^θ smooth $\implies$ $\mathcal{IC}_{M_\gamma^\theta}$ is the constant sheaf with stalk $\mathbb{Q}$
- $\phi_{\mathrm{Tr}(W)_\gamma^\theta}$: *vanishing cycle functor* for the function $\mathrm{Tr}(W)_\gamma^\theta$
  - $\phi_{\mathrm{Tr}(W)_\gamma^\theta} \mathcal{IC}_{M_\gamma^\theta}$: sheaf on the critical locus $C_\gamma^\theta \subset M_\gamma^\theta$
- See Kontsevich–Soibelman, Joyce–Song, Reineke, Davison–Meinhardt

Why are refined DT invariants of quivers interesting?



Ex: $\Omega^{\theta}_{\gamma}(y,t)$ can generally be very complicated

- The 3-Kronecker quiver appears in $\mathcal{N}=2$, $4d$ $SU(3)$ super Yang-Mills theory[1]

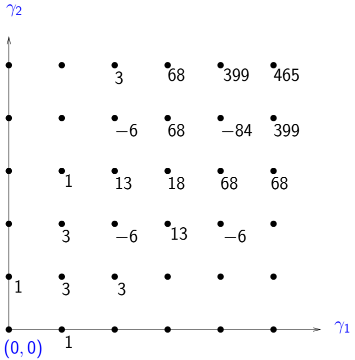


Figure: Values of $\Omega^{\theta}_{\gamma}(y=1,t=1)$ for the 3-Kronecker quiver

[1] Galakhov–Longhi–Mainiero–Moore–Neitzke, "Wild wall crossing and BPS giants." Journal of High Energy Physics 2013.

Wall crossing in the space of stability structures

- Study $\Omega^{\theta}_{\gamma}(y,t) \in \mathbb{Z}[y^{\pm 1}, t^{\pm 1}]$ for "γ-generic stability parameters"!

Definition

A stability parameter $\theta \in \gamma^{\perp}$ is called γ-**generic** if for every $\gamma' \in N$ such that $\sum_{i \in Q_0} |\gamma'_i| \leq \sum_{i \in Q_0} \gamma_i$,

$$\theta \in \gamma'^{\perp} \implies \gamma' // \gamma$$

- As long as $\theta \in \gamma^{\perp}$ is γ-generic $\Omega^{\theta}_{\gamma}(y,t)$ is constant.
- $\theta \in \gamma^{\perp}$ non-generic $\implies$ $\Omega^{\theta}_{\gamma}(y,t)$ jumps![a]

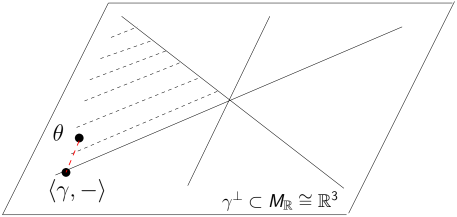
[a] Wall-crossing formula of Kontsevich–Soibelman, Joyce–Song

The attractor chamber

- Let $\{s_1, \ldots, s_{|Q_0|}\}$ be a basis for N. Define a skew symmetric form $\langle -, - \rangle$ on N by
$$\langle s_i, s_j \rangle := a_{ij} - a_{ji}.$$
where a_{ij} is the number of arrows from i to j.
- Fix $\gamma \in N$. The chamber containing $\langle \gamma, - \rangle \in \gamma^\perp \in M_{\mathbb{R}}$ is an attractor chamber for γ (generally not γ-generic).

Definition (Alexandrov–Pioline, Mozgovoy–Pioline, Kontsevich–Soibelman)

Let $\theta \in \gamma^\perp \subset M_{\mathbb{R}}$ be a small perturbation of $\langle \gamma, - \rangle$ which is γ-generic. Define the **attractor DT invariants** by $\Omega^*_\gamma(y,t) := \Omega^\theta_\gamma(y,t)$.



The attractor DT invariants

- $\Omega^*_\gamma(y,t)$ do not depend on the stability parameter θ, and are generally much simpler to compute.

Theorem (Bridgeland[a])

[a] Generalizations for some non-acyclic quivers: Lang Mou, arXiv:1910.13714

If Q is acyclic then

$$\Omega^*_\gamma(y,t) = \begin{cases} 1 & \text{if} \gamma = s_i = (0,\ldots,0,1,0,\ldots,0) \\ 0 & \text{otherwise} \end{cases}$$

The attractor DT invariants

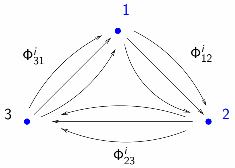
Conjecture (Beaujard–Manschot–Pioline, Mozgovoy–Pioline)

- *K_S: local del -Pezzo (canonical bundle over a del Pezzo surface S)*
- *(Q, W): quiver with potential s.t. $D^b Rep(Q, W) \cong D^b Coh(K_S)$*

$$\Omega^*_\gamma(y, t) = \begin{cases} 1, & \text{if} \gamma = s_i = (0, \dots, 0, 1, 0, \dots, 0) \\ (-y)^{-1}(1 + b_2(S)y^2 + y^4), & \text{if} \gamma = (k, \dots, k) \end{cases}$$

Example

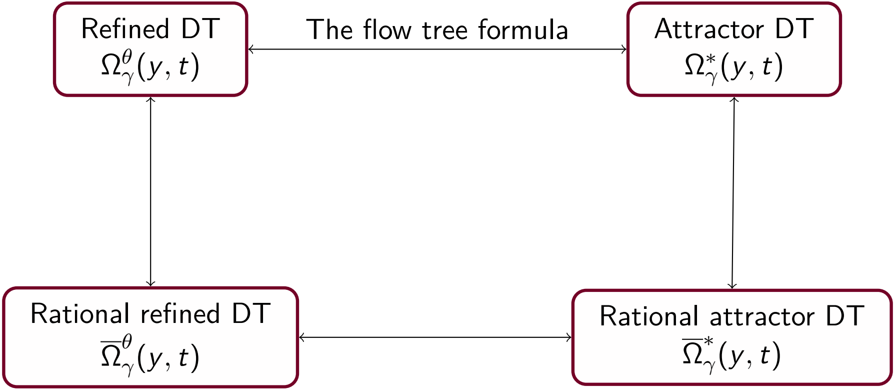
For $S = \mathbb{P}^2$, (Q, W) is illustrated below.



$$W = \Sigma_{(ijk)\in S_3} \operatorname{sgn}(ijk)\Phi^i_{12}\Phi^j_{23}\Phi^k_{31}$$

Goal: $\Omega_\gamma^\theta(y,t)$ from $\Omega_\gamma^*(y,t)$ and wall crossing

- For computational convenience we will express the flow tree formula using "rational" versions of attractor and refined DT invariants.



Definition

- (Q, W): quiver with potential
- $\gamma \in N^+$
- $\theta \in \gamma^\perp \subset M_\mathbb{R}$

The **rational refined Donaldson–Thomas (DT) invariant** $\overline{\Omega}_\gamma^\theta(y,t)$ is

$$\overline{\Omega}_\gamma^\theta(y,t) := \sum_{\substack{\gamma = k\gamma' \\ k \in \mathbb{Z}_{\geq 1}, \gamma' \in N^+}} \frac{1}{k} \frac{y - y^{-1}}{y^k - y^{-k}} \Omega_{\gamma'}^\theta(y^k, t^k) \,.$$

Refined attractor DT, $\overline{\Omega}_\gamma^\star(y,t) := \sum_{\substack{\gamma = k\gamma' \\ k \in \mathbb{Z}_{\geq 1}, \gamma' \in N^+}} \frac{1}{k} \frac{y - y^{-1}}{y^k - y^{-k}} \Omega_{\gamma'}^\star(y^k, t^k)$

The flow tree formula

Theorem (Flow tree formula (A-Bousseau, 2021))

- (Q, W): *quiver with potential*
- $\gamma \in N^+$, *and* $\theta \in \gamma^\perp$, γ*-generic.*

$$\overline{\Omega}_\gamma^\theta(y,t) = \sum_{\gamma=\gamma_1+\cdots+\gamma_r} \frac{1}{|\mathrm{Aut}((\gamma_i)_i)|} F_r^\theta(\gamma_1,\ldots,\gamma_r) \prod_{i=1}^r \overline{\Omega}_{\gamma_i}^\star(y,t)\,.$$

where

- $|\mathrm{Aut}((\gamma_i)_i)|$ *is the order of the group of permutation symmetries of the decomposition* $\gamma = \gamma_1 + \cdots + \gamma_r$, *and*
- $F_r^\theta(\gamma_1,\ldots,\gamma_r) \in \mathbb{Q}(y)$ *are defined concretely in terms of "flows" and binary "trees".*

- We will describe $F_r^\theta(\gamma_1,\ldots,\gamma_r)$ to state the theorem more precisely in a moment!

The flow tree formula - some remarks

- The flow tree formula was conjectured by Alexandrov and Pioline, partly based on physics argument (in particular, the attractor mechanism for black holes in $\mathcal{N} = 2$ supergravity).
- In a sequel paper Alexandrov-Pioline conjectured a further variant of the flow tree formula, referred to as the "attractor tree formula", which is proved recently by Mozgovoy. Direct relation between the flow tree formula with the attractor flow tree formula is unclear.

	Flow tree formula[2]	Attractor flow tree
F_r^θ	via binary trees	via arbitrary trees
Proof uses	wall-crossing / stability	operads
Phrased for	Lie algebras[3]	associative algebras

Table: Flow tree / attractor flow tree formula

[2] Computationally more efficient

[3] Stronger statement: flow tree formula for scattering diagrams

- For (Q, W), let $\gamma = \gamma = \gamma_1 + \cdots + \gamma_r \in N^+$. (repetitions allowed!)
- **Simplifying assumption for now:** $\{\gamma_1, \ldots, \gamma_r\}$ is a basis for N.

$$F_r^\theta(\gamma_1, \ldots, \gamma_r) := \sum_{T_r} \prod_{v \in V_{T_r}^\circ} \epsilon_{T_r,v}^{\theta,\omega} (-1)^{\langle e_{v'}, e_{v''} \rangle} \frac{y^{\langle e_{v'}, e_{v''} \rangle} - y^{-\langle e_{v'}, e_{v''} \rangle}}{y - y^{-1}} .$$

where the sum is over rooted binary trees T_r with r leaves (decorated by $\{\gamma_1, \ldots, \gamma_r\}$), $V_{T_r}^\circ$: set of interior vertices of of T_r, and for any $v \in V_{T_r}^\circ$; $e_v \in \mathcal{N}$ is the sum of γ_i's attached to leaves descendant from v, ω be a small generic perturbation of $\langle -, - \rangle$, and $\epsilon_{T_r,v}^{\theta,\omega} \in \{-1, 0, 1\}$ is a sign defined via "flows".

Lemma (A.–Bousseau)

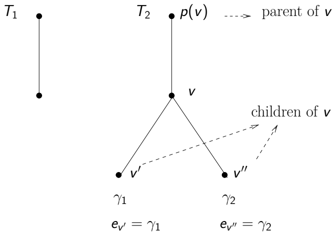
There exists a small generic perturbation ω of $\langle -, - \rangle$ making signs well-defined, and so that $F_r^\theta(\gamma_1, \ldots, \gamma_r)$ is independent of the choice of ω.

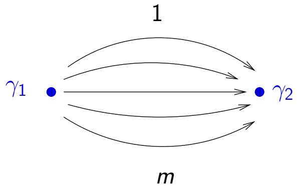
Example: the m-Kronecker quiver

- Q : m-Kronecker quiver (so $W = 0$),
- Assume $\gamma = (1, 1)$, so that $\gamma_1 = (1, 0)$ and $\gamma_2 = (0, 1)$.

$$F_1^{\theta}(\gamma_1, \gamma_2) = 1$$

$$F_2^{\theta}(\gamma_1, \gamma_2) = \epsilon_{T,v}^{\theta,\omega}(-1)^m \frac{y^m - y^{-m}}{y - y^{-1}}$$





$$\langle \gamma_1, \gamma_2 \rangle = m$$

Figure: The m-Kronecker quiver

The sign $\epsilon^{\theta,\omega}_{T,v}$ via attractor flows

- T_r: rooted binary tree with r-leaves
- Flow on T_r: orientations on edges from the root towards the leaves
- The **discrete attractor flow**: $v \mapsto \theta^{\alpha,\omega}_{T,v} \in M_{\mathbb{R}}$ defined recursively
  - v root $\implies \theta^{\alpha,\omega}_{T,v} := \theta$
  - v is not the root, with parent $p(v)$, then

$$\theta^{\alpha,\omega}_{T,v} = \theta^{\alpha,\omega}_{T,p(v)} - \frac{\theta^{\alpha,\omega}_{T,p(v)}(e_{v'})}{\omega(e_v, e_{v'})}\omega(e_v, -)\,.$$

Definition

$$\epsilon^{\alpha,\omega}_{T,v} := -\frac{1}{2}(\mathrm{sgn}(\theta^{\alpha,\omega}_{T,p(v)}(e_{v'})) + \mathrm{sgn}(\omega(e_{v'}, e_{v''})) \in \{0, 1, -1\}$$

Key technical point: for ω generic perturbation of $\langle -, - \rangle$, we have $\theta^{\alpha,\omega}_{T,p(v)}(e_{v'}) \neq 0$, $\omega(e_{v'}, e_{v''}) \neq 0$, and so the signs in the above definition make sense!

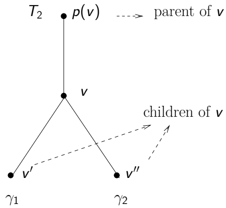
The sign $\epsilon^{\theta,\omega}_{T,v}$ via attractor flows

- Q: m-Kronecker quiver, $\gamma = (1,1)$, $\theta = (\theta_1, -\theta_1)$.
- For T_2; $\theta^{\alpha,\omega}_{T,p(v)} = \theta$ by definition the attractor flow map.

$$\theta^{\alpha,\omega}_{T,v} = \theta - \frac{\theta(\gamma_1)}{\langle \gamma_1 + \gamma_2, \gamma_1 \rangle} \langle \gamma_1 + \gamma_2, - \rangle$$

$$\epsilon^{\alpha,\omega}_{T,v} := -\frac{1}{2}(\mathrm{sgn}(\theta(\gamma_1)) + \mathrm{sgn}(\langle \gamma_1, \gamma_2 \rangle))$$

$$= -\frac{1}{2}(\mathrm{sgn}(\theta_1) + 1)$$



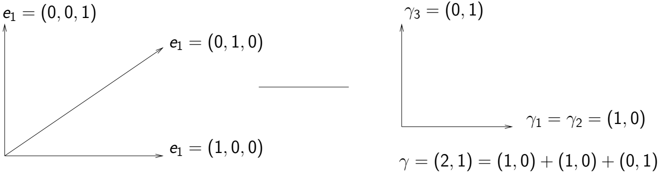
- Hence, $\theta_1 < 0 \implies \epsilon^{\alpha,\omega}_{T,v} = 0$ and $\theta_1 > 0 \implies \epsilon^{\alpha,\omega}_{T,v} = -1$

$$\overline{\Omega}^{\theta}_{\gamma}(y,t) = F^{\theta}_{1}(\gamma)\overline{\Omega}^{*}_{\gamma}(y,t) + F^{\theta}_{2}(\gamma_1,\gamma_2)\overline{\Omega}^{*}_{\gamma_1}(y,t)\overline{\Omega}^{*}_{\gamma_2}(y,t)$$

$$= 1 \cdot 0 + (-1) \cdot (-1)^m \frac{y^m - y^{-m}}{y - y^{-1}} \cdot 1 \cdot 1$$

$$= (-y)^{-(m-1)}(1 + y^2 + \dots y^{2(m-1)})$$

- Generally, $\gamma = \gamma_1 + \ldots + \gamma_r$ and $\{\gamma_1, \ldots, \gamma_r\}$ is not a basis.
- We introduce a bigger lattice $\mathcal{N} := \bigoplus_{i=1}^{r} \mathbb{Z} e_i$ and the map
- $p \colon \mathcal{N} \to N$ defined by $e_i \mapsto \gamma_i$
- Define a skew-symmetric form η on $\mathcal{N}$ by $\eta(e_i, e_j) := \langle \gamma_i, \gamma_j \rangle$.
- By duality, get a map $q : M_{\mathbb{R}} \to \mathcal{M}_{\mathbb{R}} = \mathrm{Hom}(\mathcal{N}, \mathbb{R})$.
- Denote $\alpha := q(\theta)$, and set $\theta_{T,v}^{\alpha,\omega} := \alpha$ in the discrete attractor flow.



The flow tree formula

Theorem (Flow tree formula (A-Bousseau, 2021))

Let ω be a small generic perturbation of η. Then,

$$\overline{\Omega}^{\theta}_{\gamma}(y,t) = \sum_{\gamma=\gamma_1+\cdots+\gamma_r} \frac{1}{|\mathrm{Aut}((\gamma_i)_i)|} F^{\theta}_r(\gamma_1,\ldots,\gamma_r) \prod_{i=1}^{r} \overline{\Omega}^{\star}_{\gamma_i}(y,t)$$

where $F^{\theta}_r(\gamma_1,\ldots,\gamma_r)$ is a sum over rooted binary trees

$$F^{\theta}_r(\gamma_1,\ldots,\gamma_r) := \sum_{T_r} \prod_{v\in V^{\circ}_{T_r}} \epsilon^{\alpha,\omega}_{T_r,v} (-1)^{\eta(e_{v'},e_{v''})} \frac{y^{\eta(e_{v'},e_{v''})} - y^{-\eta(e_{v'},e_{v''})}}{y-y^{-1}}$$

and the factors $\epsilon^{\alpha,\omega}_{T_r,v} \in \{0,1,-1\}$ are given in terms of the discrete attractor flow $v \mapsto \theta^{\alpha,\omega}_{T_r,v}$ by

$$\epsilon^{\alpha,\omega}_{T_r,v} := -\frac{1}{2}\left(\mathrm{sgn}(\theta^{\alpha,\omega}_{T_r,p(v)}(e_{v'})) + \mathrm{sgn}(\omega(e_{v'},e_{v''}))\right).$$

Idea of proof

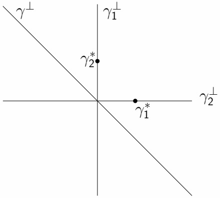
- We introduce and prove a more general **"flow tree formula for scattering diagrams"**
  - Scattering diagrams: combinatorial gadgets defined by Kontsevich–Soibelman, Gross–Siebert
  - Definition of a scattering diagrams is based on a choice of a Lie algebra
- The flow tree formula for DT invariants is then obtained by applying the general flow tree formula for scattering diagrams to the Bridgeland stability scattering diagram.

Scattering diagrams

- Let $N^+ \subset N$, $M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{R})$ and $P \subset N^+$ finite.
- $\mathfrak{g} = \bigoplus_{n \in N^+} \mathfrak{g}_n$ N^+-graded Lie algebra ($[\mathfrak{g}_{n_1}, \mathfrak{g}_{n_2}] \subset \mathfrak{g}_{n_1+n_2}$) such that $\{n \in N^+, \mathfrak{g}_n \neq 0\} \subset P$ (in particular, $\mathfrak{g}$ nilpotent).
- $\mathfrak{S}_P$: cone complex in $M_{\mathbb{R}}$ defined by the hyperplanes $n^\perp$ for $n \in P$.
- Wall_P: set of *walls*, codimension-one cones of $\mathfrak{S}_P$.
- For every wall $\mathfrak{d} \in \mathrm{Wall}_P$, denote $n_{\mathfrak{d}}$ the unique primitive element of N^+ such that $\mathfrak{d} \subset n_{\mathfrak{d}}^\perp$.
- A *joint* is a codimension-two cone of $\mathfrak{S}_P$.

Example

- $N = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$
- $P = \{\gamma_1, \gamma_2, \gamma_1 + \gamma_2\}$
- $\mathfrak{g} = \mathfrak{g}_{\gamma_1} \oplus \mathfrak{g}_{\gamma_2} \oplus \mathfrak{g}_{\gamma_1+\gamma_2}$
  - $\mathfrak{g}_{\gamma_1} = \mathbb{Q}(y,t)z^{\gamma_1}$
  - $\mathfrak{g}_{\gamma_2} = \mathbb{Q}(y,t)z^{\gamma_2}$
  - $\mathfrak{g}_{\gamma_1+\gamma_2} = \mathbb{Q}(y,t)z^{\gamma_1+\gamma_2}$



Scattering diagrams

Definition

A $(N^+, \mathfrak{g})$-scattering diagram is a map

$$\phi\colon \mathrm{Wall}_P \longrightarrow \mathfrak{g}$$

$$\mathfrak{d} \longmapsto \phi(\mathfrak{d})$$

such that $\phi(\mathfrak{d}) \in \bigoplus_{k\geq 1} \mathfrak{g}_{kn_{\mathfrak{d}}}$.

- Write $\phi(\mathfrak{d}) = \sum_{n=kn_{\mathfrak{d}}} \phi(\mathfrak{d})_n$ with $\phi(\mathfrak{d})_n \in \mathfrak{g}_n$.
- $G := \exp(\mathfrak{g})$, unipotent algebraic group defined by the nilpotent Lie algebra $\mathfrak{g}$ (product defined by the Baker-Campbell-Hausdorff formula).

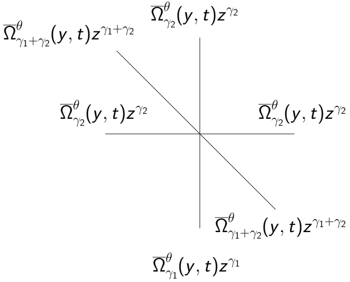
- For $\gamma \in N^+$, $P := \{n \in N^+, \sum_{i \in Q_0} n_i \leq \sum_{i \in Q_0} \gamma_i\}$.
- Define the quantum torus Lie algebra $\mathfrak{g} := \bigoplus_{n \in P} \mathbb{Q}(y,t) z^n$, where

$$[z^{n_1}, z^{n_2}] = (-1)^{\langle n_1, n_2 \rangle} \frac{y^{\langle n_1, n_2 \rangle} - y^{-\langle n_1, n_2 \rangle}}{y - y^{-1}} z^{n_1 + n_2}$$

if $n_1 + n_2 \in P$, and 0 else.

$$\phi \colon \mathrm{Wall}_P \longrightarrow \mathfrak{g}$$

$$\mathfrak{d} \longmapsto \sum_{n \in \mathbb{Z}_{\geq 1} n_{\mathfrak{d}} \cap P} \overline{\Omega}_n^{\theta}(y,t) z^n, \text{ for } \theta \in \mathfrak{d}$$



Scattering diagrams

- $\mathfrak{p} \colon [0,1] \to M_{\mathbb{R}}$, $t \mapsto \mathfrak{p}(t)$ a general loop around a joint, intersecting successively walls $\mathfrak{d}_1, \ldots, \mathfrak{d}_k$ for t equal to $t_1 < \cdots < t_k$.
- Path ordered product along $\mathfrak{p}$:

$$\Psi_{\mathfrak{p},\phi} := \exp(\epsilon_k \phi(\mathfrak{d}_k)) \ldots \exp(\epsilon_1 \phi(\mathfrak{d}_1)) \in G \,,$$

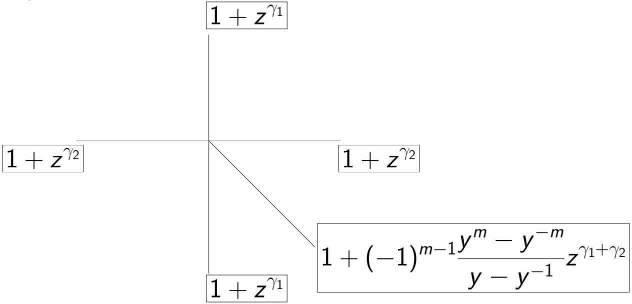
where $\epsilon_i \in \{\pm 1\}$ is the sign of the derivative of $t \mapsto -\mathfrak{p}(t)(n_{\mathfrak{d}_i})$ at $t = t_i$.

Definition

A $(N^+, \mathfrak{g})$-scattering diagram ϕ is *consistent* if $\Psi_{\mathfrak{p},\phi} = 1$ for every loop $\mathfrak{p}$.

Example: a consistent scattering diagram

- Q: m-Kronecker quiver, $\gamma = (1,1) = \gamma_1 + \gamma_2$
- $P = \{\gamma_1, \gamma_2, \gamma_1 + \gamma_2\}$

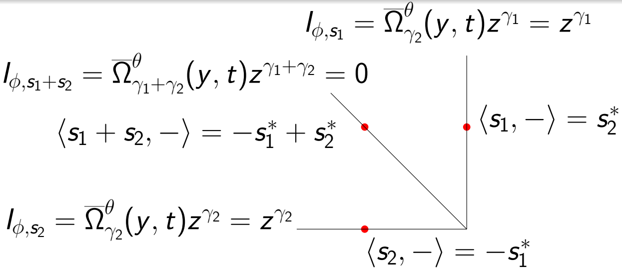


$$\text{Consistency check} : (1+z^{\gamma_1})(1+z^{\gamma_2})(1+z^{\gamma_1})^{-1}(1+z^{\gamma_2})^{-1}$$
$$= (1+z^{\gamma_1})(1+z^{\gamma_2})(1-z^{\gamma_1})(1-z^{\gamma_2})$$
$$= (1+z^{\gamma_1}+z^{\gamma_2}+z^{\gamma_1}z^{\gamma_2})(1-z^{\gamma_1}-z^{\gamma_2}+z^{\gamma_1}z^{\gamma_2})$$
$$= (1+z^{\gamma_1}z^{\gamma_2}-z^{\gamma_2}z^{\gamma_1}) = 1 + [z^{\gamma_1}, z^{\gamma_2}]$$

Definition

Initial data of a scattering diagram ϕ:

- Set of walls $\mathfrak{d} \in \mathrm{Wall}_P$ be a wall such that $\langle n, - \rangle \in \mathfrak{d}$, for $n \in P$,
- $I_{\phi,n} := \phi(\mathfrak{d})_n$



Theorem (Gross–Siebert, Kontsevich–Soibelman)

A $(N^+, \mathfrak{g})$-scattering diagram ϕ is uniquely determined by its initial data $(I_{\phi,n})_{n \in P}$.

The stability scattering diagram is consistent

- $\gamma \in N^+$, $P := \{n \in N^+, \sum_{i \in Q_0} n_i \leq \sum_{i \in Q_0} \gamma_i\}$.
- The quantum torus Lie algebra $\mathfrak{g} := \bigoplus_{n \in P} \mathbb{Q}(y,t) z^n$, where

$$[z^{n_1}, z^{n_2}] = (-1)^{\langle n_1, n_2 \rangle} \frac{y^{\langle n_1, n_2 \rangle} - y^{-\langle n_1, n_2 \rangle}}{y - y^{-1}} z^{n_1 + n_2}$$

  if $n_1 + n_2 \in P$, and 0 else.
- ϕ: $\mathrm{Wall}_P \to \mathfrak{g}$ by $\phi(\mathfrak{d}) = \sum_{n \in \mathbb{Z}_{\geq 1} n_{\mathfrak{d}} \cap P} \overline{\Omega}_n^\theta(y,t) z^n$, where $\theta \in \mathfrak{d}$

Theorem (Bridgeland)

ϕ is a consistent scattering diagram, and for all $n \in P$, $I_{\phi,n} = \overline{\Omega}_n^\star(y,t) z^n$.

This is a reformulation of the Kontsevich-Soibelman wall-crossing formula for DT invariants.

The flow tree formula for scattering diagrams

- ϕ: consistent $(N^+, \mathfrak{g})$-scattering diagram, $\mathfrak{d} \in \mathrm{Wall}_P$, $n \in \mathbb{Z}_{\geq 1} n_{\mathfrak{d}}$.
- The flow tree formula for ϕ expresses any $\phi(\mathfrak{d})_n$ in terms of initial data
  - We proved the flow tree formula for DT invariants by applying this formula for the stability scattering diagram.

$\phi(\mathfrak{d})_n = \sum_{n=n_1+\ldots+n_r} \frac{1}{|\mathrm{Aut}(n_1,\ldots,n_r)|} A^{\alpha,\omega}(I_{\phi,n_1}, \ldots, I_{\phi,n_r})$ where $A^{\alpha,\omega} : \prod_{i=1}^r \mathfrak{g}_{n_i} \to \mathfrak{g}_n$ is the flow tree map for scattering diagrams defined by

$$A^{\alpha,\omega} = \sum_{T_r} \Big(\prod_{v \in V_{T_r}^{\circ}} \epsilon_{T_r,v}^{\alpha,\omega} \Big) \cdot \mathcal{I}$$

where $\mathcal{I}$ is defined recursively by iterated Lie brackets of initial data.

The flow tree formula for scattering diagrams

- Naive idea: start at some $\theta \in \mathfrak{d} \subset \gamma^{\perp}$ and move along a line in the direction $\langle \gamma, - \rangle$.

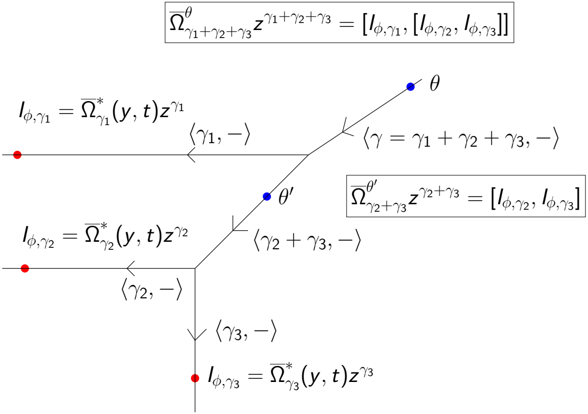
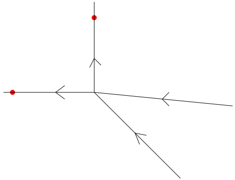


Figure: "Nice part" of a consistent scattering diagram

The proof flow tree formula for scattering diagrams

- Issue: when moving along lines we may hit non-trivalent vertices (bad situation).
- Technical heart of the proof: go to a bigger space and perturb the skew–symmetric form to avoid bad situations.



- In available literature on scattering (eg “the tropical vertex” [GPS]) such bad situations are handled by doing a “global perturbation” of the tree. Perturbing the skew symmetric form is more "local" (when somewhere along the flow the line we hit a triple or higher intersection of walls, we can avoid that by slightly perturbing the components of $<$-,-$>$ corresponding to that direction). In other words, one does not move the starting point of the flow as in [GPS] but the directions of the edges of the embedded trees.