Combinatorics of the Quantum Symmetric Simple Exclusion Process, associahedra and free cumulants

Algebra, Geomerty and Physics Seminar
Humboldt University Berlin / MPIM Bonn /Zoom
17 may 2022

Philippe Biane
CNRS, IGM
Université Gustave-Eiffel

Main Topic: Loop polynomials

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-Encode the asymptotic fluctuations of the *Quantum Symmetric Simple Exclusion Process*.

-Goal of the talk: Give a combinatorial formula for Q_σ; explain that they are *free cumulants*

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5. Exchange relation:

$$[x_i](Q_\sigma + Q_{s_i\sigma s_i})|_{x_i = x_{i+1}} - [x_{i+1}](Q_\sigma + Q_{s_i\sigma s_i})|_{x_i = x_{i+1}} =$$

$$2([x_i]Q_{\sigma^-}(x^-))([x_{i+1}]Q_{\sigma^+}(x^+))$$

($s_i\sigma = \sigma^+\sigma^-$, σ^+ moves $i+1$ and σ^- moves i.)

Exclusion Process

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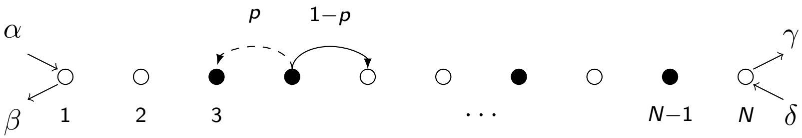
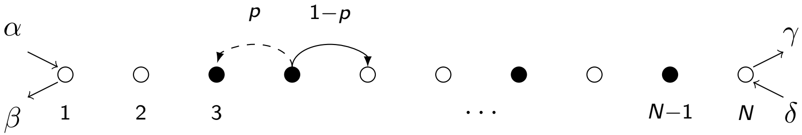
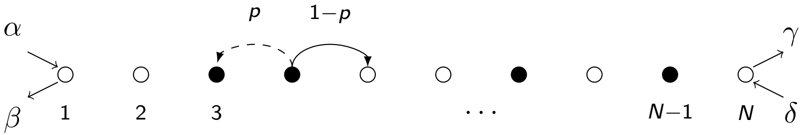


Particles can jump to neighbouring sites if empty and may exit or enter the interval from the boundary points 1 and N with rates $\alpha, \beta, \gamma, \delta$.



In the large time limit a current is established and the configuration of particles converges to a *stationary measure* μ which is a probability measure on the set of configurations

$$\Omega = \{0,1\}^N$$



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The exclusion process is a paradigm of non-equilibrium statistical mechanics. It is simple enough to be solvable yet sufficiently complex to exhibit non-trivial behaviour. It has been the object of numerous studies in physics, probability and combinatorics literature.

Quantum Symmetric Simple Exclusion Process

Fermionic particles on $\{1, 2, \ldots, N\}$ are subject to a Hamiltonian

$$H_t = \sum_{j=1}^{N} c_{j+1}^\dagger c_j W_t^j + c_j^\dagger c_{j+1} \bar{W}_t^j$$

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$c_i^{\dagger}, c_i$=fermionic creation and annihilation operators, satisfying

$$c_i c_j^{\dagger} + c_j^{\dagger} c_i = \delta_{ij}$$

acting on

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V is the quantum version of $\Omega = \{0, 1\}^N$.
A state of the form $e_{i_1} \otimes \ldots \otimes e_{i_N}$ corresponds to a classical configuration (e_0 =empty, e_1=occupied).

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$$d\rho_t = -i[dH_t, \rho_t] - \frac{1}{2}[dH_t, [dH_t, \rho_t]] + \mathcal{L}_{bdry}(\rho_t)dt$$

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ρ_t is a random matrix, if the initial configuration is diagonal on the classical states the expected value $\bar{\rho}_t$ satisfies the same evolution as the classical SSEP.

Asymptotics and loop polynomials

As $t \to \infty$ one has $\rho_t \to \rho$ in distribution
ρ is the *stationary state* (a random $2^N \times 2^N$ matrix)

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The two-point functions $G_{ij} = Tr(\rho c_i c_j^\dagger)$ form a random matrix

$$\mathsf{G} = (G_{ij})_{1 \leq i,j \leq N}$$

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The fluctuations of G are measured by their cumulants (connected correlation functions in physics)

$$E[G_{i_1 j_1} G_{i_2 j_2} \dots G_{i_p j_p}]^c = C_p(G_{i_1 j_1}, G_{i_2 j_2}, \dots, G_{i_p j_p})$$

These are the quantities of interest.

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Only the ones for which $j_1, \dots, j_p$ is a cyclic permutation of $i_1, \dots, i_p$ have a nonzero limit.
If $i_1/N, i_2/N, \dots, i_p/N \to u_1, u_2, \dots, u_p \in [0,1]$ as $N \to \infty$, then

$$E[G_{i_1 i_p} G_{i_p i_{p-1}} \dots G_{i_2 i_1}]^c = \frac{1}{N^{p-1}} g_p(u_1, \dots, u_p) + O(\frac{1}{N^p})$$

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The g_p are piecewise polynomial functions, polynomial in each sector corresponding to an ordering of the u_i.

Loop polynomials (Bernard and Jin, 2021)

Define $Q_\sigma(x_1, \ldots, x_p)$ for $0 \le x_1 \le x_2 \le \ldots \le x_p \le 1$, indexed by circular permutations σ of $1, \ldots, p$ by

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The Q_σ are the loop polynomials. They give the values of the functions g_p in each sector.

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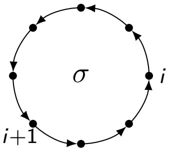
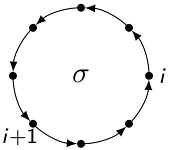

σ
i
i+1



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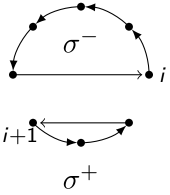

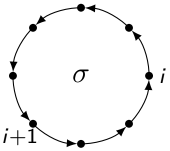

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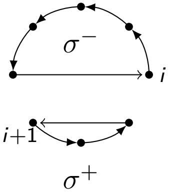

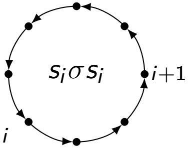

How to use the defining relations

$$Q_\sigma = A + x_i B + x_{i+1} C + x_i x_{i+1} D,$$

$$Q_{s_i \sigma s_i} = A' + x_i B' + x_{i+1} C' + x_i x_{i+1} D',$$

where $A, B, C, D, A', B', C', D'$ do not depend on x_i, x_{i+1}.

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Let $s_i \sigma = \sigma^- \sigma^+$ and

$$\Delta := ([x_i] Q_{\sigma^-}(x^-))([x_{i+1}] Q_{\sigma^+}(x^+)).$$

By the exchange condition

$$B - C' = B' - C = \Delta.$$

One can obtain Q_σ for all n-cycles if one knows Q_σ for one of the cycles. Bernard and Jin prove that the conditions above completely determine the loop polynomials.

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Examples: $n = 5$

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\begin{aligned}
Q_{12345} &= x_1(1-4x_2-3x_3-2x_4+9x_2x_3+7x_2x_4+5x_3x_4-14x_2x_3x_4)(1-x_5) \\
Q_{13245} &= x_1(1-6x_2-x_3-2x_4+9x_2x_3+10x_2x_4+2x_3x_4-14x_2x_3x_4)(1-x_5) \\
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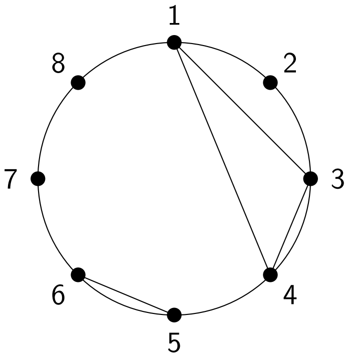
The goal is to give an explicit combinatorial formula for the Q_σ.

Non-crossing partitions

Set partitions of $\{1, 2, \ldots, n\}$ without crossing:



$$\pi = \{1, 3, 4\} \cup \{2\} \cup \{5, 6\}, \cup \{7\} \cup \{8\}$$

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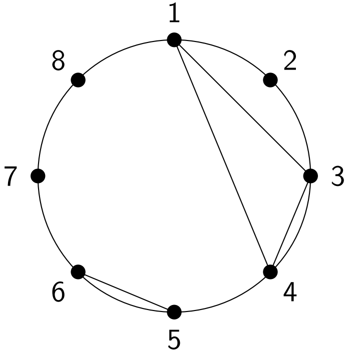

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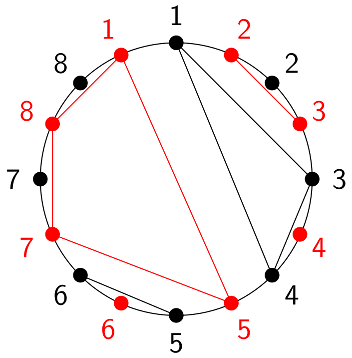
$$|NC(n)| = \mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$$

Kreweras complement



$$\pi = \{1,3,4\} \cup \{2\} \cup \{5,6\}, \cup \{7\} \cup \{8\}$$

$$K(\pi) = \{1,5,7,8\} \cup \{2,3\} \cup \{4\} \cup \{6\}$$

Free cumulants (R. Speicher)

A=unital algebra, $\varphi : A \to C$ such that $\varphi(1) = 1$.

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$$\text{where} \quad \kappa_\pi(a_1, \ldots, a_n) = \prod_{p \text{ part of } \pi} \kappa_{|p|}(a_{i_1}, a_{i_2}, \ldots, a_{i_{|p|}})$$

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One has

$$\boxed{\kappa_n(a_1, a_2, \ldots, a_n) = \sum_{\pi \in NC(n)} \mu(\pi) \varphi_\pi(a_1, \ldots, a_n)}$$

where μ is the Möbius function of $NC(n)$:

$$\mu(\pi) = \prod_{p \text{ part of } K(\pi)} (-1)^{|p|-1} \mathrm{Cat}_{|p|-1}$$

Examples:

$$\varphi(a_1) = \kappa_1(a_1) \qquad \{1\}$$

$$\begin{aligned} \varphi(a_1 a_2) = \quad & \kappa_2(a_1, a_2) && \{1, 2\} \\ & + \kappa_1(a_1)\kappa_1(a_2) && \{1\} \cup \{2\} \end{aligned}$$

hence

$$\begin{aligned} \kappa_1(a) &= \varphi(a) \\ \kappa_2(a_1, a_2) &= \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2) \end{aligned}$$

$$
\begin{array}{rcl}
\varphi(a_1 a_2 a_3) = & \kappa_3(a_1, a_2, a_3) & \{1,2,3\} \\
& +\kappa_1(a_1)\kappa_2(a_2, a_3) & \{1\} \cup \{2,3\} \\
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\end{array}
$$

$$
\begin{aligned}
\kappa_3(a_1, a_2, a_3) &= \varphi(a_1 a_2 a_3) - \varphi(a_1 a_2)\varphi(a_3) - \varphi(a_1 a_3)\varphi(a_2) \\
&\quad -\varphi(a_1)\varphi(a_2 a_3) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3)
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Schröder trees

Plane, rooted trees such that each internal vertex has at least two descendants.

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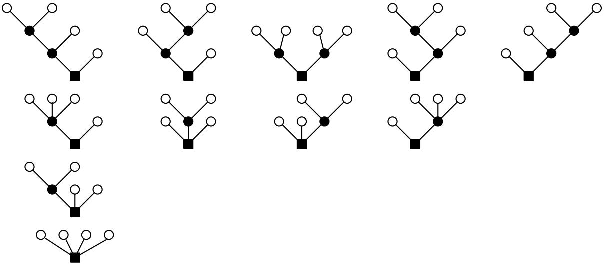

Schröder trees with 4 leaves

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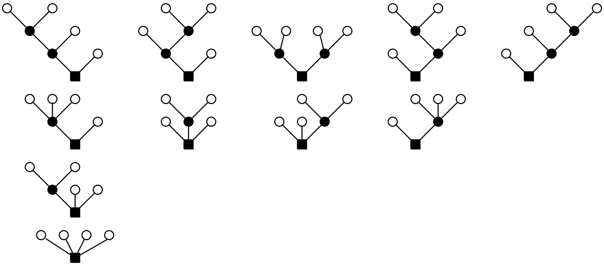
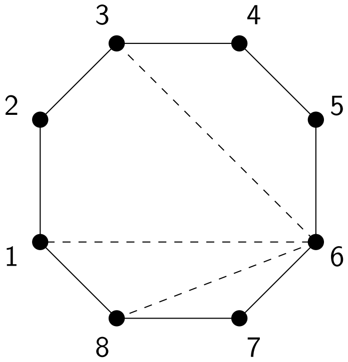


Schröder trees with 4 leaves

Counted, in terms of the number of leaves, by the small Schröder numbers $s_n = 1, 1, 3, 11, 45, \ldots$ for $n = 1, 2, 3, \ldots$ (A001003 in OEIS) with generating series $\frac{1+x-\sqrt{1-6x+x^2}}{4x}$.

Associahedra

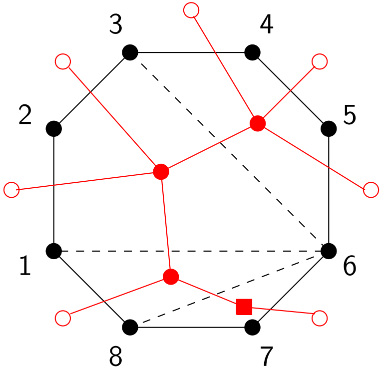
A *dissection* of a polygon is a collection of non-crossing diagonals



The dissections of a polygon form a simplicial complex, the *associahedron* which can be realized as a polytope.

Schröder trees and associahedra

There is a natural bijection between Schröder trees and dissections of polygons.



Prime Schröder trees

A Schröder tree is *prime* if the righmost edge of its root is a leaf.
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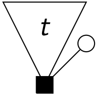
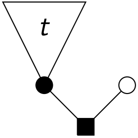
For each Schröder tree t, with $n-1$ leaves, we can build two prime Schröder trees t_1 and t_2, with n leaves



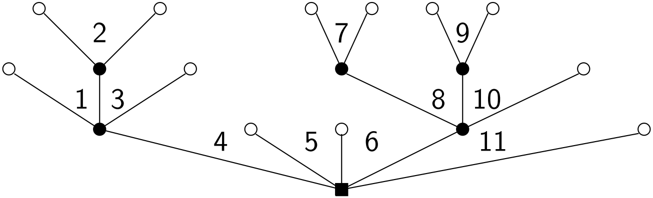
t_1



t_2

Corners

corner=angle between pair of consecutive edges.
A Schröder tree with n leaves has exactly $n-1$ corners, numbered from left to right.



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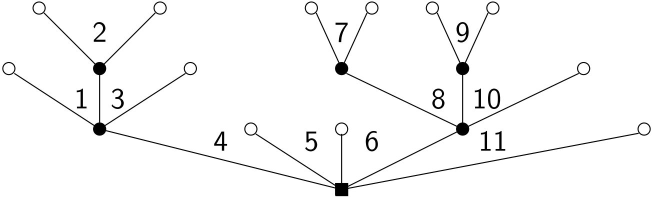

A Schröder tree determines a non-crossing partition of its corners:

$$\pi(t) = \{1,3\}, \{2\}, \{4,5,6,11\}, \{7\}, \{8,10\}, \{9\}$$

Prime Schröder trees and Möbius function on $NC(n)$ (Josuat-Vergès, Menous, Novelli, Thibon, 2017)

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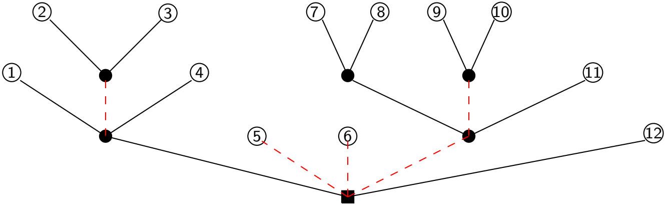
In a prime Schröder tree remove, for each internal vertex, the internal edges pointing out of this vertex, except the leftmost and rightmost ones.



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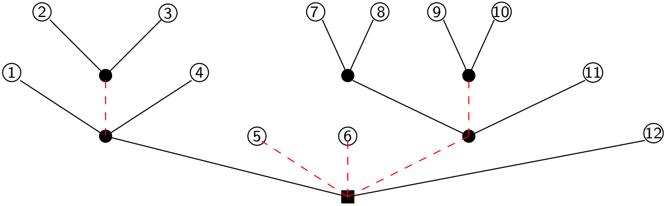

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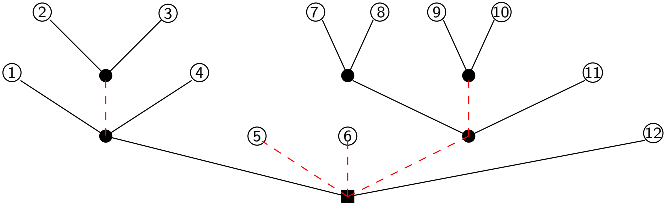

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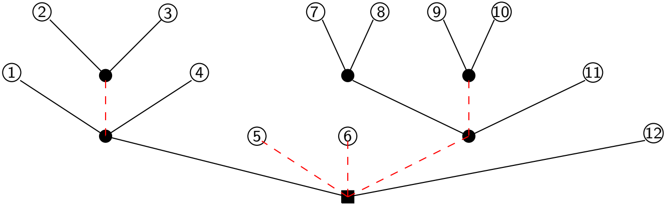

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$$\prod_{p \text{ part of } K(\pi)} \mathrm{Cat}_{|p|-1} = |\mu(\pi)|$$

is the number of prime Schröder trees with $\pi(t) = \pi$.

A formula for the loop polynomials

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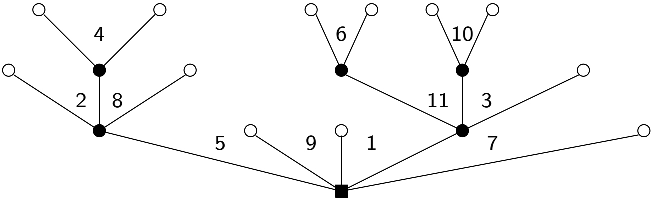
t= prime Schröder tree t, with $n+1$ leaves, $k \in [1, n]$ and σ circular permutation.
Label the corners of t, by the numbers $\sigma(k), \sigma^2(k), \ldots, \sigma^{n-1}(k), k$.

$$\sigma = 2, 4, 8, 5, 9, 1, 6, 11, 10, 3, 7, \qquad k = 7$$



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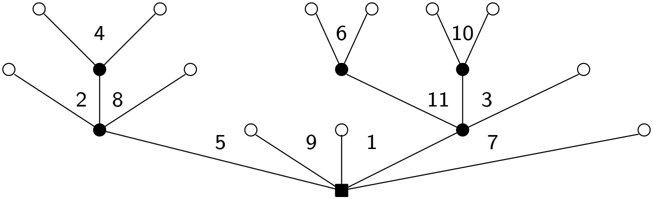

For each internal vertex $i(v)$=the smallest label of corners of v.

$$x^{t,k,\sigma} = \prod_v (-x_{i(v)}) \qquad \text{(here } x^{t,7,\sigma} = (-1)^6 x_2 x_4 x_1 x_6 x_3 x_{10}\text{)}$$

Theorem: for each $k \in [1, n]$ one has

$$\boxed{Q_\sigma(x_1, \ldots, x_n) = -\sum_{t \in pS_{n+1}} x^{t,k,\sigma}}$$

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Then check the continuity and the exchange conditions (uses simple properties of Schröder trees).

Proof of the exchange condition by cutting a prime Schröder tree into two prime Schröder trees

$$[x_i](Q_\sigma + Q_{s_i \sigma s_i})|_{x_i = x_{i+1}} - [x_{i+1}](Q_\sigma + Q_{s_i \sigma s_i})|_{x_i = x_{i+1}} =$$

$$2([x_i]Q_{\sigma^-}(x^-))([x_{i+1}]Q_{\sigma^+}(x^+))$$

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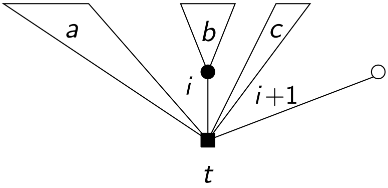

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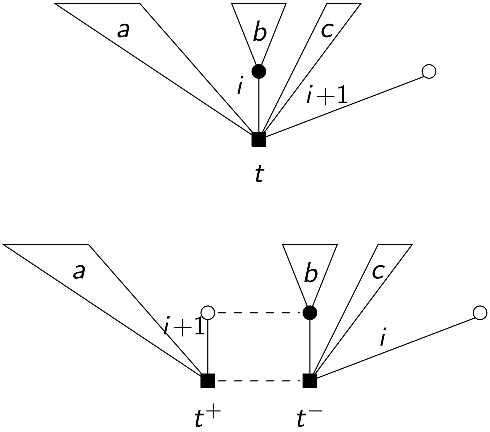

The loop polynomials as free cumulants

On $[0,1] \subset \mathrm{R}$ with Lebesgue measure let for $x \in [0,1]$

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Proof: use the connection between Schröder trees, $NC(n)$ and the Möbius function.

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The loop polynomials are free cumulants:

$$\Pi_x = 1_{[0,x]}; \quad x \in [0,1]$$

$$Q_\sigma(x_1, \dots, x_n) = \kappa_n(\Pi_{x_1}, \Pi_{x_{\sigma(1)}}, \Pi_{x_{\sigma^2(1)}}, \Pi_{x_{\sigma^{n-1}(1)}})$$

THANK YOU