Interacting particle systems and random walks on Hecke algebras

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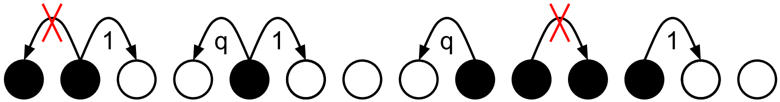
Definition of ASEP



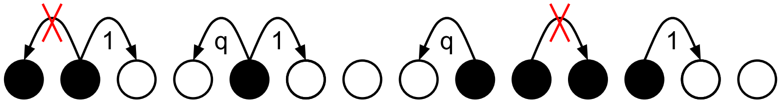
Collection of particles on $\mathbb{Z}$ which evolves in time.

There are two Poisson processes of rates 1 and $q < 1$ associated with each particle.

Each particle jumps one step to the right with rate 1, and jumps one step to the left with rate q, if the neighboring positions are vacant. If the position is occupied by another particle, the jump does not happen.

All Poisson processes are independent.

Stationary measures

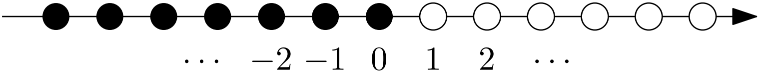


In general, the configuration can be described by a collection of random variables. Let $\eta_t^{asep}(z)$ be the number of particles in position $z \in \mathbb{Z}$ after time t. This is a random variable with values 0 and 1.

Stationary measure Fix $0 \leq p \leq 1$, and let $\{\eta_t^{asep}(z)\}_{z\in\mathbb{Z}}$ be a collection of independent, identically distributed Bernoulli random variables:

$$\mathrm{Prob}\left(\eta_t^{asep}(z) = 1\right) = p, \qquad \mathrm{Prob}\left(\eta_t^{asep}(z) = 0\right) = 1 - p.$$

Step initial condition

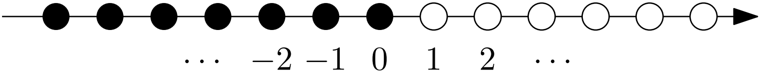


Highly non-stationary initial condition.

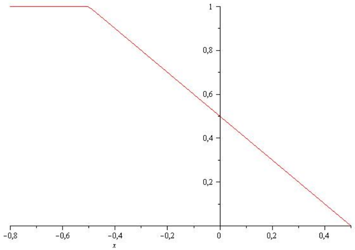
Asymptotic behavior in time ? It will NOT converge to a stationary measure.

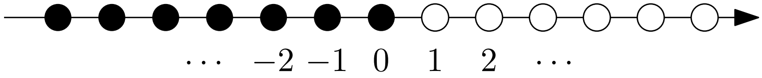
This type of questions: Harris, Liggett, Rost, . . . , . . . , . . .

Step initial condition



Evolution in time ? Density :





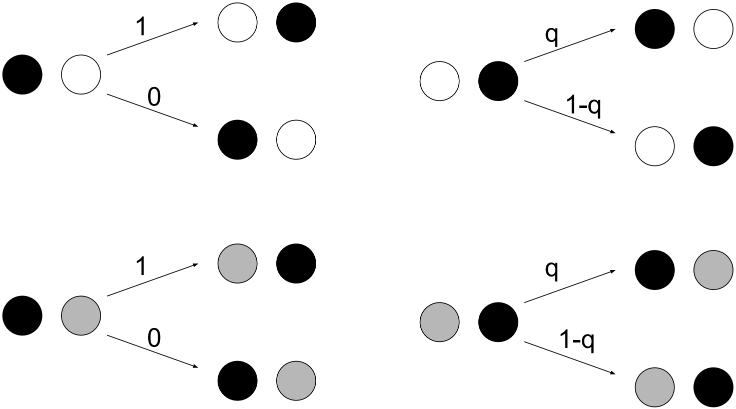
Theorem (Andjel-Vares, Benassi-Fouque, 87)

Let $m = m(t)$, $t \in \mathbb{R}_{\geq 0}$, *be a collection of integers such that* $\lim_{t\to\infty} \frac{m(t)}{t} = y, \qquad y \in \mathbb{R}$. *Then*

$$\lim_{t\to\infty} P(\eta_t^{asep}(m(t)) = 1) = d(y) :=$$

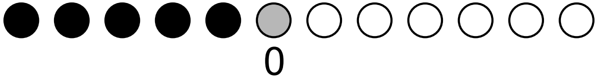
$$\begin{cases} 0, & y \geq (1-q), \\ \frac{1}{2}\left(1 - \frac{y}{1-q}\right), & -(1-q) < y < (1-q), \\ 1, & y \leq -(1-q). \end{cases}$$

Moreover, for any fixed $L \in \mathbb{Z}_{>0}$ *the random variables* $\{\eta_t^{asep}(m(t)+i)\}_{i=-L,\dots,L}$ *converge, as* $t \to \infty$, *to i.i.d. Bernoulli distributions with probability of 1 equal to* $d(y)$.



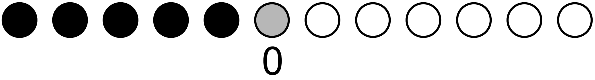
We consider particles of various types (=classes, colors, species).

Set of types is linearly ordered, and a particle of a smaller type interacts with a particle of a larger type as a particle with a hole.



Let us start with this initial condition. Let $S_1(t)$ be the position of the second class particle at time t.

Asymptotics of $S_1(t)$?

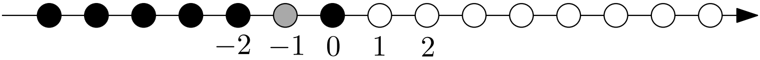


Let us start with this initial condition. Let $S_1(t)$ be the position of the second class particle at time t.

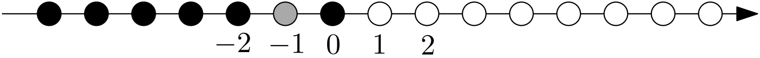
$$\lim_{t\to\infty} \mathrm{Prob}\left(\frac{S_1(t)}{t} < x\right) = d(-x) = \frac{1}{2}\left(1 + \frac{x}{1-q}\right).$$

Uniform distribution on $[-(1-q); (1-q)]$.

P.A. Ferrari-Kipnis'95, P.A. Ferrari-Goncalves-Martin'08.



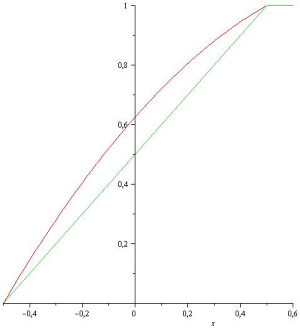
The asymptotic distribution of the second class particle ?

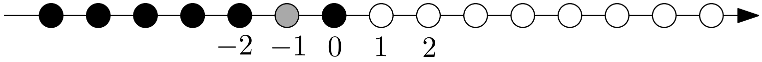


(Borodin-Bufetov'19) The asymptotic distribution of the second class particle

$$\lim_{t\to\infty} \mathrm{Prob}\left(\frac{S_1(t)}{t} < x\right) = d(-x) + (1-q)d(-x)(1-d(-x)).$$

Note the nontrivial dependence on q.





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- $q = 0$: TASEP, Cator-Pimentel'13
- for a class of initial configurations and general q: Borodin-Bufetov'19

Hecke algebra

$W = S_n$, $s_i = (i, i+1)$.

$L(w) :=$ number of inversions in $w \in W$.

Hecke algebra: $\{T_w\}_{w \in W}$ — linear basis

$$\begin{cases} T_s T_w = T_{sw}, & \text{if } L(sw) = L(w) + 1 \\ T_s T_w = (1-q) T_w + q T_{sw}, & \text{if } L(sw) = L(w) - 1. \end{cases}$$

The linear map $I : \mathcal{H} \to \mathcal{H}$

$$I : \sum_w a_w T_w \to \sum_w a_w T_{w^{-1}}$$

satisfies

$$I(h_r h_{r-1} \dots h_2 h_1) = I(h_1) I(h_2) \dots I(h_r), \qquad h_i \in \mathcal{H}.$$

Random walk on Hecke algebra

Generators $\{G_1, \ldots, G_k\}$, each of these generators has an independent exponential clock. When the clock s rings, we multiply G_s to the current position of the random walk $P \in \mathcal{H}$ — our new position is $G_s P$. This is a *random walk on Hecke algebra*.

An element of Hecke algebra

$$h := \sum_w \kappa_w T_w, \qquad \kappa_w \geq 0, \ \sum_w \kappa_w = 1,$$

can be interpreted as a random element of W. Random walk on Hecke algebra generates the random walk on W.

Multi-species ASEP / Hecke algebra

$W = S_n$, generators: $\{T_{s_i}\}_{i=1}^{n-1}$. Equivalent language to for the description of ASEP: Vocabulary

- Random multi-species configuration — element of Hecke algebra
- Update —- multiplication by T_s
- ASEP evolution — element of S_n generated by random walk on Hecke algebra
- Projection to fewer colors — projection to cosets of parabolic subgroups
- Class-position symmetry — involution I swaps w and w^{-1}.

Other Coxeter groups generate ASEP with a source (hyperoctahedral group), ASEP on a ring (affine Weyl group $\tilde{A}_n$).

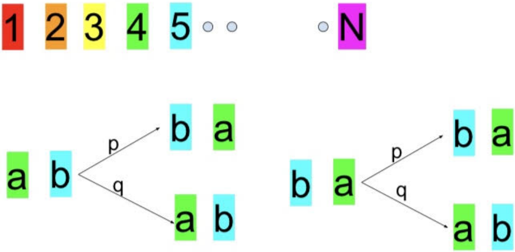
Multi-species ASEP / Hecke algebra

$W = S_n$, generators: $\{T_{s_i}\}_{i=1}^{n-1}$. Equivalent language for the description of ASEP.

- Multi-species ASEP is generated by Hecke algebra: Alcaraz-Rittenberg'93, Alcaraz-Droz-Henkel-Rittenberg'93, ..., Lam'11 , Cantini-de Gier-Wheeler'15, ...
- Class-position symmetry and applications for asymptotic analysis: Angel-Holroyd-Romik'08 (TASEP, $q = 0$), Amir-Angel-Valko'08 (ASEP), Borodin-Bufetov'19 (inhomogeneous stochastic six vertex model)

What happens if we consider other generators of the random walk on Hecke algebra ?
In fact, a variety of other processes appear (stochastic six vertex model, K-exclusion processes, coalescence processes, ...).

Biased Card Shuffling

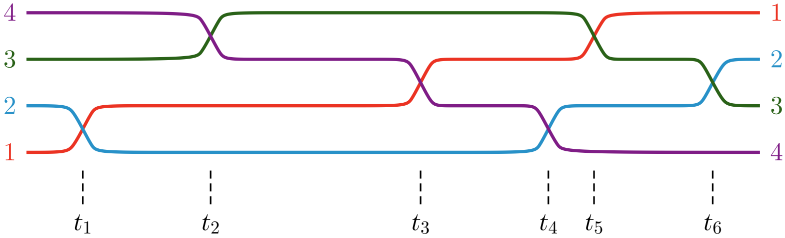


$a < b$, $p = 1$, $q = 0$.

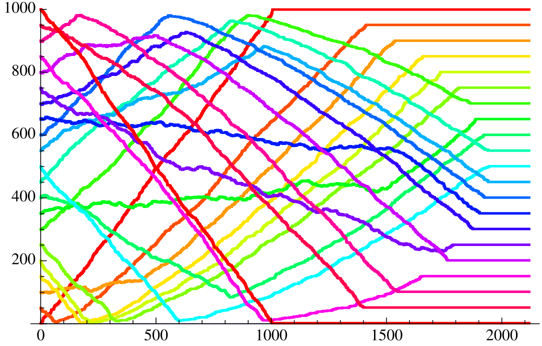
Continuous time: Updates happen according to independent Poisson processes on $\mathbb{R}_{\geq 0}$ attached to each pair of neighboring positions.
Question: When the sorting stops?

Multispecies TASEP on an interval

- Interval $\{1, 2, \ldots, N\}$. Symmetric group S_N.
- Each transposition $(i, i+1)$ has independent exponential clock.
- When the clock rings, we swap particles at i and $i+1$, but only **if** it will **increase** the number of color-position inversions.



Angel-Holroyd-Romik-08: What's happening as N becomes large?

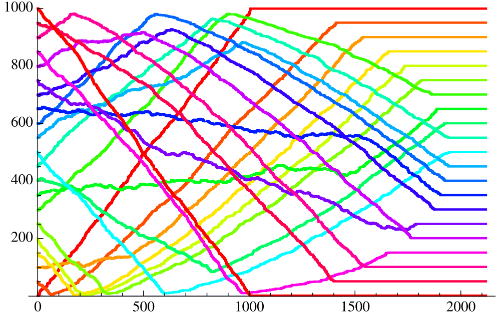


Picture from Angel-Holroyd-Romik-08.
Only 21 out of 1000 trajectories shown.

Theorem. [Angel-Holroyd-Romik] Set $\gamma_y = 1 + 2\sqrt{y(1-y)}$.
If $U_N(k)$ is the **last time the swap** $(k, k+1)$ **happens**, then

$$\frac{U_N(k) - N\gamma_{k/N}}{N^{1/3}(\gamma_{k/N})^{2/3}\left(\frac{k}{N}(1-\frac{k}{N})\right)^{-1/6}} \xrightarrow[N\to\infty]{d} F_2, \qquad \text{(Tracy-Widom distribution)}$$

Proof is based on coupling with TASEP with step initial condition and the result of Johansson'00.



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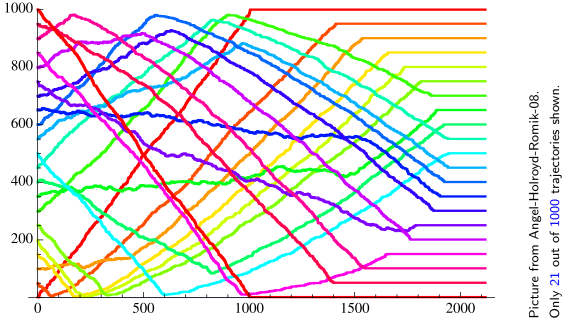
Question. Set T_N^{OSP} — the time when the systems **stops**

[AHR-08]: We have $T_N^{\mathrm{OSP}} \approx 2N$. What are **the fluctuations**?



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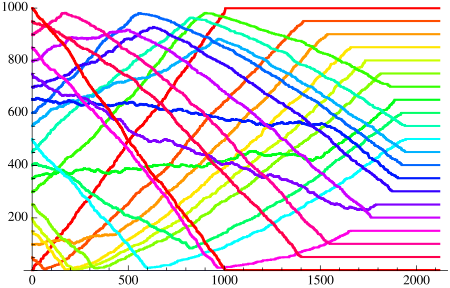
Theorem. Bufetov-Gorin-Romik'20

$$\frac{T_N^{\mathrm{OSP}} - 2N}{2^{1/3}\,N^{1/3}} \xrightarrow[N\to\infty]{d} F_1,$$

where F_1 is another Tracy-Widom distribution.



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Question. Set T_N^{OSP} — the time when the systems **stops**

Theorem. (Bufetov-Gorin-Romik-20)

$$\frac{T_N^{\mathrm{OSP}} - 2N}{2^{1/3}\, N^{1/3}} \xrightarrow[N\to\infty]{d} F_1,$$

Proof is based on symmetries of interacting particle systems Borodin-Gorin-Wheeler'19, Galashin'20; also we prove some of conjectures from Bisi-Cunden-Gibbons-Romik'20.

Mallows measure on S_n

S_n — symmetric group, $L(w)$ — number of inversions in w, and $0 \le q < 1$.

$$\mathrm{Prob}\,(w) = q^{n(n-1)/2 - L(w)} Z.$$

For $q = 0$ this measure is concentrated on one word (longest element), for general q it is "not far" from it for large n.

If we run multi-species ASEP on a finite interval of length n for a long time, it converges to this measure.

Mallows'53

$n \to \infty$: Gnedin-Olshanski'09, Gnedin-Olshanski'11

Other sets of generators also lead to interesting particle systems.

$[a; b] := \{j \in \mathbb{Z} : a \leq j \leq b\}$ the interval between a and b. $S_{a;b} \subset S_n$ permutes the elements from $[a; b]$ only.

Mallows element

$$\mathcal{M}_{a;b} := \sum_{w \in S_{a;b}} Zq^{(b-a+1)(b-a)/2 - L(w)} T_w, \qquad \mathcal{M}_{a;b} \in \mathcal{H}(S_n),$$

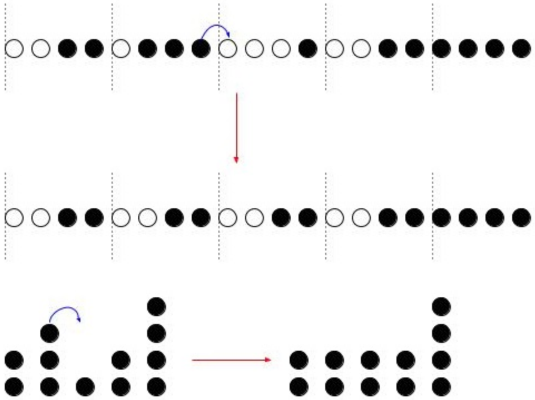
where $L(w)$ is the number of inversions in w. The main property of the element $\mathcal{M}_{a;b}$ is

$$T_w \mathcal{M}_{a;b} = \mathcal{M}_{a;b} T_w = \mathcal{M}_{a;b}, \qquad \text{for any } w \in S_{a;b}.$$

Let $n = NM$, with $M, N \in \mathbb{Z}_{>0}$, and consider the following set of generators of a random walk on the Hecke algebra :

$$\left\{ \mathcal{M}_{(x-1)M+1;xM} \mathcal{M}_{xM+1;(x+1)M} \, T_{(xM,xM+1)} \mathcal{M}_{(x-1)M+1;xM} \mathcal{M}_{xM+1;(x+1)M} \right\}_{x=1}^{N-1} .$$

This dynamics generates a multi-species $ASEP(q, M)$.
$q = 0$: M-exclusion TASEP.



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This dynamics generates a multi-species $ASEP(q, M)$.

- Construction is related to the notion of fusion: Kulish-Reshetikhin-Sklyanin'81, Corwin-Petrov'15.
- Single species version of ASEP(q,M) was introduced by Carinci-Giardina-Redig-Sasamoto'15
- Multi-species version of ASEP(q,M) was introduced by Kuan'16
- $M \to \infty$: q-TAZRP (single species version introduced by Sasamoto-Wadati'98).

Instead of just $T_{(xM,xM+1)}$ we can have arbitrary interaction between two blocks. This leads to a variety of processes and possible interactions, and one obtains multi-species versions of all these processes. In particular, in $M \to \infty$ limit one recovers the models of Povolotsky'13.

Summary

- Many multi-species interacting particle systems can be interpreted as random walks on Hecke algebras
- The structure of Hecke algebras can be useful for the asymptotic analysis. In particular, the involution $T_w \to T_{w^{-1}}$ is a powerful tool.
- There is a lot of information available about Hecke algebras. There is a potential for more applications of this point of view on interacting particle systems.