r-th roots: better negative than positive

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I Motivation

Witten's conjecture '91 / Kontsevich Thm '92

Descendant potential of trivial CohFT = a certain solution of the KdV hierarchy

r-spin CohFT $\qquad$ r-KdV

$r \geq 2$

'93 Witten's r-spin conjecture

Theorem (Polischuk-Vaintrob '00): $\exists$ a CohFT (with unit) (not s.s.)

pure degree

$$W^r_{g,n}(a_1,\dots,a_n) \in H^*(\overline{\mathcal{M}}_{g,n}) \qquad 1 \le a_i \le r-1$$

<u>Theorem</u> (Faber–Shadrin–Zvonkine '06, Givental)

Descendant potential of $W^r_{g,n}$ is a τ-function of the r-KdV heirarchy

Q) What about r "negative"?

II <u>The CohFT</u>

$$\mathcal{C} \xrightarrow{f} \overline{\mathcal{M}}^{(r,-1)}_{g,n} = \left\{ (C, p_1, \dots, p_n, L) \;\middle|\; L^{\otimes r} \simeq \omega_{c,\log}^{-1}\left(-\textstyle\sum a_i p_i\right) \right\}$$

$$\downarrow \pi$$

$$\overline{\mathcal{M}}_{g,n}$$

$a_i \ge 0 \qquad a_i \ge 1$

$\omega_{c,\log} = \omega_c(\sum p_i).$

$g=0$: $\pi_* c_{top}(R^1 f_* \mathcal{L}) =: W_{0,n}(a_1, \dots, a_n)$

$g \geq 0$: $R^1 f_* \mathcal{L}$ is a v.b

$|a| := \sum a_i$

Def: $\Theta^r_{g,n}(a_1, \dots, a_n) := (-1)^n r^{\frac{2g-2+n+|a|}{r}} \pi_* c_{top}(R^1 f_* \mathcal{L})$

Remarks. $\deg \Theta^r_{g,n} = \frac{(r+2)(g-1)+n+|a|}{r}$

For $r=2$, was considered by Norbury (Chiodo)

Theorem: $\Theta^r_{g,n}$ is a CohFT of rank $r-1$

$V = \bigoplus_{i=1}^{r-1} \mathbb{C}\langle v_i \rangle$ $\quad 1 \leq a_i \leq r-1 \; (a_i \neq 0)$

- $\Theta^r_{g,n}$ does not have a unit

$$\Theta^r_{g,n+1}(v_{i_1}, v_{i_2}, \ldots, v_{i_n}, v_{r-1})$$

$$= \psi_{n+1} \, p^* \, \Theta^r_{g,n}(v_{i_1}, v_{i_2}, \ldots, v_{i_n})$$

modified unit

- not semi-simple

– semi-simple CohFTs are completely classified (Givental, Teleman)

<u>Remark</u>: A CohFT defines a Dubrovin-Frobenius manifold:

$$F^r(t_1, \ldots, t_{r-1}) = \sum_{k_1+\cdots+k_{r-1}=n} \int \Theta^r_{0,n}\left(v_1^{k_1} \otimes \cdots \otimes v_{r-1}^{k_{r-1}}\right) \prod_i \frac{t_i^{k_i}}{k_i!}$$

$$\uparrow \qquad \overline{\mathcal{M}}_{0,n}$$

Frobenius potential

- DF manifold for $\Theta^{s}_{g,n}$ is not semi-simple

$$F^2(t_1) \equiv 0$$

<u>Problem</u>: We can't use Teleman's result.

<u>Solution</u>: deformation along $a_i = 0$

$$\Theta^{s,\epsilon}_{g,n}(\vec{v}) := \sum_{m\geq 0} \frac{\epsilon^m}{m!} p_{m*} \Theta^{s}_{g,n+m}(\vec{v}, \underbrace{v_0, \dots, v_0}_{m\text{-times}})$$

finite

$\vec{v} \in V^{\otimes n}$

<u>Prop</u>: $\Theta^{s,\epsilon}_{g,n}$ is a CohFT s.t.

- $\Theta^{s,\epsilon}_{g,n} = \Theta^{s}_{g,n} + \epsilon\,(\text{lower degree terms})$

- for $\epsilon \neq 0$, $\Theta^{r,\epsilon}_{g,n}$ is semi-simple.

<u>Remark</u>: $F^{2,\epsilon}(t_1) = \frac{\epsilon^2}{2} \log(1 - t_1)$

<u>Thm</u>: Applying Givental-Teleman

$\Theta^{r,\epsilon}_{g,n}$ = explicit expression in terms of ψ, κ & boundary strata.

<u>for $r=2$</u>

$$\Theta^{2,\epsilon}_{g,n}(1,\dots,1) = (\epsilon^2)^{2g-2+n} \times \exp\left(\sum_{m>0} (\epsilon^{-2})^m s_m \kappa_m\right)$$

explicit $s_m \in \mathbb{C}$

" $\epsilon \to 0$ " gives tautological relations

Cor: $(r=2)$

$$\left[\exp\left(\sum_{m>0} s_m \kappa_m\right)\right]_d = 0 \quad d > 2g-2+n$$

$$\Theta^2_{g,n}(1,\dots,1) = \left[\exp\left(\sum_{m>0} s_m \kappa_m\right)\right]_{2g-2+n}$$

conjectured by Kazarian - Norbury '21

III Integrability

$1 \le a_i \le r-1$

define $F^{\Theta^r}_{g,n}[a_1+rd_1, \dots, a_n+rd_n]$

$$:= \int_{\overline{\mathcal{M}}_{g,n}} \Theta^r_{g,n}(a_1,\dots,a_n)\,\psi_1^{d_1}\cdots\psi_n^{d_n}$$

$$Z^{\Theta^r} = \exp\left(\sum_{g\ge 0} \frac{\hbar^{g-1}}{n!} \sum_{k_i \ge 1} F^{\Theta^r}_{g,n}[k_1,\dots,k_n]\, x_{k_1}\cdots x_{k_n}\right)$$

$n \geq 1$

$\uparrow$

descendant potential of Θ^r $\quad 2g-2+n>0$

Recall Thm (FSZ) $Z_{r\text{-spin}}$ is a τ-fn for the r-KdV hierarchy.

$\Updownarrow$ Adler – van Moerbeke

$Z_{r\text{-spin}}$ is the unique soln to a set of W-algebra constraints

$$\tilde{H}^i_k Z_{r\text{-spin}} = 0 \qquad i=1,\dots,r \quad k \geq -i+1$$

diff op in x_i

Thm: Z^{Θ^r} is the unique soln to a set of W-algebra constraints

$$H^i_k Z^{\textcircled{H}^r} = 0 \qquad i=1,\dots,r \quad k \geq -i+2$$

Proof: uses Eynard – Orantin topological recursion.

$\textcircled{H}^{r,\epsilon}$ s.s $\longleftrightarrow$ TR on a spectral curve.

$\nearrow$ Dunin-Barkowsky – Orantin-Shadrin-Spitz

$\epsilon \to 0$ limit.

$\textcircled{H}^r \longleftrightarrow$ TR on the limit curve

Borot - Bouchard - Chidambaram - Creutzig - Noshchenko

$\updownarrow$

W-algebra constraints

Remark: W-constraints $\Rightarrow$ a recursive formula (on $2g-2+n$) for the descendant integrals $F_{g,n}$

<u>Theorem</u>: for $r = 2, 3$ Z^{Θ} is a τ-function of the r-KdV heirarchy.

$\boxed{\text{for } r=2}$ $Z^{\Theta^2} = Z_{BGW}$ ← KdV tau function associated to $u(t_1, 0, 0, \ldots) = \dfrac{1}{8(1-t_1)^2}$

conj by Norbury