Universal operations on the Tate-Hochschild complex

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Plan

References

Background

1. Kaufmann, Ralph M. “Moduli space actions on the Hochschild cochain complex I: cell models”. Journal of Noncommutative Geometry 1, 3 (2007) 333-384. arXiv:math/0606064

   “Moduli space actions on the Hochschild cochain complex II: correlators”. Journal of Noncommutative Geometry 2, 3 (2008), 283-332. arXiv:math/0606065
2. Kaufmann, Ralph M. ”Open/Closed String Topology and Moduli Space Actions via Open/Closed Hochschild Actions”. SIGMA 6 (2010) 036, 33 pages. arXiv:0910.5929
3. Kaufmann, Ralph M. “A detailed look on actions on Hochschild complexes especially the degree 1 co-product and actions J. of Noncommutative Geometry, 16 (2022), no. 2, 677-716. on loop spaces” . arXiv:1807.10534.

References

Preprint

Berger, Clemens and Kaufmann, Ralph M. “Trees, graphs and aggregates: a categorical perspective on combinatorial surface topology, geometry, and algebra.” arXiv:2201.10537

Ongoing

Kaufmann, Ralph M, Rivera, Manuel and Wang, Zhengfang. ”The Algebraic Structure of the Tate-Hochschild Complex Via A Geometric Moduli Space Action”

Hochschild, (co)homology for an associative unital A

Hochschild chains $CH_n(A, M) = M \otimes A^{\otimes n} \rightsquigarrow HH_*(A, M)$

$$
(1) \quad
\begin{aligned}
d_i(m \otimes a_1 \otimes \cdots \otimes a_n) &= m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\
d_0(m \otimes a_1 \otimes \cdots \otimes a_n) &= (m a_1 \otimes \cdots \otimes a_n) \\
d_n(m \otimes a_1 \otimes \cdots \otimes a_n) &= (a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1})
\end{aligned}
$$

$CH_*(A) = CH_*(A, A) = C_*(A)$

Hochschild cochains $CH^*(A, M) = Hom(A^{\otimes n}, M) \rightsquigarrow HH^*(A, M)$

$$
(2) \quad
\begin{aligned}
d_i(f(a_1 \otimes \cdots \otimes a_n) &= f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\
d_0(f(a_1 \otimes \cdots \otimes a_n) &= a_1 f(a_2 \otimes \cdots \otimes a_n) \\
d_n(m \otimes a_1 \otimes \cdots \otimes a_n) &= f(a_1 \otimes \cdots \otimes a_{n-1}) a_n
\end{aligned}
$$

$CH^*(A) = CH^*(A, A) = C^*(A)$

Gerstenhaber [Ger63]

Operations on $CH^*(A, A)$

- Cup product:
  $f \cup g(a_1 \otimes \cdots \otimes a_{n+m}) = f(a_1 \otimes \cdots \otimes a_n)g(a_{n+1} \otimes \cdots \otimes a_{n+m})$
  Non–commutative if A is not commutative.
- Insertion $f \circ_i g(a_1 \otimes \cdots \otimes a_{n+m-1}) =$
  $f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+m}) \otimes a_{i+m+1} \otimes \cdots \otimes a_{n+m-1})$
- Pre–Lie: $f \circ g = \sum (-1)^{i-1} f \circ_i g$
- Gerstenhaber bracket $\{f \bullet g\} = f \circ g \mp g \circ f$.
  Note this is Lie for degrees shifted by 1. It is also odd Poisson for $\cup$.

Theorem (Gerstenhaber)

$\cup$ is commutative on $HH^*(A)$. The homotopy is given by $\circ$. The operation $\{- \bullet -\}$ is a cohomology operation.

Deligne's conjecture

Operads

- The $\circ_i$ operations define a pseudo–operad structure. Their properties actually yield a definition. $\{\mathcal{O}(n)\}$.
- An algebra over an operad is given by a linear space V together with operations from the operads. $\rho : \mathcal{O}(n) \to Hom(V^{\otimes n}, V)$
- The operations $\cup, \{- \bullet -\}$ are such operations. The operad is called e_2.
- The homology of an operad is an operad. In particular $e_2 = H_*(E_2)$, where E_2 are the little two cubes, which is a topological operad. One can also use little discs D_2. The operad structure is plugging discs into discs.
- What about the $\circ_i$. These are chain level operations, as is $\cup$.

Deligne's conjecture

Deligne's conjecture/question

Is there a operad chain model $C_*(D_2)$ with operations induced by the operad structure on D_2 which acts on $CH^*(A, A)$ inducing the operadic Gerstenhaber structure on $HH^*(A, A)$.

Short answer

Yes.

Deligne's conjecture

Some details

Need to fix signs carefully. (Not today)

The statement has several parts

1. A chain lift that provides the operations.
2. An operadic chain lift that provides operadic operations.
3. Showing that the chain model is a chain model for the little discs.
4. Showing that the chain model has an operadic structure induced from that of the little discs.

Levels

Interrelations/hierarchies/levels

- Spaces/geometry $\leftrightarrow$ Cell models/chains/dg $\leftrightarrow$ (co)homology.
- weak/dg operations $\leftrightarrow$ algebraic structures
- linear operations $\leftrightarrow$ operadic operations $\leftrightarrow$ operadic operations induces from geometric level.
- dg–operations $\leftrightarrow$ dg–operadic operations$\leftrightarrow$ dg–operadic operations induces from geometric level.
- dg on chain $\leftrightarrow$ compactifications and geometry.

Deligne's conjecture: History

Answers, sometimes almost or partial, good sources of structures

- Getzler–Jones '94. (Cell model not operadic)
- Tamarkin-Kontsevich '97-'98
- Voronov '98 (works over $\mathbb{Q}$)
- Kontsevich-Soibelman '99 (A_∞, Minimal operad M.)
- McClure-Smith '99 (Proof has wrong recognition. Surjection operad)
- McClure-Smith '02 (Not really proven to be operadic. New monoidal product $\boxtimes$, new operations $\sqcup, \square_i$)
- Berger-Fresse '02 (uses McClure Smith)
- Kaufmann '03 (New proof. Polysimplicial cell complex. Uses cacti a new version for D_2. Also fixes recognition in McClure-Smith)

. . .

More operations

Generalizations

- Nest, Tsygan, '94 , operations on (cyclic) chains and cochains
- Khalkhali '99, operations on cyclic chains.
- Menichi '03 BV on cohomology. Assume A is Frobenius
- Kaufmann '04 Proof of cyclic Deligne conjecture (all parts) conjectured by Tamarkin-Tsygan in '00.
- Tradler-Zeinalian. '04 combinatorial operations on chain level.
- Tradler-Zeinalian. '06 Algebraic string operations. Includes open/closed linear operations
- Kaufmann '06 Package of *operadic* actions from surfaces, fills gap in McClure–Smith '02
- Kaufmann-Schwell '07 proof of A_∞ using polytopes.
- Kaufmann '09 Open/closed *operadic* operations
- Ward '11. Cyclic A_∞.

Even more

More

1. Wahl-Westerland '11. Same closed part as '06, but A_∞, open part (D-brane, vs. open string), saturated for closed surfaces.
2. Wang '16 operations Tate-Hochschild complex using Cacti.
3. Rivera–Wang '17
4. K.–Rivera–Wang, today
5. Nest-Tsygan. Animation and a whole book forthcoming.

Dualizing and special cases

Dualization

1. The action is most naturally on the reduced tensor algebra $\bar{T}A$.
2. Dualizing yields operations on $CH^*(A, A)$
3. Dualizing more yields operations on CH^* and $CH_*(A, A)$

Specializations (it pays to compute them)

1. Retaining special operations only, one obtains the various Deligne conjectures.
2. Adding a specific $(1, 2)$ operation yields the algebraic version of GH.
3. Under dualization this gives the operations of [RW19]
4. Adding other relations yields higher brackets (bi–, tri–) etc.
5. Dually the bi–bracket is the m_3 operation of [RW19].

Fast track

Categorical version

Naturality of operations of [Kau08] according to [BK22]

Input

1. There is a double category of graphs.
2. There is a natural cyclic *Hom* operad $Cor_{A,P}$ for pair of a vector spaces with a propagator P (this is what is needed to talk about algebras over a cyclic operad).
3. Open TFT correlation functions are given by a functor Y which is induced up by a left Kan extension coming from the inclusion i of trees into graphs pushing forward algebras over the cyclic associative operad. (This follows from an adjuction of functors.)

Categories of graphs

Categories

- $\mathcal{G}r$, the Borisov–Manin category of graphs; monoidal category wrt. $\amalg$.
- $\mathcal{A}gg$ the full subcategory of aggregates of corollas.
- $\mathcal{A}gg^{ctd}$ the wide subcategory of $\mathcal{A}gg$ whose morphisms "have" connected graphs as fibers.
- $\mathcal{A}gg^{tree}$ the wide subcategory of $\mathcal{A}gg^{agg}$ whose morphisms "have" trees as fibers. Let $i : \mathcal{A}gg^{tree} \to \mathcal{A}gg^{ctd}$ be the inclusion.

Here "has" means that every morphisms ϕ has an underlying ghost graph $\mathbb{\Gamma}(\phi)$.

Results [BK22]

Theorem

All of these are Feynman categories.

Consequence

There are pull–backs f^* (restriction) and push–forwards $f_!$ (induction) on functors out of them for morphisms f between these categories.

Theorem

There is a double category $(s,t) : \mathcal{G}r \rightrightarrows \mathcal{A}gg$ with internal identities $i : \mathcal{A}gg \to \mathcal{G}r$ and internal composition ${}_t\circ_s : \mathcal{G}r\ {}_t\times_s \mathcal{G}r \to \mathcal{G}r$ given by graph insertion.
NB: The source map sends Γ to its underlying aggregate (this is called atomization in [BM08] and the target map to $\Gamma/E(\Gamma)$ the full contraction of all edges.

Results [BK22]

Remark

- $\mathcal{A}gg^{tree}$ co–represents cyclic operads.
- $\mathcal{A}gg^{ctd}$ co–represents non–genus graded modular operads.
- $\mathcal{A}gg$ co–represents nc-ng modular operads.

Theorem

The element category $\mathcal{A}gg^{ctd}_{dec\, j_!(\mathcal{T})} = \mathcal{A}gg^{ctd}_g$ co–represents modular operads.
Here $\mathcal{T}(X) = \mathbb{1} = \{*\}$ is the trivial functor and $j : \mathcal{A}gg^{tree} \to \mathcal{A}gg^{ctd}$ is the inclusion.

Application

Consequence

There is an adjuction between Frobenius algebras and OTFTs in the form

$$Nat(\mathcal{O}_{CycAss}, Cor_{V,P}^{\mathcal{A}gg^{tree}}) = Nat(\mathcal{O}_{CycAss}, j^* Cor_{V,P}^{\mathcal{A}gg^{ctd}})$$
$$\leftrightarrow Nat(j_!(\mathcal{O}_{CycAss}), Cor_{V,P}^{\mathcal{A}gg^{ctd}})$$

Frobenius algebras

If P is non–degenerate and one postulates that cyclic structure is given by a non–degenerate form defines the notions of unital algebra. Unital natural transformations are equivalent to the category of *Frob* the category of Frobenius algebras.

String topology correlators from adjunction

In words

The algebraic string topology correctors are directly induces from the fact that unital algberas over the cyclic associative operad are Frobenius algebras.

Theorem [BK22]

The correlators Y of [Kau08] also in the non–commutative case of [Kau18] are given by on $\bar{T}A$ given are given by the natural transformation $s^*(Y) \in Nat[s^*(j_!(CAss)), Cor_{A,P}]$.

Operations

Purely algebraic/combinatorial definition

Frobenius algebras

Notation

If A is a Frobenius algebra then it is isomorphic as an A-A-bimod to its dual.

- $\langle a, b\rangle, \epsilon(a) = \langle a, 1\rangle, \int_n a_1 \cdots a_n = \epsilon(a_1 \cdots a_n)$
- $\Delta = \mu^\dagger$, $\langle a, bc\rangle = \sum \langle a^{(1)}, b\rangle\langle a^{(2)}, c\rangle$.
- $\Delta(1) = C = \sum C^{(1)} \otimes C^{(2)}$ (Casimir), $\mu\Delta(1) = \sum C^{(1)}C^{(2)} = e$ (Euler class).

Main isomorphisms

- $CH^n(A) \simeq A \otimes \check{A}^{\otimes n} \simeq \check{A}^{\otimes n+1} \simeq A^{\otimes n+1}$
- $CH_n(A) = A^{\otimes n+1}$
- So $CH^* \simeq CH_* \simeq \bar{T}A = A \otimes TA$.
- Moreover the canonical pairing is given by $\langle -, -\rangle^{\otimes n}$ and the differentials are adjoint.

Basic algebraic operations

Monoidal structure $\otimes$ and coproduct

For A-A bimodules consider the two monoidal products
$M \otimes N = M \otimes_k N, \Delta : \bar{T}A \to \bar{T}A \otimes \bar{T}A$

$$\Delta(a_0 \otimes \cdots \otimes a_n) = \sum (a_0 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n)$$

The iterates are $\Delta^{[n]} : \bar{T}A \to \bar{T}A^{\otimes n}$

Non–unital monoidal structure $\boxtimes$ and coproduct $\Diamond$

$M \boxtimes N := M \otimes_k A \otimes_k N, \Diamond : TA \to TA \boxtimes TA$

$$\Diamond(a_0 \otimes \cdots \otimes a_n) = \sum (a_0 \otimes \cdots \otimes a_{i-1}) \otimes a_i \otimes (a_{i+1} \otimes \cdots \otimes a_n)$$

The iterates are $\Diamond^{[n]} : TA \to TA^{\boxtimes n}$.

Relations

Inserting units

$id \otimes \eta \otimes id : M \otimes N \simeq M \otimes k \otimes N \to M \otimes A \otimes N = M \boxtimes N.$

We will use:

$\eta_{tot} : A \otimes M_1 \otimes \cdots \otimes M_n \to k \boxtimes M_1 \boxtimes \cdots \boxtimes M_n$ inserts 1 into every slot of A except the first.

Dual using the coaugmentation $\epsilon : A \to k$

$id \otimes \epsilon \otimes id : N \boxtimes M = M \otimes A \otimes N \to M \otimes N$

Algebraic universal operations after [Kau08]

Notation

- Note $\bar{T}A = k \boxtimes TA = A \otimes TA$.
- For $\bar{T}A$ by abuse of notation we write $\lozenge^{[n]}$ for

$$id \otimes \lozenge^{[n]} : k \boxtimes TA \to k \boxtimes TA^{\boxtimes n}$$

  Elements are alternating words in A and $\bar{T}A$:
  $a_0 \otimes w_1 \otimes a_1 \otimes w_2 \cdots a_{n-1} \otimes w_n \in A \otimes \bar{T}A \otimes A \otimes \bar{T}A \otimes \cdots A \otimes \bar{T}A$.
- We will call the factors of $\bar{T}A$ blocks and the factors A interstices and the first factor of A the module variable.

Algebraic universal operations after [Kau08]

Universal formula (non-boundary/ribbon case)

- $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{N}^n$. Let $p = \sum_i p_i$ and think of $\mathbf{p}$, as an ordered n partition of $\bar{p} = \{1, \dots, p\}$.
- $i : \bar{p} \to \bar{p}$ a fixed point free involution, thus $p = 2e$.

$$(3) \quad Y(\mathbf{p}, i) : \bar{T}A^{\otimes n} \to k = (\bigotimes_{l=1}^{c(i)} \int_{m_l(i)}) \otimes \langle -, - \rangle_{\bar{T}A}^{\otimes e} \circ \sigma(i) \circ \bigotimes_{i=1}^{n} \Diamond^{[p_i]}$$

- Where $\sigma(i)$ is a block permutation permuting the factors of A and $\bar{T}A$ that is fixed by i.
- The factors $\langle -, - \rangle_{\bar{T}A}$ only pairs two factors of $\bar{T}A$ paired by i.
- The number $c(i)$ and the $m_l(i)$ are also fixed by i. In fact ,i determines a Ribbon graph structure and $c(i)$ is the number of cycles and $m_l(i)$ are the length of these cycles.

Moduli space operations

Theorem [Kau08]

These are linear operations which define operadic correlation functions for open cells of the moduli space $M_{g,n}$ which form an opread under composition. Using the associated graded of a filtration on the cellular chains, this induces a cyclic dg–operadic action for the cyclic operad structure of [KLP03].

Remark

This is actual a modular operad action and extends to an open/closed action [KP06, Kau10].

Graph interpretation

Graph Γ associated to data

The graph will have n vertices. There will be p_i half edges at each vertex. These correspond to the blocks $\bar{T}A$.
The involution i defined edges via its orbits.

Marked ribbon graph structure from data

The half edges have a linear order at each vertex. The associated cyclic order defines a ribbon graph structure.

Cycles determine the $\int_{m_l}$

Each ribbon graph has cycles given by the involution followed by the successor in the cyclic order.
The integrals are along the angles of the cycles.

Generalization

Associated surface with arcs

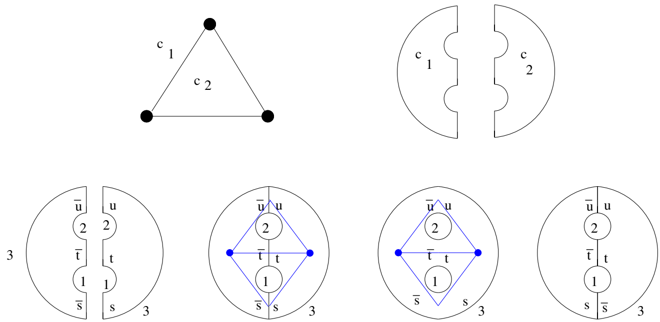
Given such a ribbon graph.

1. Take a 2l–gon for each cycle of length l. Where the sides alternate between angles and edges.
2. Glue together these polygons along the sides in the orbit of the involution.
3. The result is a surface with boundary which is made up out of the angles. There will be n such boundaries.
4. The image of the edges are arcs running from boundary to boundary. These partition each boundary into sectors. There is exactly one sector marked by the module variable in each boundary.

Dual graphs

Proposition [Kau08]

The dual ribbon graph of the surface is also dual to the graph Γ.



Extension to boundary

Kontsevich/Penner compactification

One can extend the correlation functions to the Kontsevich/Penner/combinatorial compactification.
This can be done with polycyclic genus and puncture marked graphs, cf. [BK22].

For Arcs

Let a be a system of arcs and p be a function $a \to \mathbb{N}$.

$$Y(a^p) = \Big(\bigotimes_{S \in Comp(\alpha^p)} Y_A(S) \otimes \bigotimes_{i=1}^{k} \langle -, - \rangle^{n_i - 1}\Big) \circ \sigma \circ \bigotimes_{i=0}^{n} \Diamond^{n_i} \tag{4}$$

Here $Y_A(S)$ is the value of the open TFT associated to the FA A on S, cf. [Kau18].

Prop type operations

Input/output

The correlation functions $Y : \bar{T}A^{\otimes N}$ are of the form:

$$Y(c)_{p_i,q_j} \in Hom(A^{\otimes p_1+1} \otimes \cdots \otimes A^{\otimes p_n+1} \otimes A^{\otimes q_1+1} \otimes \cdots \otimes A^{\otimes q_m+1}, k)$$

where we partitioned N into $n+m$ and c is a cell/decorated graph. Dualizing the $A^{\otimes q_i+1}$ one obtains a PROP action on $\bar{T}A$:

$$\hat{Y}(c)_{p_i,q_j} \in Hom(A^{\otimes p_1+1} \otimes \cdots \otimes A^{\otimes p_n+1}, A^{\otimes q_1+1} \otimes \cdots \otimes A^{\otimes q_m+1})$$

Finally identifying the $A^{\otimes k+1} \simeq CH^k(A,A)$, one obtains operations

$$op_{CH}(c)_{p_i,q_j} \in \mathrm{Hom}(\bigotimes_{i=1}^{n} CH^{p_i}, \bigotimes_{j=1}^{m} CH^{q_j})$$

Chas–Sullivan string topology type operations

Sullivan space [Kau07]

1. These spaces retracts to a CW complex.
2. The cells are cells of the Kontsevich/Penner compactification, where the graph allows for a partition into input and output vertices (arcs only from in to out) and all inputs contain arcs plus a choice of such an input/output designation. Note there may be none, one, two or several such choices.
3. There is a kind of duality switching all in's and out's. This is a time reversal in the physics picture.
   NB: This might violate the input condition.
4. The Sullivan space is the subspace of $(\overline{M}^{K/P}_{g,n})^{i/o}$ in which all inputs have incident arcs.

Actions

Action with in/out distinction

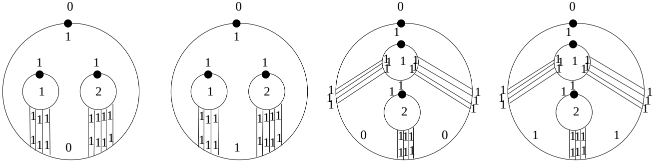
Given an input/output distinction the operation is defined by

$$\hat{Y}(c, i/o) := \hat{Y}(c)(id^{in} \otimes \eta^{out}_{tot})$$

All angles on the output except the module angle contribute $1 \in A$.

Theorem [Kau07, Kau08]

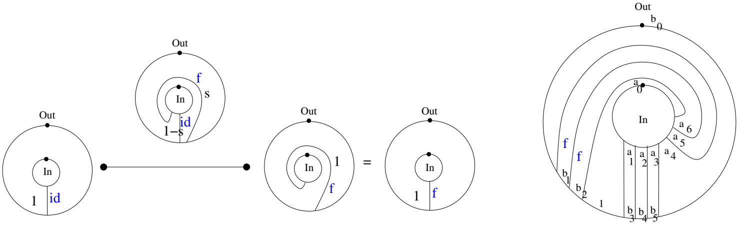
The Sullivan space has a weak prop structure and a chain model that is a dg-Prop.
The operations above give a dg–prop action of this chain level prop on $CH^*(A, A)$ for a Frobenius algebra A which lifts as operadic correlation functions to a weak Frobenius algebra.



Examples of the angle marked partitioned families yielding $\cup$, $\sqcup$, $\circ_i$ and $\square_i$

$$\square_i(f,g)(a_1,\dots,a_{n+m+2}) = f(a_1,\dots,a_{i-1},a_i g(a_{i+1},\dots a_{i_m})a_{i+m+1},a_{i+m+2},\dots,a_{n+m+2}).$$

Animation and BV



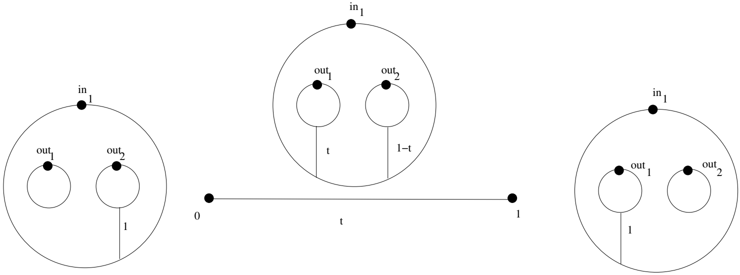
- The i/o version gives Connes' operator B. Without inserting 1 this is the operator N.
- There is the option to insert a function into the factors $\langle -, - \rangle$. This realizes the animation of Nest-Tsygan. The animated operation on $\bar{T}A$ is

$$a_0 \otimes \cdots \otimes a_n \mapsto \pm \sum \epsilon(a_0)\, a_p \otimes f(a_{p+1}) \otimes \cdots \otimes f(a_n) \otimes a_1 \otimes \cdots \otimes a_{p-1}$$

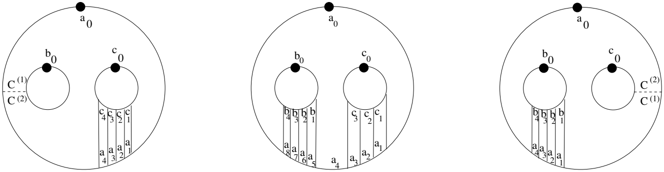
This is a homotopy from $id_{A \otimes TA}$ to $id_A \otimes f^{\otimes}$.

GH coproduct

The 1–dimensional cell for the coproduct and its two boundary points



Coproduct (algebraic Goresky-Hingston)



Middle: The $(8,3)$–summand of the component of the degree 1 operation on CH^* corresponding to $CH^8 \to CH^3 \otimes CH^4$. Cutting at the arcs yields one octagon P_8 and 7 quadrilaterals (P_4s).

Left: the component of the ∂_0 boundary operation $CH^4 \to CH^4 \otimes CH^0$.

Right: The component of the ∂_1 boundary operation $CH^4 \to CH^0 \otimes CH^4$. The extra cut for the annulus is the dotted line and decorated by $C = \Delta(1) = C^{(1)} \otimes C^{(2)}$.

GH coproduct

Theorem [Kau18]

Given a Frobenius algebra A consider $CH := CH^*(A, A)$. The cell for the coproduct given above acts, as a coproduct morphism

$$\Delta_{CH} \in Hom(CH, CH^{\otimes 2})$$

The formulas for its non–zero components $\Delta_{CH}(f) \in \bigoplus_{p+q=n-1} CH^p \otimes CH^q$ are explicitly given by

$$\Delta_{CH}(f)[(a_1 \otimes \cdots \otimes a_p) \otimes (a_{p+1} \otimes \cdots \otimes a_{n-1})]$$
$$= (-1)^p \sum_{C_1, C_2} C_1^{(1)} f(a_1 \otimes \cdots \otimes a_p \otimes C_2^{(1)} C_1^{(2)} \otimes a_{p+1} \otimes \cdots \otimes a_{n-1}) \otimes C_2^{(2)}$$

GH coproduct

Theorem continued

The boundary of this chain operation is given by the operation of the boundary of the cell. It has two components and the operations corresponding to these are

$$\partial_0\Delta_{CH} : CH^n \to CH^n \otimes CH^0 \text{ and } \partial_1\Delta_{CH} : CH^n \to CH^0 \otimes CH^n$$

which are given by the following explicit formulas: using $CH^0(A,A) = Hom(k,A)$, choosing $a_i \in A$ and $\lambda \in k$

$$\partial_0\Delta_{CH}(f)(\lambda \otimes (d_1 \otimes \cdots \otimes d_n)) = \lambda(1 \otimes f(a_1,\ldots,a_n))\Delta(1)^2$$
$$\partial_1\Delta_{CH}(f)((a_1 \otimes \cdots \otimes a_n) \otimes \lambda) = \lambda\Delta(1)^2(f(a_1,\ldots,a_n) \otimes 1)$$

Consequences

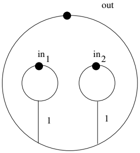
Corollary

If A is graded Gorenstein, then the boundary correlation functions vanish unless $a_0, b_0, c_0 \in A_0$.
Dually, $\partial_{0/1}\Delta_{CH}(f) = 0$ unless $f : A^{\otimes n+1} \to A_0 \simeq k$ is a constant map and $\partial_{0/1}\Delta_{CH}(f)$ has image

$$Im(\partial_0\Delta_{CH}) \subset CH^n(A, A_0) \otimes CH^0(A, A_0) \subset CH^n(A, A) \otimes CH^0(A, A)$$

$$Im(\partial_1\Delta_{CH}) \subset CH^0(A, A_0) \otimes CH^n(A, A_0) \subset CH^0(A, A) \otimes CH^n(A, A)$$

Duality and Moving strings



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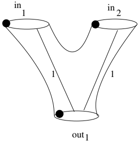

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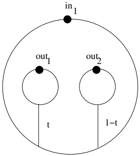

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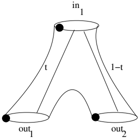
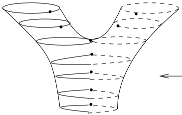
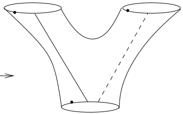


IV



V



VI

Two–colored versions

Actions on Hochschild cochains (co) and Hochshild chains (ho)

Dualizations

Yoga

$$Hom(V,W) \simeq W \otimes V^* \simeq (V^* \otimes W)^* \simeq Hom(V \otimes W^*, k) \simeq Hom(W^*, V^*)$$

Yoga II

$$Hom(\bar{T}A^{n+m}, k) \simeq Hom((CH^*)^{\otimes n}, (CH^*)^{\otimes m}) \simeq CH^{\otimes m} \otimes (CH^{\otimes n})^* \simeq Hom(CH_*^{\otimes m}, CH_*^{\otimes n})$$

Mantra for duality

Hochschild chain inputs are Hochschild cochain outputs
Hochschild chain outputs are Hochschild cochain inputs.
NB: This looks like right and left movers.

Action on Hochschild chains and cochains

ho/co

To implement this, we need to add a new labelling ho/co to the cells/graphs marking the cycles/boundaries. That is each boundary has a double designation in/out and ho/co.

Theorem K-Rivera-Wang

The correlation functions $\hat{Y}$ act on $CH^* \otimes CH_*$ via

$$\hat{Y}(id^{in\,co} \otimes \eta_{tot}^{in\,ho} \otimes \eta_{tot}^{out\,co} \otimes id^{in\,ho})$$

This is a 2–colored dg–prop action extending the operations of [RW19].

GH product

Example

The degree 1, $(2,1)$ product on CH_* given by the formula

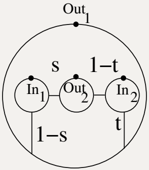
$$
\begin{aligned}
& b_0 \otimes \cdots \otimes b_p \cup c_0 \otimes \cdots \otimes c_{n-p-1} \\
& = \pm \sum b_0 C^{(1)} \otimes c_1 \otimes \cdots \otimes c_{n-p-1} \otimes c_0 C^{(2)} \otimes b_1 \otimes \cdots \otimes b_p
\end{aligned}
$$

Double Poisson bracket

Proposition

In particular the mixed triple products of [RW19] correspond to a map $CH^* \otimes CH^* \to CH^* \otimes CH^*$ which is a double Poisson/Gerstenhaber bracket.

Graph and operation



$\{\{\ ,\ \}\}$

$$(a_0 \otimes \cdots \otimes a_n) \otimes (b_0 \otimes \cdots \otimes b_m) \mapsto$$

$$\sum_{p,q} \pm \langle a_p, b_q \rangle (C^{(2)} b_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} \otimes b_{q+1} \otimes \cdots \otimes b_m)$$

$$\otimes (C^{(1)} a_0 \otimes b_1 \otimes \cdots \otimes b_{q-1} \otimes a_{p+1} \otimes \cdots \otimes a_n)$$

Double Poisson

Comments

1. This is a homotopy between different iterations of $\mu, \mu^{op}, \Delta, \Delta^{op}$
2. This is independent, but related to Iyudu-Kontsevich-Vlassopoulos.
3. This is part of a series of even higher brackets in the sense of van den Bergh, Turaev.

Higher operations

Theorem [K-Rivera-Wang]

Defining μ_{2k+1} via the figure below and setting the $\mu_{2k} = 0$ the corresponding multiplications satisfy the equations

$$\mu_k \circ \mu_l + \mu_k \circ \mu_l = 0 \tag{5}$$

$$\mu_k \circ \mu_k = 0 \tag{6}$$

NB: These equations appeared in [CS99].

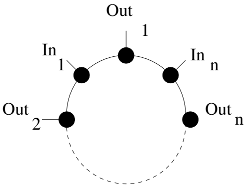


Figure: The arc graph for the n–bracket

Proof using arc graphs

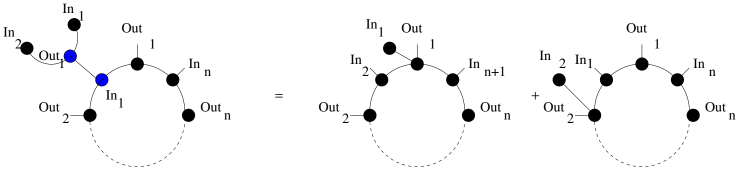


Figure: The Leibniz identity in the first slot.

The composition of and n bracket and an m bracket is given by Figure 3

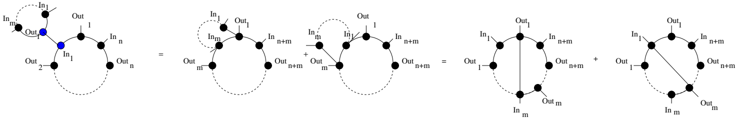


Figure: The composition of the m bracket and the n bracket in the first slot

Trijacobi

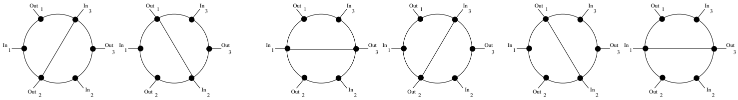


Figure: Terms for the tri-Jacobi identity

Action on TH,chains

Theorem K.-Rivera-Wang

- There is a differential on the two colored version of the Sullivan operad such that the above action becomes a dg action of a dg–PROP on the Tate-Hochschild complex $(\mathcal{D}^*(A,A),\delta) = \mathrm{Cone}(\gamma)$.

$$CH_*(A,A)[1] \oplus CH^*(A,A)$$

$$\gamma(a) = C^{(1)} a C^{(2)} : A = CH_0(A) \to A = CH^0(A)$$

- In particular, this action extends the cyclic A_∞-algebra and the BV-operator defined on $\mathcal{D}^*(A,A)$ in [RW19].
- Note if A is commutative $\gamma(a) = ae$ where $e = \mu\Delta(1)$.

Action on TH, space

Theorem K.-Rivera-Wang

- There is a differential on a two colored version of the Sullivan space making it into a CW complex.
- The new action may be obtained via a cellular chain complex of a new compactification of the Sullivan space, which can be identified with a cover of the Kontsevich–Penner compactification.

Plus construction

Remark

- There is a new theory of hereditary UFCs and plus constructions [KM22].
- Cospans play the role of cobordisms.
- The decoration by arcs can be thought of as a functor from the plus construction.
- This should tie to Master Equations and many other structures.

The End

Thank you!

Extra Page

Extra Page

Extra Page

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