Topological recursion, BPS structures, and quantum curves

Algebra, Geometry & Physics Seminar, MPIM / HU Berlin

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Joint w/ K. Iwaki

1. Introduction
2. Quadratic differentials
3. BPS structures and spectral networks
4. Topological recursion for hypergeometric spectral curves
5. Riemann-Hilbert problem via quantum curves

Two theories

Topological recursion [Eynard-Orantin, Chekhov-Eynard-Orantin]:

- Matrix models, loop equations
- Enumerative geometry (Kontsevich-Witten, Gromov-Witten, Hurwitz, Mirzakhani-Weil-Petersson...)
- Differential equations, WKB analysis

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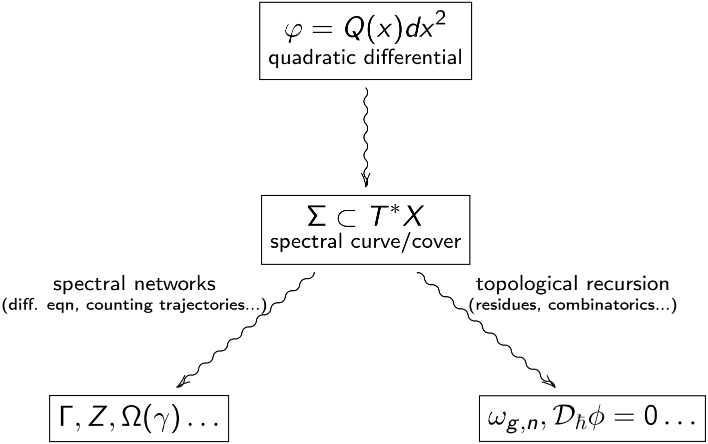
BPS structures [Gaiotto-Moore-Neitzke, Bridgeland, Kontsevich-Soibelman]:

- 4d $\mathcal{N} = 2$ QFT, hyperkähler geometry / Hitchin system
- Stability conditions on triangulated CY_3 categories
- Generalized DT invariants, wall-crossing

Let X cpt Riemann surface, usually $\mathbb{P}^1$

Summary


$\varphi = Q(x)dx^2$
quadratic differential
$\Sigma \subset T^*X$
spectral curve/cover
spectral networks
(diff. eqn, counting trajectories...)
topological recursion
(residues, combinatorics...)
$\Gamma, Z, \Omega(\gamma) \dots$
$\omega_{g,n}, \mathcal{D}_\hbar \phi = 0 \dots$

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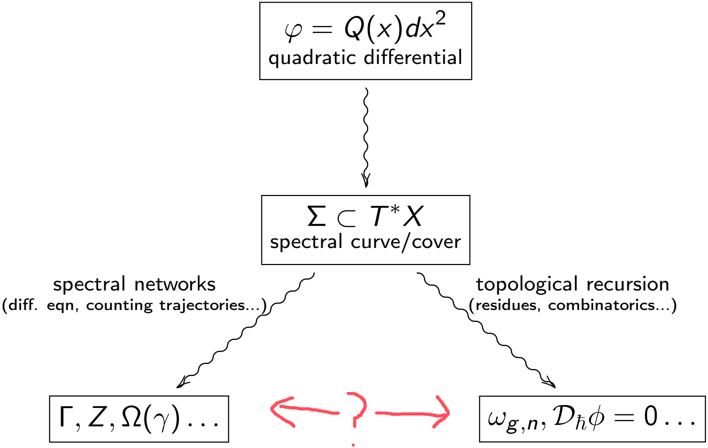
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$\omega_{g,n}, \mathcal{D}_\hbar \phi = 0 \ldots$

A formula

$$F_g(\boldsymbol{m}) = \frac{B_{2g}}{2g(2g-2)} \sum_{\substack{\gamma \in \Gamma \\ Z(\gamma) \in \mathbb{H}}} \Omega(\gamma) \left(\frac{2\pi i}{Z(\gamma)} \right)^{2g-2}.$$

What we do

- Prove for "hypergeometric" example

$$\varphi_{\mathrm{HG}} = \frac{{m_\infty}^2 x^2 - ({m_\infty}^2 + {m_0}^2 - {m_1}^2)x + {m_0}^2}{x^2(x-1)^2} dx^{\otimes 2}$$

  $+$ 8 other examples arising from limits/confluence.
- In particular,
  - Extend GMN construction of BPS structures
  - Compute BPS invariants (existence, location, classification of saddles)
  - Show Borel-resummed Voros symbols solve a natural "BPS Riemann-Hilbert problem"

Spectral curves of hypergeometric type

All are genus 0, degree two curves,

$$y^2 = Q_\bullet(x)$$

Name	$Q_\bullet(x)$	Assumption
Gauss (HG)	$\frac{{m_\infty}^2x^2-({m_\infty}^2+{m_0}^2-{m_1}^2)x+{m_0}^2}{x^2(x-1)^2}$	$m_0, m_1, m_\infty \neq 0$, $m_0 \pm m_1 \pm m_\infty \neq 0$.
Degenerate Gauss (dHG)	$\frac{{m_\infty}^2x+{m_1}^2-{m_\infty}^2}{x(x-1)^2}$	$m_1, m_\infty \neq 0$, $m_1 \pm m_\infty \neq 0$.
Kummer (Kum)	$\frac{x^2+4m_\infty x+4{m_0}^2}{4x^2}$	$m_0 \neq 0$, $m_0 \pm m_\infty \neq 0$.
Legendre (Leg)	$\frac{m_\infty^2}{x^2-1}$	$m_\infty \neq 0$.
Bessel (Bes)	$\frac{x+4m_0^2}{4x^2}$	$m_0 \neq 0$.
Whittaker (Whi)	$\frac{x-4m_\infty}{4x}$	$m_\infty \neq 0$.
Weber (Web)	$\frac{1}{4}x^2-m_\infty$	$m_\infty \neq 0$.
Degenerate Bessel (dBes)	$\frac{1}{x}$	–
Airy (Ai)	x	–

Meromorphic quadratic differentials

Let X compact Riemann surface, ω_X canonical bundle.

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Call $P :=$ set of *poles* of φ, $T :=$ set of *turning points* (zeroes + simple poles).

Spectral cover

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$(\overline{\Sigma}, \pi, \lambda)$ is a branched double cover with a meromorphic one-form smooth away from $\pi^{-1}(P)$.

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such that

- $\Omega(\gamma) = \Omega(-\gamma)$
- For some (any) norm $||\cdot||$ on $\Gamma \otimes \mathbb{R}$, there is > 0 s.t.

$$\Omega \neq 0 \implies |Z(\gamma)| > C \cdot ||\gamma||$$

BPS structures

Terminology:

- finite - only finitely many $\Omega(\gamma) \neq 0$
- uncoupled - $\Omega(\gamma_1), \Omega(\gamma_2) \neq 0 \implies \langle \gamma_1, \gamma_2 \rangle = 0$
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Note: all our BPS structures will be finite, uncoupled and integral

GMN construction

Gaiotto-Moore-Neitzke constructed BPS structures – we consider rank 2 case.

Choose a sufficiently nice meromorphic $\varphi = Q(x)dx^{\otimes 2}$ (say, hypergeometric type).

Let $\widetilde{\Sigma}$ denote Σ with simple poles filled in.

GMN construction

Define:

- $\Gamma := \{\gamma \in H_1(\widetilde{\Sigma}, \mathbb{Z}) \,|\, \iota_* \gamma = -\gamma\}$, ι the sheet-exchange
- $Z(\gamma) := \oint_\gamma \sqrt{\varphi} = \oint_\gamma \sqrt{Q(x)} dx$

(in all our examples, Σ is genus 0, Γ is easy to determine and $Z(\gamma)$ is easily computed as linear combinations of parameters m_i.

Now, to define $\Omega : \Gamma \to \mathbb{Z}$.

Spectral networks

Fix $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$. The *foliation of phase* ϑ, $\mathcal{F}_\vartheta(\varphi)$ is given by

$$\mathrm{Im}\, e^{-i\vartheta} \int^x \sqrt{Q(x)}dx = \mathrm{const}$$

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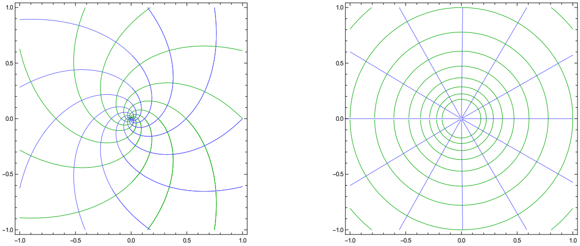
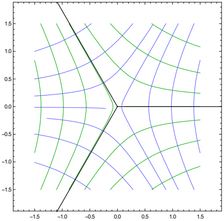
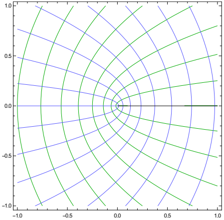


Figure 1: $Q(x) = r/x^2$

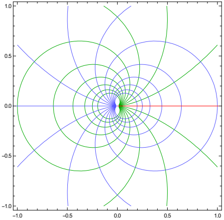
Spectral networks



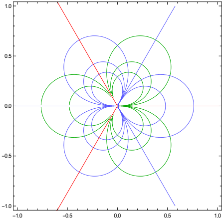
a $Q(x) = x$



b $Q(x) = 1/x$



c $Q(x) = 1/x^3$



d $Q(x) = 1/x^5$

Spectral networks

Fact: Trajectory pentachotomy:

1. (i) saddle
2. (ii) separating
3. (iii) generic
4. (iv) closed
5. (v) recurrent

Spectral networks

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5. ~~recurrent~~ for us, by Jenkins

Spectral networks

Definition. The *spectral network (Stokes graph)* $\mathcal{W}_\vartheta(\varphi)$ *of phase* ϑ is the collection of all separating and saddle trajectories in $\mathcal{F}_\vartheta$.

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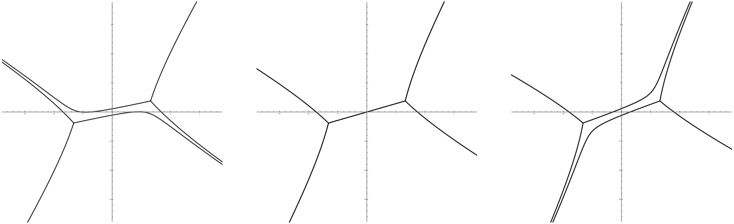
Example: $Q_{\mathrm{Web}}(x) = \frac{1}{4}x^2 - m_\infty^2$



a $\vartheta < \vartheta_c$ b $\vartheta = \vartheta_c$ c $\vartheta > \vartheta_c$

Spectral networks

Fact: Every saddle trajectory or closed trajectory has a *canonical lift* $\gamma \in \Gamma$ (up to sign)

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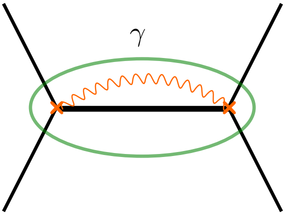
For example, if both endpoints simple zeroes:



Spectral networks

Definition. The *BPS invariants* $\Omega(\gamma)$ of φ are defined below for $\gamma \in \Gamma$ appearing as canonical lifts of saddles in $\mathcal{W}_\vartheta(\varphi)$

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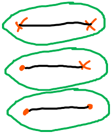
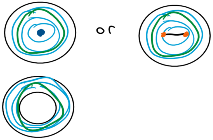
$$\Omega(\gamma) = \begin{cases} +1 & \text{type I} \\ +2 & \text{type II} \\ +4 & \text{type III} \\ -1 & \text{deg. ring domain} \\ -2 & \text{nondeg. ring domain} \end{cases}$$





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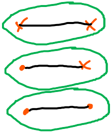

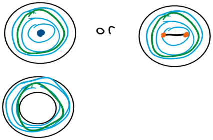

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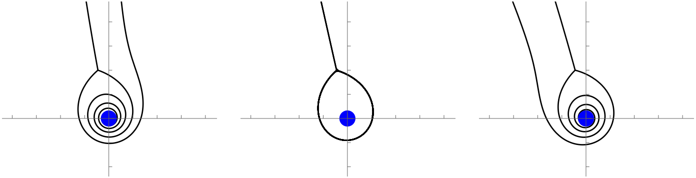
Note, in the original (type I and nondeg r.d.) case, these are Euler characteristics of certain moduli spaces of quiver representations.

BPS spectrum

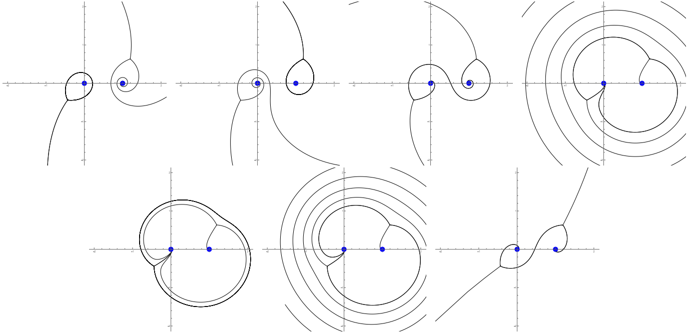
Simple example (Bessel): $Q_{\mathrm{Bes}}(x) = \dfrac{x + 4m_0^2}{4x^2}$



$$\Omega(\gamma_{\mathrm{BPS}}) = -1$$

BPS structure

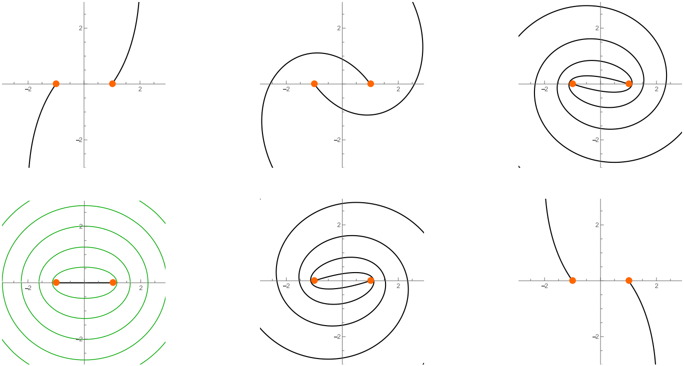
Main example: $Q_{\mathrm{HG}}(x) = \dfrac{m_\infty^2 x^2 - (m_\infty^2 - m_1^2 + m_0^2)x + m_0^2}{x^2(x-1)^2}$



$$\Omega(\gamma_{\mathrm{BPS}}) = \pm 1$$

BPS structure

Weird example (Legendre): $Q_{\mathrm{Bes}}(x) = \dfrac{m_\infty}{(x-1)(x+1)}$



$$\Omega(\gamma_{\mathrm{BPS}}) = 4,\ \Omega(\gamma_{\mathrm{BPS}}) = -1$$

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- $x, y : \mathcal{C} \to \mathbb{P}^1$ nonconstant meromorphic functions, dx and dy do not vanish simultaneously
- Bidifferential: meromorphic section

$$B(z_1, z_2) \in p_1^*(T^*\mathcal{C}) \otimes p_2^*(T^*\mathcal{C})$$

with some properties ($p_i : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ projection).
For us, $\mathcal{C} = \mathbb{P}^1$ so there is a canonical B,

$$B(z_1, z_2) := \frac{dz_1 dz_2}{(z_1 - z_2)^2}$$

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Zeroes or poles order ≥ 3 of dx: *ramification points*, denoted $r \in \mathcal{R}$

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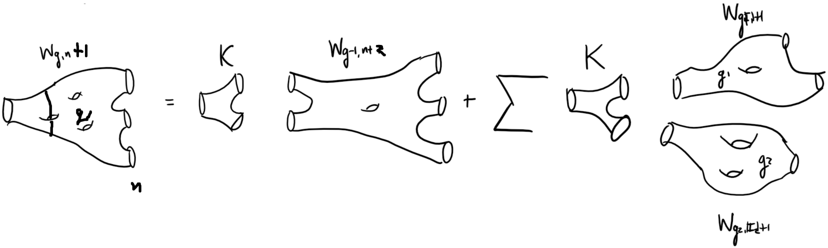
Note: Given q.d. φ on $X = \mathbb{P}^1$ with corresponding spectral cover (Σ, π, λ) of genus 0, we can obtain a TR spectral curve by taking

$$\mathcal{C} := \overline{\Sigma},\ x := \pi,\ y := \frac{\lambda}{dx},\ B := \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$$

Topological recursion

Start with $\omega_{0,1}(z_0) := y(z_0)dx(z_0)$, $\omega_{0,2}(z_0, z_1) = B(z_0, z_1)$.



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$$\omega_{g,n+1}(z_0, z_1, \cdots, z_n) := \sum_{r\in\mathcal{R}} \operatorname*{Res}_{z=r} K_r(z_0, z) \Bigg[\omega_{g-1,n+2}(z, \overline{z}, z_1, \cdots, z_n) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{1,2,\cdots,n\}}}^{\prime} \omega_{g_1,|I_1|+1}(z, z_{I_1}) \omega_{g_2,|I_2|+1}(\overline{z}, z_{I_2}) \Bigg]$$

for $2g + n \geq 2$,

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for $2g + n \geq 2$,where

$$K_r(z_0, z_1) = \frac{1}{(y - \bar{y})dx} \int_{\zeta=\bar{z}}^{\zeta=z} B(z_0, \zeta)$$

$\bar{z}$ is "local conjugation" near ramification point r.

Topological recursion

Definition. Let $\Phi(z)$ be any primitive of $y(z)dx(z)$. The *gth free energy* $(g \geq 2)$ is

$$F_g = \frac{1}{2-2g} \sum_{r \in \mathcal{R}} \operatorname*{Res}_{z=r} [\Phi(z)\omega_{g,1}(z)]$$

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[Iwaki-Koike-Takei] showed (for example):

$$F_g^{\mathrm{HG}}(\boldsymbol{m}) = \frac{B_{2g}}{2g(2g-2)} \Bigg(\frac{1}{(m_0+m_1+m_\infty)^{2g-2}} + \frac{1}{(m_0+m_1-m_\infty)^{2g-2}} + \frac{1}{(m_0-m_1+m_\infty)^{2g-2}} + \frac{1}{(m_0-m_1-m_\infty)^{2g-2}} - \frac{1}{(2m_0)^{2g-2}} - \frac{1}{(2m_1)^{2g-2}} - \frac{1}{(2m_\infty)^{2g-2}} \Bigg).$$

+ formulas for the other 8 examples.

Result

Theorem. [Iwaki-K] *For the spectral curves of hypergeometric type, $\boldsymbol{m}$ generic, we have*

$$F_g(\boldsymbol{m}) = \frac{B_{2g}}{2g(2g-2)} \sum_{\substack{\gamma\in\Gamma \\ Z(\gamma)\in\mathbb{H}}} \Omega(\gamma)\left(\frac{2\pi i}{Z(\gamma)}\right)^{2g-2}, \quad g \geq 2$$

where $\mathbb{H}$ is any generic half-plane.

Conjecture. [Iwaki-K] This holds in higher rank too, under the assumption the BPS structure is *uncoupled* (some evidence presented, more in progress).

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- TR can tell us something about spectral networks. Practically speaking, TR can help us compute information about BPS counts without ever having to draw a spectral network!
- On the other hand, spectral networks can teach us about the structure of TR. Can we predict new examples?
- Our formula is one simple example of this. What can we learn in more complicated or exotic cases?

More results

We can upgrade these results to the analytic setting.

- Solve natural "BPS Riemann-Hilbert problem" associated to the BPS structure using Voros symbols of quantum curves
- Natural TR intepretation of Bridgeland's *BPS τ-function* which generates the solution.

BPS Riemann-Hilbert problem [Bridgeland, GMN]

- Fix (Γ, Z, Ω). Seek functions X_γ (one for each γ) in the $\hbar$-plane, prescribed jumping across BPS rays.
- Define *twisted torus*

$$\mathbb{T}_- := \left\{ g : \Gamma \to \mathbb{C}^* \,|\, g(\gamma_1 + \gamma_2) = (-1)^{\langle \gamma_1, \gamma_2 \rangle} g(\gamma_1) g(\gamma_2) \right\}$$

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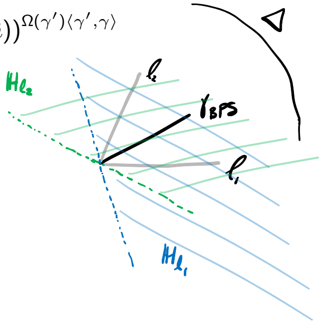
1. Δ sector, ∂ rays ℓ_1, ℓ_2 (not BPS). For $\gamma \in \Gamma$, $\hbar \in \mathbb{H}_{\ell_1} \cap \mathbb{H}_{\ell_2}$

$$X_{\ell_2,\gamma} = X_{\ell_1,\gamma}(\hbar) \prod_{\substack{\gamma' \in \Gamma \\ Z(\gamma') \in \Delta}} (1 - X_{\ell_1,\gamma'}(\hbar))^{\Omega(\gamma')\langle \gamma', \gamma \rangle}$$



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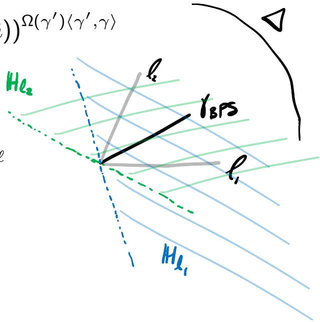

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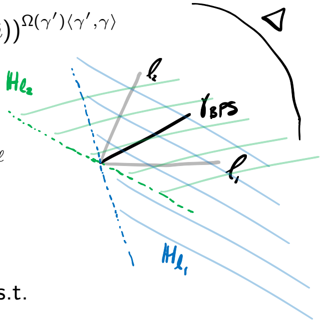

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3. For $\gamma \in \Gamma$, whenever ℓ is not BPS, there exists k s.t.

$$|\hbar|^{-k} < |X_{\ell,\gamma}(\hbar)| < |\hbar|^k$$

for $|\hbar| \gg 0$ in $\mathbb{H}_\ell$.

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$$S(x) := \sum_{k=-1}^{\infty} \hbar^k \sum_{\substack{2g-2+n=k \\ g\geq 0,\, n\geq 1}} \frac{1}{n!} \int_{\zeta_1 \in D(z;\boldsymbol{\nu})} \cdots \int_{\zeta_n \in D(z;\boldsymbol{\nu})} \left(\omega_{g,n}(\zeta_1, \ldots \zeta_n) - \delta_{g,0}\delta_{n,2} \frac{dx(\zeta_1)dx(\zeta_2)}{(x(\zeta_1) - x(\zeta_n))^2} \right)$$

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A quantum curve is a (formally depending on $\hbar$) differential operator $\mathcal{D}_\hbar(\boldsymbol{\nu})$ (geometrically, an sl_2-oper) such that

$$\mathcal{D}_\hbar(\boldsymbol{\nu})\varphi = 0,$$

and classically limits to the spectral curve.

Quantum curves

Example (Weber):

$$y^2 - (\frac{x^2}{4} - m_\infty^2) = 0, \quad \boldsymbol{\nu} = (\nu_{\infty_+}, \nu_{\infty_-}), \quad \nu_{\infty_+} + \nu_{\infty_-} = 1$$

Quantum curve: $$\left(\hbar^2 \frac{d^2}{dx^2} - \left(\frac{x^2}{4} - m_\infty^2\right) - \hbar\left(\frac{\nu_{\infty_+} - \nu_{\infty_-}}{2}\right)\right)\varphi = 0$$

Voros coefficients

Let $dS^{\mathrm{odd}} := \dfrac{dS(x) - \iota^* dS(x)}{2}$.

For any $\gamma \in H_1(\overline{\Sigma}, \mathbb{Z})$, $\beta \in H_1(\overline{\Sigma}, P \setminus T, \mathbb{Z})$, the *Voros coefficients* are

$$V_\gamma := \oint_\gamma dS^{\mathrm{odd}}(x)dx, \qquad V_\beta := \int_\beta dS^{\mathrm{odd}}_{\geq 1}(x)dx$$

where ≥ 1 denotes truncation of the $\hbar^{-1}$ and $\hbar^0$ terms.

[Iwaki-Koike-Takei] computed V_γ and V_β explicitly, which can be written in terms of the BPS spectrum, in a similar formula as F_g.

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If all goes well, the Borel sum is a piecewise analytic function in $\hbar$ that jumps certain rays, and asymptotic to the original f. We are able to compute the Borel sums of V_γ, V_β, more or less by hand (see results of Aoki, Takei, Koike, Kamimoto and others).

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By Poincare-Lefschetz duality, we may identify elements in Γ^* with elements of $H_1(\overline{\Sigma}, P \setminus T, \mathbb{Z})$ using the intersection pairing.

Result

Theorem. [Iwaki-K] *Let $Q(x)$ be of hypergeometric type, and V_γ, V_β denote the Voros coefficients of $\mathcal{D}_\hbar(\boldsymbol{\nu})$. Then*

$$X_{\ell,\gamma}(\hbar) := \sigma_\gamma \cdot \mathcal{S}_\ell e^{-V_\gamma(\hbar)}, \quad X_{\ell,\beta}(\hbar) := \sigma_\beta \cdot \mathcal{S}_\ell e^{V_\beta(\hbar)}$$

where σ is a sign, solves the BPS Riemann-Hilbert problem for the corresponding almost-doubled BPS structure, with constant term $\xi = \xi(\boldsymbol{\nu})$ given explicitly.

Further

- Higher rank: conjecture + a few experiments (ongoing)
- Relation to Joyce structures / Joyce function
- Relation to Nekrasov partition function
- Coupled case?
- β-deformed case (ongoing w/ K. Osuga)
- q-deformed case (5d BPS states)
- TBA equations
- ...