Bimodules over categories enriched over closed monoidal categories

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Filtered abelian groups

We consider a partially ordered commutative monoid $\mathbb{L}$ be with the operation $+$ and neutral element 0. Of course, we assume that $a \leqslant b$, $c \leqslant d$ imply $a + c \leqslant b + d$. We assume that $\mathbb{L}$ satisfies the following conditions:

1. for all $a, b \in \mathbb{L}$ there is $c \in \mathbb{L}$ such that $a \leqslant c$, $b \leqslant c$ (that is, $(\mathbb{L}, \leqslant)$ is directed);
2. for all $a, b \in \mathbb{L}$ there is $c \in \mathbb{L}$ such that $c \leqslant a$, $c \leqslant b$ (that is, $\mathbb{L}^{\mathrm{op}}$ is directed);
3. for all $a \in \mathbb{L}$ there is $c \in \mathbb{L}$ such that $a + c \geqslant 0$.

If $\mathbb{L}$ is a directed group (satisfies (i)), then $\mathbb{L}$ satisfies (ii) and (iii) as well for obvious reasons.

An $\mathbb{L}$-filtered abelian group is an abelian group M together with, for every $l \in \mathbb{L}$, a subgroup $\mathcal{F}^l M$ such that $a \leqslant b \in \mathbb{L}$ implies that $\mathcal{F}^a M \supset \mathcal{F}^b M$ and $\cup_{l \in \mathbb{L}} \mathcal{F}^l M = M$.

The category $\mathbf{Ab}_{\mathbb{L}}$ of $\mathbb{L}$-filtered abelian groups is a symmetric monoidal category. The tensor product of a family $M_1, \ldots,$ M_n, $n \geqslant 1$, is the tensor product of abelian groups M_i, equipped with the filtration

$$\mathcal{F}^l(\otimes_{i=1}^n M_i) = \mathrm{Im}(\oplus_{l_1+\cdots+l_n=l} \otimes_{i=1}^n \mathcal{F}^{l_i} M_i \to \otimes_{i=1}^n M_i).$$

Definition

A category $\mathcal{V}$ is *monoidal* when it is equipped with a functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, tensor product, associative in the sense:

– there is a natural isomorphism, associator,

$\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$,

and unital:

– there are an object $\mathbb{1}$, the unit object, and natural isomorphisms $\mathbb{1} \otimes A \to A$, left unitor,

and $A \otimes \mathbb{1} \to A$, right unitor,

such that Mac Lane's pentagon for α commutes and an equation involving the above data holds.

A monoidal category $\mathcal{V}$ is *symmetric* when it is equipped with a natural isomorphism $c : X \otimes Y \to Y \otimes X$, satisfying appropriate equations.

In practice it is simpler to say that $\mathbb{L}$-filtered abelian groups constitute a symmetric multicategory $\widehat{\mathbf{Ab}_{\mathbb{L}}}$. It is formed by multilinear maps preserving the filtration:

$$\widehat{\mathbf{Ab}_{\mathbb{L}}}(M_1, \dots, M_n; N) = \{\text{multilinear maps } f : M_1 \times \cdots \times M_n \to N \\ \mid (\mathcal{F}^{l_1} M_1 \times \cdots \times \mathcal{F}^{l_n} M_n) f \subset \mathcal{F}^{l_1 + \cdots + l_n} N\}, \quad n \geqslant 1.$$

Notice that $\widehat{\mathbf{Ab}_{\mathbb{L}}}(M_1, \dots, M_n; N)$ is naturally isomorphic to $\mathbf{Ab}_{\mathbb{L}}(M_1 \otimes \cdots \otimes M_n, N)$ for $n \geqslant 1$.

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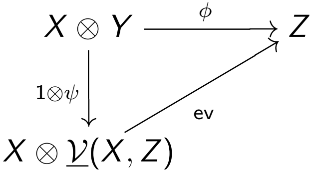
Notice that $\widehat{\mathbf{Ab}_{\mathbb{L}}}(M_1, \dots, M_n; N)$ is naturally isomorphic to $\mathbf{Ab}_{\mathbb{L}}(M_1 \otimes \cdots \otimes M_n, N)$ for $n \geqslant 1$.
We recall the definition of a closed monoidal category, leaving aside that of a closed multicategory.

Definition

A monoidal category $\mathcal{V}$ is *closed* if for each pair X, Z of objects of $\mathcal{V}$ there is an object $\underline{\mathcal{V}}(X, Z)$ of $\mathcal{V}$ and an evaluation element

$$\mathrm{ev}^{\mathcal{V}}_{X,Z} \in \mathcal{V}\big(X \otimes \underline{\mathcal{V}}(X, Z), Z\big)$$

such that for an arbitrary morphism $\phi : X \otimes Y \to Z \in \mathcal{V}$ there exists a unique morphism $\psi : Y \to \underline{\mathcal{V}}(X, Z)$ such that



commutes.

The unit object is $\mathbb{Z}$, equipped with the filtration

$$\mathcal{F}^l\mathbb{Z} = \begin{cases} \mathbb{Z}, & \text{if } l \leqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

The monoidal category $\mathbf{Ab}_{\mathbb{L}}$ is symmetric. Furthermore, it is closed. Associate with $M, N \in \mathbf{Ab}_{\mathbb{L}}$ an $\mathbb{L}$-filtered abelian group $\underline{\mathbf{Ab}_{\mathbb{L}}}(M, N) \subset \mathbf{Ab}(M, N)$ with

$$\mathcal{F}^l\underline{\mathbf{Ab}_{\mathbb{L}}}(M, N) = \{f \in \underline{\mathbf{Ab}}(M, N) \mid \forall \lambda \in \mathbb{L} \quad (\mathcal{F}^\lambda M)f \subset \mathcal{F}^{\lambda+l}N\}.$$

The internal hom is obtained as $\underline{\mathbf{Ab}_{\mathbb{L}}}(M, N) = \cup_{l\in\mathbb{L}}\mathcal{F}^l\underline{\mathbf{Ab}_{\mathbb{L}}}(M, N)$. The evaluation

$$\mathrm{ev} : M \otimes \underline{\mathbf{Ab}_{\mathbb{L}}}(M, N) \to N, \quad m \otimes f \mapsto (m)f,$$

is a morphism of $\mathbf{Ab}_{\mathbb{L}}$, and it turns $\mathbf{Ab}_{\mathbb{L}}$ into a closed symmetric monoidal category.

The multicategory $\widehat{\mathbf{Ab}_{\mathbb{L}}}$ is closed. Namely, for each family of $\mathbb{L}$-filtered abelian groups M_1, …, M_n, N there is internal hom object $\underline{\widehat{\mathbf{Ab}_{\mathbb{L}}}}(M_1, \dots, M_n; N) \subset \underline{\widehat{\mathbf{Ab}}}(M_1, \dots, M_n; N) \cong \operatorname{Multi}(M_1 \times \dots \times M_n, N)$. It is filtered by subgroups

$$\mathcal{F}^l \underline{\widehat{\mathbf{Ab}_{\mathbb{L}}}}(M_1, \dots, M_n; N) = \{f \in \operatorname{Multi}(M_1 \times \dots \times M_n, N) \mid \\ \mid \forall (\lambda_k) \in \mathbb{L}^n \quad (\mathcal{F}^{\lambda_1} M_1 \times \dots \times \mathcal{F}^{\lambda_n} M_n) f \subset \mathcal{F}^{\lambda_1 + \dots + \lambda_n + l} N\}.$$

The internal hom is defined as
$\underline{\widehat{\mathbf{Ab}_{\mathbb{L}}}}(M_1, \dots, M_n; N) = \cup_{l \in \mathbb{L}} \mathcal{F}^l \underline{\widehat{\mathbf{Ab}_{\mathbb{L}}}}(M_1, \dots, M_n; N)$.
The evaluation

$$\operatorname{ev}: M_1 \times \dots \times M_n \times \underline{\widehat{\mathbf{Ab}_{\mathbb{L}}}}(M_1, \dots, M_n; N) \to N, \\ (m_1, \dots, m_n, f) \mapsto (m_1, \dots, m_n) f,$$

is a morphism of $\widehat{\mathbf{Ab}_{\mathbb{L}}}$, and it turns $\widehat{\mathbf{Ab}_{\mathbb{L}}}$ into a closed multicategory.

The category $\mathcal{V}$ is assumed to be additive, closed symmetric monoidal, complete (has all limits) and cocomplete (has all colimits). Furthermore, it is set-like, which means that there is an isomorphism of internal hom with coproduct as a first argument with the product of internal homs. It is obvious that coend in the first argument of the hom-functor can be moved outside as an end. It is not clear to me that the same can be done for enriched coend, internal hom and enriched end. We work with set-like categories $\mathcal{V}$ which have this property.

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Prior to defining A_∞-categories we study the $\mathcal{V}$-bicategory of bimodules over $\mathcal{V}$-categories. Actually, we exhibit isomorphisms which are a part of the structure of a $\mathcal{V}$-bicategory but do not prove equations between them, like the pentagon relation *etc*.

Properties of products and coproducts

Proposition

For all $\mathcal{V}$ (complete and cocomplete locally small symmetric monoidal closed) there is a natural isomorphism

$$\underline{\mathcal{V}}(X, \prod_{i \in I} Y_i) \cong \prod_{i \in I} \underline{\mathcal{V}}(X, Y_i).$$

Part of proof.

Define $\chi_i = X \otimes \mathrm{pr}_i : X \otimes \prod_{j \in I} \underline{\mathcal{V}}(X, Y_j) \to X \otimes \underline{\mathcal{V}}(X, Y_i)$. Combined together they give
$\chi = (\chi_i) : X \otimes \prod_{j \in I} \underline{\mathcal{V}}(X, Y_j) \to \prod_{i \in I} [X \otimes \underline{\mathcal{V}}(X, Y_i)]$.
A natural morphism $\phi : \prod_{j \in I} \underline{\mathcal{V}}(X, Y_j) \to \underline{\mathcal{V}}(X, \prod_{i \in I} Y_i)$ is obtained from

$$\begin{array}{ccc} X \otimes \prod\limits_{j \in I} \underline{\mathcal{V}}(X, Y_j) & \xrightarrow{\chi} & \prod\limits_{i \in I} [X \otimes \underline{\mathcal{V}}(X, Y_i)] \\ {\scriptstyle X \otimes \phi} \downarrow & & \downarrow {\scriptstyle \prod \mathrm{ev}} \\ X \otimes \underline{\mathcal{V}}(X, \prod\limits_{i \in I} Y_i) & \xrightarrow{\quad \mathrm{ev} \quad} & \prod\limits_{i \in I} Y_i \end{array}$$

Define $\eta_j = \underline{\mathcal{V}}(X, \mathrm{pr}_j) : \underline{\mathcal{V}}(X, \prod_{i \in I} Y_i) \to \underline{\mathcal{V}}(X, Y_j)$. Combined together they give a natural morphism
$\eta = (\eta_j)_{j \in I} : \underline{\mathcal{V}}(X, \prod_{i \in I} Y_i) \to \prod_{j \in I} \underline{\mathcal{V}}(X, Y_j)$. □

Proposition

For all $\mathcal{V}$ *there are natural morphisms*
$\psi : \prod_{i \in I} \underline{\mathcal{V}}(X_i, Y) \to \underline{\mathcal{V}}(\coprod_{i \in I} X_i, Y)$ *and*
$\xi : \underline{\mathcal{V}}(\coprod_{i \in I} X_i, Y) \to \prod_{i \in I} \underline{\mathcal{V}}(X_i, Y)$ *such that* $\psi \centerdot \xi = 1$.

Part of proof.

A natural morphism $\psi : \prod_{i \in I} \underline{\mathcal{V}}(X_i, Y) \to \underline{\mathcal{V}}(\coprod_{i \in I} X_i, Y)$ is obtained from

$$
\begin{array}{ccc}
\left(\coprod_{i \in I} X_i\right) \otimes \prod_{j \in I} \underline{\mathcal{V}}(X_j, Y) & \xrightarrow{\cong} & \coprod_{i \in I} \left[X_i \otimes \prod_{j \in I} \underline{\mathcal{V}}(X_j, Y)\right] \\
{\scriptstyle 1 \otimes \psi} \downarrow & & \downarrow {\scriptstyle \coprod [1 \otimes \mathrm{pr}_i]} \\
\left(\coprod_{i \in I} X_i\right) \otimes \underline{\mathcal{V}}\left(\coprod_{i \in I} X_i, Y\right) & & \coprod_{i \in I} [X_i \otimes \underline{\mathcal{V}}(X_i, Y)] \\
{\scriptstyle \mathrm{ev}} \downarrow & & \downarrow {\scriptstyle \coprod \mathrm{ev}} \\
Y & \xleftarrow{(1)_{i \in I}} & \coprod_{i \in I} Y
\end{array}
$$

Natural morphisms $\xi_j = \underline{\mathcal{V}}(\mathrm{in}_j, Y) : \underline{\mathcal{V}}(\coprod_{i \in I} X_i, Y) \to \underline{\mathcal{V}}(X_j, Y)$ combine to $\xi : \underline{\mathcal{V}}(\coprod_{i \in I} X_i, Y) \to \prod_{j \in I} \underline{\mathcal{V}}(X_j, Y)$. □

Lemma (obvious)

For any $X \in \operatorname{Ob}\mathcal{V}$ *the functors* $\underline{\mathcal{V}}(X, \text{-}) : \mathcal{V} \to \mathcal{V}$, $\underline{\mathcal{V}}(\text{-}, X) : \mathcal{V}^{\mathrm{op}} \to \mathcal{V}$ *are left exact, that is, for any diagram* $I \to \mathcal{V}$, $i \mapsto Y_i$, *there are natural isomorphisms*

$$\lim_{i \in I} \underline{\mathcal{V}}(X, Y_i) \cong \underline{\mathcal{V}}(X, \lim_{i \in I} Y_i),$$

and

$$\lim_{i \in I^{\mathrm{op}}} \underline{\mathcal{V}}(Y_i, X) = \underline{\mathcal{V}}(\operatorname{colim}_{i \in I} Y_i, X).$$

Definition

A small $\mathcal{V}$-category $\mathcal{C}$ consists of

- a set $\mathrm{Ob}\,\mathcal{C}$ of objects of $\mathcal{C}$,
- an object $\mathcal{C}(X, Y)$ of $\mathcal{V}$ for every pair of objects X, Y in $\mathcal{C}$,
- an arrow $id_X : \mathbb{1} \to \mathcal{C}(X, X)$ in $\mathcal{V}$ called an identity for every object X in $\mathcal{C}$,
- an arrow $\centerdot : \mathcal{C}(X, Y) \otimes \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$ in $\mathcal{V}$, a composition for each triple of objects X, Y, Z in $\mathcal{C}$,

such that the composition is associative and unital.
$\mathcal{V}$-functors are supposed to preserve the identities and the composition.

Example

When $\mathcal{V}$ is a closed monoidal category, the commutative diagrams

$$\begin{CD}
X \otimes \underline{\mathcal{V}}(X,Y) \otimes \underline{\mathcal{V}}(Y,Z) @>{\mathrm{ev} \otimes 1}>> Y \otimes \underline{\mathcal{V}}(Y,Z) \\
@V{1 \otimes .}VV @VV{\mathrm{ev}}V \\
X \otimes \underline{\mathcal{V}}(X,Z) @>{\mathrm{ev}}>> Z
\end{CD}$$

$$\begin{CD}
X \otimes \mathbb{1} @>{\cong}>> X \\
@V{1 \otimes \mathrm{id}_X}VV @| \\
X \otimes \underline{\mathcal{V}}(X,X) @>{\mathrm{ev}}>> X
\end{CD}$$

make $\underline{\mathcal{V}} = \big(\mathrm{Ob}\,\mathcal{V}, (\underline{\mathcal{V}}(X,Y))_{X,Y \in \mathrm{Ob}\,\mathcal{V}}\big)$ into a $\mathcal{V}$-category.

Example

When $\mathcal{V}$ is a closed monoidal category, the commutative diagrams

$$\begin{array}{ccc}
X \otimes \underline{\mathcal{V}}(X,Y) \otimes \underline{\mathcal{V}}(Y,Z) & \xrightarrow{\mathsf{ev}\otimes 1} & Y \otimes \underline{\mathcal{V}}(Y,Z) \\
{\scriptstyle 1\otimes\cdot}\downarrow & & \downarrow{\scriptstyle \mathsf{ev}} \\
X \otimes \underline{\mathcal{V}}(X,Z) & \xrightarrow{\qquad \mathsf{ev} \qquad} & Z
\end{array}$$

$$\begin{array}{ccc}
X \otimes \mathbb{1} & \xrightarrow{\quad \cong \quad} & X \\
{\scriptstyle 1\otimes \mathrm{id}_X}\downarrow & & \Vert \\
X \otimes \underline{\mathcal{V}}(X,X) & \xrightarrow{\mathsf{ev}} & X
\end{array}$$

make $\underline{\mathcal{V}} = \big(\mathrm{Ob}\,\mathcal{V}, (\underline{\mathcal{V}}(X,Y))_{X,Y\in\mathrm{Ob}\,\mathcal{V}}\big)$ into a $\mathcal{V}$-category.
External tensor product $\mathcal{A} \boxtimes \mathcal{B}$ of $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ has by definition $\mathrm{Ob}\,\mathcal{A} \boxtimes \mathcal{B} = \mathrm{Ob}\,\mathcal{A} \times \mathrm{Ob}\,\mathcal{B}$ and $(\mathcal{A} \boxtimes \mathcal{B})\big((A,B),(C,D)\big) = \mathcal{A}(A,C) \otimes \mathcal{B}(B,D)$.

Definition (Kelly)

let $\mathcal{E}$ be a $\mathcal{V}$-category. Let B be an $\mathcal{E}$-bimodule = $\mathcal{V}$-functor $B : \mathcal{E}^{\mathrm{op}} \boxtimes \mathcal{E} \to \underline{\mathcal{V}}$. Enriched end of B is the equalizer

$$\int_{E\in\mathcal{E}} B(E,E) \xrightarrow{\iota}$$
$$\prod_{E\in\mathcal{E}} B(E,E) \underset{(\mathrm{pr}_{E}\cdot\gamma)_{E,E'}}{\overset{(\mathrm{pr}_{E'}\cdot\beta)_{E,E'}}{\rightrightarrows}} \prod_{E,E'\in\mathcal{E}} \underline{\mathcal{V}}(\mathcal{E}(E,E'),B(E,E')),$$

where $\beta : B(E',E') \to \underline{\mathcal{V}}\big(\mathcal{E}(E,E'),B(E,E')\big)$ is adjunct to

$$^{B}_{\mathcal{E}}\alpha : \mathcal{E}(E,E') \otimes B(E',E') \to B(E,E')$$

and $\gamma : B(E,E) \to \underline{\mathcal{V}}\big(\mathcal{E}(E,E'),B(E,E')\big)$ is adjunct to composition

$$\mathcal{E}(E,E') \otimes B(E,E) \xrightarrow{c} B(E,E) \otimes \mathcal{E}(E,E') \xrightarrow{\alpha^{B}_{\mathcal{E}}} B(E,E').$$

Properties of enriched ends and coends

Lemma

Let B be an $\mathcal{E}$-bimodule, $B : \mathcal{E}^{\mathrm{op}} \boxtimes \mathcal{E} \to \underline{\mathcal{V}}$. For $X \in \operatorname{Ob} \mathcal{V}$ there is a natural isomorphism of $\mathcal{V}$-functors
$\underline{\mathcal{V}}\big(X, \int_{E\in\mathcal{E}} B(E,E)\big) \to \int_{E\in\mathcal{E}} \underline{\mathcal{V}}\big(X, B(E,E)\big)$.

Definition

Let $\mathcal{E}$ be a small $\mathcal{V}$-category and let $B : \mathcal{E}^{\mathrm{op}} \boxtimes \mathcal{E} \to \underline{\mathcal{V}}$ be an $\mathcal{E}$-bimodule. The enriched coend of B is the coequalizer

$$\coprod_{E,F\in\mathcal{E}} \mathcal{E}(E,F) \otimes B(F,E) \underset{({}^{B}_{\mathcal{E}}\alpha . \mathrm{in}_E)_{E,F}}{\overset{(c . \alpha^{B}_{\mathcal{E}} . \mathrm{in}_F)_{E,F}}{\rightrightarrows}} \coprod_{E\in\mathcal{E}} B(E,E) \to \int^{E\in\mathcal{E}} B(E,E).$$

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$$\coprod_{E,F\in\mathcal{E}} \mathcal{E}(E,F) \otimes B(F,E) \underset{({}^{B}_{\mathcal{E}}\alpha \centerdot \mathrm{in}_E)_{E,F}}{\overset{(c\centerdot\alpha^{B}_{\mathcal{E}}\centerdot\mathrm{in}_F)_{E,F}}{\rightrightarrows}} \coprod_{E\in\mathcal{E}} B(E,E) \to \int^{E\in\mathcal{E}} B(E,E).$$

Proposition

Let B be an $\mathcal{E}$-bimodule, $B : \mathcal{E}^{\mathrm{op}} \boxtimes \mathcal{E} \to \underline{\mathcal{V}}$. For $Y \in \mathrm{Ob}\,\mathcal{V}$ there are natural morphisms of $\mathcal{V}$-functors
$\Xi : \underline{\mathcal{V}}\big(\int^{E\in\mathcal{E}} B(E,E), Y\big) \to \int_{E\in\mathcal{E}} \underline{\mathcal{V}}\big(B(E,E), Y\big)$,
$\Psi : \int_{E\in\mathcal{E}} \underline{\mathcal{V}}\big(B(E,E), Y\big) \to \underline{\mathcal{V}}\big(\int^{E\in\mathcal{E}} B(E,E), Y\big)$, *that satisfy*
$\Psi \centerdot \Xi = 1$.

Corollary

$$\Big[\int_{E\in\mathcal{E}} \underline{\mathcal{V}}\big(B(E,E),Y\big) \xrightarrow{\Psi} \underline{\mathcal{V}}\Big(\int^{E\in\mathcal{E}} B(E,E),Y\Big)$$

$$\xrightarrow[\Xi_{\bullet}p_E]{\underline{\mathcal{V}}(i_E,1)} \underline{\mathcal{V}}\big(B(E,E),Y\big)\Big] = p_E.$$

Corollary

$$\Big[\int_{E\in\mathcal{E}} \underline{\mathcal{V}}\big(B(E,E),Y\big) \xrightarrow{\Psi} \underline{\mathcal{V}}\Big(\int^{E\in\mathcal{E}} B(E,E),Y\Big) \xrightarrow[\Xi . p_E]{\underline{\mathcal{V}}(i_E,1)} \underline{\mathcal{V}}\big(B(E,E),Y\big)\Big] = p_E .$$

Definition

A complete and cocomplete locally small symmetric monoidal closed category $\mathcal{V}$ is called *set-like* if the morphism ψ or ξ is invertible.

If one of ψ or ξ is invertible, then so is the other and, moreover, ψ and ξ are inverse to each other.

Lemma

Assuming that $\mathcal{V}$ is set-like we find that natural morphisms of $\mathcal{V}$-functors

$$\Xi : \underline{\mathcal{V}}(\int^{E\in\mathcal{E}} B(E,E), Y) \to \int_{E\in\mathcal{E}} \underline{\mathcal{V}}(B(E,E), Y),$$

$$\Psi : \int_{E\in\mathcal{E}} \underline{\mathcal{V}}(B(E,E), Y) \to \underline{\mathcal{V}}(\int^{E\in\mathcal{E}} B(E,E), Y)$$

are inverse to each other.

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$$\Psi : \int_{E\in\mathcal{E}} \underline{\mathcal{V}}\big(B(E,E), Y\big) \to \underline{\mathcal{V}}\Big(\int^{E\in\mathcal{E}} B(E,E), Y\Big)$$

are inverse to each other.

Example

The category $\mathcal{V} = \mathcal{S}\mathrm{et}$ is complete, cocomplete and Cartesian closed with the internal hom $\underline{\mathcal{S}\mathrm{et}}(X, Y) = \mathcal{S}\mathrm{et}(X, Y)$. The morphisms
$\xi = (\mathcal{S}\mathrm{et}(\mathrm{in}_j, Y))_{j\in I} : \mathcal{S}\mathrm{et}(\coprod_{i\in I} X_i, Y) \to \prod_{j\in I} \mathcal{S}\mathrm{et}(X_j, Y)$ are invertible by the definition of $\coprod_{i\in I} X_i$. Thus, $\mathcal{S}\mathrm{et}$ is non-additive set-like.

Example

Let $\Bbbk$ be a commutative ring. The category $\mathcal{V} = \Bbbk\text{-Mod}$ is complete, cocomplete, additive and closed symmetric monoidal with the tensor product $M \otimes_{\Bbbk} N$ and internal hom $\underline{\Bbbk\text{-Mod}}(M, N) = \Bbbk\text{-Mod}(M, N)$ equipped with the obvious action of $\Bbbk$. The morphisms

$$\xi = (\Bbbk\text{-Mod}(\mathrm{in}_j, N))_{j \in I} : \Bbbk\text{-Mod}(\coprod_{i \in I} M_i, N) \to \prod_{j \in I} \Bbbk\text{-Mod}(M_j, N)$$

are invertible by the definition of $\coprod_{i \in I} M_i$. Thus, $\Bbbk$-Mod is set-like.

Lemma

The category $\mathbf{Ab}_{\mathbb{L}}$ *has small colimits and the forgetful functor* $\mathbf{Ab}_{\mathbb{L}} \to \mathbf{Ab}$ *preserves colimits.*

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The category $\mathbf{Ab}_{\mathbb{L}}$ *has small limits. The* l*-th filtration term of* $\lim_{i\in I}(M_i \in \mathbf{Ab}_{\mathbb{L}})$ *is*

$$\mathcal{F}^l \lim_{i\in I}(M_i \in \mathbf{Ab}_{\mathbb{L}}) = \lim_{i\in I}(\mathcal{F}^l M_i \in \mathbf{Ab}) \subset \lim_{i\in I}(M_i \in \mathbf{Ab}).$$

Thus,

$$\lim_{i\in I}(M_i \in \mathbf{Ab}_{\mathbb{L}}) = \bigcup_{l\in\mathbb{L}} \lim_{i\in I}(\mathcal{F}^l M_i \in \mathbf{Ab}) \subset \lim_{i\in I}(M_i \in \mathbf{Ab}).$$

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Thus,

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Proposition

The category $\mathbf{Ab}_{\mathbb{L}}$ *is set-like.*

Filtered graded abelian groups

We extend the consideration to the graded setting.
An $\mathbb{L}$-filtered graded abelian group is a $\mathbb{Z}$-graded abelian group M together with, for every $l \in \mathbb{L}$, a graded subgroup $\mathcal{F}^l M$ such that $a \leqslant b \in \mathbb{L}$ implies that $\mathcal{F}^a M \supset \mathcal{F}^b M$ and $\cup_{l \in \mathbb{L}} \mathcal{F}^l M = M$. The symmetric multicategory $\widehat{\mathbf{grAb}_{\mathbb{L}}}$ of $\mathbb{L}$-filtered graded abelian groups is formed by multilinear maps of certain degree preserving the filtration:

$$\widehat{\mathbf{grAb}_{\mathbb{L}}}(M_1, \dots, M_n; N)^d = \\ \{(\text{multilinear maps } f : M_1^{k_1} \times \dots \times M_n^{k_n} \to N^{k_1+\dots+k_n+d})_{k_i \in \mathbb{Z}} \mid \\ \mid (\mathcal{F}^{l_1} M_1^{k_1} \times \dots \times \mathcal{F}^{l_n} M_n^{k_n}) f \subset \mathcal{F}^{l_1+\dots+l_n} N^{k_1+\dots+k_n+d}\},$$

$n \geqslant 1$. There is a sign for composition!
This multicategory is representable by a symmetric monoidal category which we denote $\mathbf{grAb}_{\mathbb{L}}$.

One deduces the tensor product of a family $M_1, \ldots, M_n$, $n \geqslant 1$, as the tensor product of $\mathbb{Z}$-graded abelian groups M_i, equipped with the filtration

$$\mathcal{F}^l(\otimes_{i=1}^n M_i) = \operatorname{Im}(\oplus_{l_1+\cdots+l_n=l} \otimes_{i=1}^n \mathcal{F}^{l_i} M_i \to \otimes_{i=1}^n M_i).$$

Thus, $\widehat{\mathbf{grAb}_{\mathbb{L}}}(M_1, \ldots, M_n; N)$ is naturally isomorphic to $\mathbf{grAb}_{\mathbb{L}}(M_1 \otimes \cdots \otimes M_n, N)$ for $n \geqslant 1$.
The unit object is $\mathbb{Z}$, concentrated in degree 0, equipped with the mentioned filtration.
The monoidal category $\mathbf{grAb}_{\mathbb{L}}$ is symmetric with the signed symmetry of $\mathbb{Z}$-graded abelian groups.

Furthermore, $\mathbf{grAb}_{\mathbb{L}}$ is closed.
In fact, let $M, N \in \mathbf{grAb}_{\mathbb{L}}$. Associate with them a new graded $\mathbb{L}$-filtered abelian group $\underline{\mathbf{grAb}_{\mathbb{L}}}(M,N) \subset \underline{\mathbf{grAb}}(M,N)$ with

$$\mathcal{F}^l \underline{\mathbf{grAb}_{\mathbb{L}}}(M,N)^d = \{f \in \underline{\mathbf{grAb}}(M,N)^d \mid \forall \lambda \in \mathbb{L}\ \forall k \in \mathbb{Z}\ (\mathcal{F}^\lambda M^k) f \subset \mathcal{F}^{\lambda+l} N^{k+d}\}.$$

The internal hom is defined as $\underline{\mathbf{grAb}_{\mathbb{L}}}(M,N)^\bullet = \cup_{l \in \mathbb{L}} \mathcal{F}^l \underline{\mathbf{grAb}_{\mathbb{L}}}(M,N)^\bullet$. The evaluation

$$\mathrm{ev} : M \otimes \underline{\mathbf{grAb}_{\mathbb{L}}}(M,N) \to N, \quad m \otimes f \mapsto (m)f,$$

is a morphism of $\mathbf{grAb}_{\mathbb{L}}$, and it turns $\mathbf{grAb}_{\mathbb{L}}$ into a closed symmetric monoidal category.

A commutative $\mathbb{L}$-filtered graded ring Λ is a commutative monoid (commutative algebra) in $\mathbf{grAb}_{\mathbb{L}}$. Modules over Λ in $\mathbf{grAb}_{\mathbb{L}}$ are called $\mathbb{L}$-filtered $\mathbb{Z}$-graded Λ-modules and are identified with commutative Λ-bimodules (for short, Λ-modules). The category of them with grading and filtration preserving Λ-module maps is denoted $\Lambda\text{-Mod}_{\mathbb{L}}$. It is symmetric monoidal with the tensor product $M \otimes_{\Lambda} N$. The unit object $\mathbb{1}$ is Λ with its filtration.

The category $\Lambda\text{-Mod}_{\mathbb{L}}$ is closed.
In fact, let $M, N \in \Lambda\text{-Mod}_{\mathbb{L}}$. Associate with them a new graded $\mathbb{L}$-filtered Λ-module $\underline{\Lambda\text{-Mod}_{\mathbb{L}}}(M,N) \subset \underline{\Lambda\text{-Mod}}(M,N)$ with

$$\mathcal{F}^l \underline{\Lambda\text{-Mod}_{\mathbb{L}}}(M,N)^d = \{f \in \underline{\Lambda\text{-Mod}}(M,N)^d \mid \forall \lambda \in \mathbb{L}\ \forall k \in \mathbb{Z} \quad (\mathcal{F}^\lambda M^k)f \subset \mathcal{F}^{\lambda+l} N^{k+d}\}.$$

The internal hom is defined as $\underline{\Lambda\text{-Mod}_{\mathbb{L}}}(M,N)^\bullet = \cup_{l \in \mathbb{L}} \mathcal{F}^l \underline{\Lambda\text{-Mod}_{\mathbb{L}}}(M,N)^\bullet$. The evaluation

$$\text{ev} : M \otimes_\Lambda \underline{\Lambda\text{-Mod}_{\mathbb{L}}}(M,N) \to N, \quad m \otimes f \mapsto (m)f,$$

is a morphism of $\Lambda\text{-Mod}_{\mathbb{L}}$, and it turns this category into a closed symmetric monoidal one.

Lemma

The category $\Lambda\text{-Mod}_{\mathbb{L}}$ *has small colimits and the forgetful functor* $\Lambda\text{-Mod}_{\mathbb{L}} \to \Lambda\text{-Mod}$ *preserves colimits. Namely,* $\mathrm{colim}_{i\in I}(M_i \in \Lambda\text{-Mod}_{\mathbb{L}}) = \mathrm{colim}_{i\in I}(M_i \in \Lambda\text{-Mod})$ *with the filtration*

$$\begin{aligned}
&\mathcal{F}^l \operatorname*{colim}_{i\in I}(M_i \in \Lambda\text{-Mod}_{\mathbb{L}}) \\
&\quad = \mathrm{Im}\Big[q_l : \operatorname*{colim}_{i\in I}(\mathcal{F}^l M_i \in \Lambda\text{-Mod}) \to \operatorname*{colim}_{i\in I}(M_i \in \Lambda\text{-Mod})\Big].
\end{aligned}$$

Lemma

The category $\Lambda\text{-Mod}_{\mathbb{L}}$ *has small limits. The* l*-th filtration term of* $\lim_{i\in I}(M_i \in \Lambda\text{-Mod}_{\mathbb{L}})$ *is*

$$\mathcal{F}^l \lim_{i\in I}(M_i \in \Lambda\text{-Mod}_{\mathbb{L}}) = \lim_{i\in I}(\mathcal{F}^l M_i \in \Lambda\text{-Mod}) \subset \lim_{i\in I}(M_i \in \Lambda\text{-Mod}).$$

Thus,

$$\lim_{i\in I}(M_i \in \Lambda\text{-Mod}_{\mathbb{L}}) = \bigcup_{l\in\mathbb{L}} \lim_{i\in I}(\mathcal{F}^l M_i \in \Lambda\text{-Mod}) \subset \lim_{i\in I}(M_i \in \Lambda\text{-Mod}).$$

Lemma

The category $\Lambda\text{-Mod}_{\mathbb{L}}$ *has small limits. The* l*-th filtration term of* $\lim_{i\in I}(M_i \in \Lambda\text{-Mod}_{\mathbb{L}})$ *is*

$$\mathcal{F}^l \lim_{i\in I}(M_i \in \Lambda\text{-Mod}_{\mathbb{L}}) = \lim_{i\in I}(\mathcal{F}^l M_i \in \Lambda\text{-Mod}) \subset \lim_{i\in I}(M_i \in \Lambda\text{-Mod}).$$

Thus,

$$\lim_{i\in I}(M_i \in \Lambda\text{-Mod}_{\mathbb{L}}) = \bigcup_{l\in\mathbb{L}} \lim_{i\in I}(\mathcal{F}^l M_i \in \Lambda\text{-Mod}) \subset \lim_{i\in I}(M_i \in \Lambda\text{-Mod}).$$

Proposition

The category $\Lambda\text{-Mod}_{\mathbb{L}}$ *is set-like.*

Symmetric monoidal $\mathcal{V}$-bicategories

We shall encounter bicategories $\mathscr{B}$ enriched over the symmetric monoidal 1-category $\mathcal{V}$ at the level of 2-cells only. Thus it has a collection of objects, with 1-morphisms between the objects, and for any parallel 1-morphisms $f, g : x \to y$, a hom-object $\mathscr{B}(x, y)(f, g) \in \mathcal{V}$. This can be identified with a $\mathcal{V}$-$\mathcal{C}at$-enriched bicategory. Symmetric monoidal bicategories were defined by Schommer-Pries. Definition of a symmetric monoidal $\mathcal{V}$-bicategory can be obtained from the $\mathcal{S}$et case by requiring that the tensor product be a $\mathcal{V}$-functor, in place of natural transformations were natural $\mathcal{V}$-transformations, which are elements of a hom-set: the functor $\mathcal{V}(\mathbb{1}, \text{-})$ applied to the object of $\mathcal{V}$-transformations. Similarly with natural $\mathcal{V}$-modifications. Equations between pastings of those are the same as in the $\mathcal{S}$et case.

Bicategories of $\mathcal{V}$-categories

We suppose that $\mathcal{V}$ is set-like complete and cocomplete locally small symmetric monoidal closed category. Let us denote by $\mathcal{V}$-$\mathcal{C}at$ the bicategory, whose objects=0-cells are small $\mathcal{V}$-categories, 1-morphisms=1-cells are $\mathcal{V}$-functors, 2-morphisms=2-cells are natural transformations $\lambda : F \to G : \mathcal{A} \to \mathcal{B}$ which consist of collections $\big(\lambda_X \in \mathcal{V}(\mathbb{1}, \mathcal{B}(FX, GX))\big)_{X \in \mathrm{Ob}\,\mathcal{A}}$ such that

$$\begin{array}{ccc} \mathcal{A}(X,Y) & \xrightarrow{G} & \mathcal{B}(GX,GY) \\ {\scriptstyle F}\downarrow & = & \downarrow{\scriptstyle \mathcal{B}(\lambda_X,1)} \\ \mathcal{B}(FX,FY) & \xrightarrow{\mathcal{B}(1,\lambda_Y)} & \mathcal{B}(FX,GY) \end{array}$$

$\mathcal{V}$-$\mathcal{C}at$ is a symmetric monoidal bicategory.
Dropping the 2-cells we get a symmetric monoidal category, which we denote again by $\mathcal{V}$-$\mathcal{C}at$ by abuse of notation.

The unit object of the monoidal category $\mathcal{V}$-$\mathcal{C}at$ is the $\mathcal{V}$-category $\mathbb{1}$ with $\operatorname{Ob}\mathbb{1} = \{*\}$; $\mathbb{1}(*, *) = \mathbb{1}_{\mathcal{V}}$ is the unit object of the monoidal category $\mathcal{V}$. The composition is an obvious isomorphism $\mathbb{1}_{\mathcal{V}} \otimes \mathbb{1}_{\mathcal{V}} \to \mathbb{1}_{\mathcal{V}}$ and the unit is $\mathrm{id}_{\mathbb{1}_{\mathcal{V}}}$.

For small $\mathcal{V}$-categories $\mathcal{E}$ and $\mathcal{C}$ the (right) internal hom $\underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{E},\mathcal{C})$ is given (following Kelly) by

- $\operatorname{Ob}\underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{E},\mathcal{C}) = \{\mathcal{V}\text{-functors } \mathcal{E} \to \mathcal{C}\}$,
- $\underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{E},\mathcal{C})(F,G) = \int_{E\in\mathcal{E}} \mathcal{C}(FE,GE)$, the object of $\mathcal{V}$-transformations $F \to G : \mathcal{E} \to \mathcal{C}$, this enriched end is explicitly defined as the enriched equalizer

$$\int_{E\in\mathcal{E}} \mathcal{C}(FE,GE) \xrightarrow{\iota} \prod_{E\in\mathcal{E}} \mathcal{C}(FE,GE) \underset{(\mathrm{pr}_E\cdot\gamma)}{\overset{(\mathrm{pr}_{E'}\cdot\beta)}{\rightrightarrows}} \prod_{E,E'\in\mathcal{E}} \underline{\mathcal{V}}^r\big(\mathcal{E}(E,E'),\mathcal{C}(FE,GE')\big),$$

where $\beta : \mathcal{C}(FE',GE') \to \underline{\mathcal{V}}^r\big(\mathcal{E}(E,E'),\mathcal{C}(FE,GE')\big)$ is adjunct to composition

$$\mathcal{E}(E,E')\otimes\mathcal{C}(FE',GE') \xrightarrow{F\otimes 1} \mathcal{C}(FE,FE')\otimes\mathcal{C}(FE',GE') \xrightarrow{\cdot} \mathcal{C}(FE,GE')$$

and $\gamma : \mathcal{C}(FE,GE) \to \underline{\mathcal{V}}^r\big(\mathcal{E}(E,E'),\mathcal{C}(FE,GE')\big)$ is adjunct to composition

$$\mathcal{E}(E,E') \otimes \mathcal{C}(FE,GE) \xrightarrow{c} \mathcal{C}(FE,GE) \otimes \mathcal{E}(E,E') \xrightarrow{1\otimes G} \mathcal{C}(FE,GE) \otimes \mathcal{C}(GE,GE') \xrightarrow{\cdot} \mathcal{C}(FE,GE').$$

- the identity transformation $1_F : \mathbb{1} \to \underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{E},\mathcal{C})(F,F)$ is $1_F = \left(1_{FE} : \mathbb{1} \to \mathcal{C}(FE,FE)\right)_{E\in\mathcal{E}}$,
- the composition of objects of $\mathcal{V}$-transformations $\underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{E},\mathcal{C})(F,G)$ comes from the composition in $\mathcal{C}$:

$$\begin{array}{ccc}
\underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{E},\mathcal{C})(F,G) \otimes \underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{E},\mathcal{C})(G,H) & \xrightarrow{\exists!\,\cdot} & \underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{E},\mathcal{C})(F,H) \\
\downarrow{\scriptstyle \iota\otimes\iota} & = & \downarrow{\scriptstyle \iota} \\
\prod_{E\in\mathcal{E}} \mathcal{C}(FE,GE) \otimes \prod_{E\in\mathcal{E}} \mathcal{C}(GE,HE) & \xrightarrow{m} & \prod_{E\in\mathcal{E}} \mathcal{C}(FE,HE) \\
\downarrow{\scriptstyle \mathrm{pr}_E\otimes\mathrm{pr}_E} & = & \downarrow{\scriptstyle \mathrm{pr}_E} \\
\mathcal{C}(FE,GE) \otimes \mathcal{C}(GE,HE) & \xrightarrow{\cdot} & \mathcal{C}(FE,HE)
\end{array}$$

The evaluation $\mathcal{V}$-functor ev : $\mathcal{E} \boxtimes \underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{E}, \mathcal{C}) \to \mathcal{C}$, $(X, F) \mapsto F(X)$, consists of morphisms which are diagonal maps in the following commutative diagram

$$\begin{array}{ccc}
\mathcal{E}(X,Y) \otimes [\mathcal{E},\mathcal{C}](F,G) & \xrightarrow{F \otimes \iota \mathrm{pr}_Y} & \mathcal{C}(FX,FY) \otimes \mathcal{C}(FY,GY) \\
\Big\downarrow {\scriptstyle G \otimes \iota \mathrm{pr}_X} & \searrow & \Big\downarrow {\scriptstyle \cdot} \\
\mathcal{C}(GX,GY) \otimes \mathcal{C}(FX,GX) & & \\
\Big\downarrow {\scriptstyle c} & & \\
\mathcal{C}(FX,GX) \otimes \mathcal{C}(GX,GY) & \xrightarrow{\quad \cdot \quad} & \mathcal{C}(FX,GY)
\end{array}$$

In the non-enriched case this diagram corresponds to familiar

$$\begin{array}{ccc}
FX & \xrightarrow{Ff} & FY \\
{\scriptstyle \lambda_X} \Big\downarrow & = & \Big\downarrow {\scriptstyle \lambda_Y} \\
GX & \xrightarrow{Gf} & GY
\end{array}$$

The evaluation is a $\mathcal{V}$-functor.
The fact that $(\underline{\mathcal{V}\text{-}\mathcal{C}at}, \mathrm{ev})$ is the right internal hom in $(\mathcal{V}\text{-}\mathcal{C}at, \boxtimes)$ is proven by Kelly.

The evaluation is a $\mathcal{V}$-functor.
The fact that $(\underline{\mathcal{V}\text{-}\mathcal{C}at}, \mathrm{ev})$ is the right internal hom in $(\mathcal{V}\text{-}\mathcal{C}at, \boxtimes)$ is proven by Kelly.
Consider a $\mathcal{V}$-bicategory $\mathcal{V}$-**Cat**, whose objects are small $\mathcal{V}$-categories, 1-morphisms are $\mathcal{V}$-functors, 2-cells $F \to G : \mathcal{E} \to \mathcal{C}$ are the objects $\underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{E}, \mathcal{C})(F, G) = \int_{E \in \mathcal{E}} \mathcal{C}(FE, GE) \in \mathrm{Ob}\, \mathcal{V}$. The $\mathcal{V}$-categories of morphisms $\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{E}, \mathcal{C}) = \underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{E}, \mathcal{C})$ have a prescribed vertical composition of 2-cells given above. The horizontal composition $\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A}, \mathcal{B}) \boxtimes \mathcal{V}\text{-}\mathbf{Cat}(\mathcal{B}, \mathcal{C}) \to \mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A}, \mathcal{C})$ is the composition of $\mathcal{V}$-functors on objects.

Remark

The set of natural transformations $\{\lambda : F \to G : \mathcal{E} \to \mathcal{C}\}$ in the sense of usual diagram is in bijection with the set $\mathcal{V}\big(\mathbb{1}, \underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{E}, \mathcal{C})(F, G)\big)$ of morphisms of the unit object of $\mathcal{V}$ to the object of $\mathcal{V}$-transformations $F \to G : \mathcal{E} \to \mathcal{C}$.

Since the associators and unitors of $\mathcal{V}$-**Cat** are natural transformations of $\mathcal{V}$-functors, $\mathcal{V}$-**Cat** is enriched in $\mathcal{V}$-$\mathcal{C}at$, that is, it is a $\mathcal{V}$-bicategory. Another example of a $\mathcal{V}$-bicategory is the bicategory of bimodules, aka profunctors, aka distributors:

$\mathcal{V}$-bicategory of bimodules

For us a module over a $\mathcal{V}$-category $\mathcal{A}$ will mean a right module, that is, a $\mathcal{V}$-functor $M : \mathcal{A} \to \underline{\mathcal{V}}$. We use right operators instead of left ones. The $\mathcal{V}$-category of $\mathcal{A}$-modules $\underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{A}, \underline{\mathcal{V}})$ is denoted also Mod-$\mathcal{A}$. Similarly the $\mathcal{V}$-category of left $\mathcal{A}$-modules $\underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{A}^{\mathrm{op}}, \underline{\mathcal{V}})$ is denoted $\mathcal{A}$-Mod. One may represent an $\mathcal{A}$-module M as a collection $(M(X))_{X \in \mathrm{Ob}\,\mathcal{A}}$ of objects of $\mathcal{V}$ together with the action $M(X) \otimes \mathcal{A}(X, Y) \to M(Y)$, associative and unital in the obvious sense.

For $\mathcal{V}$-categories $\mathcal{A}$, $\mathcal{B}$ we consider $\mathcal{A}$-$\mathcal{B}$-bimodules, which are by definition $\mathcal{A}^{\mathrm{op}} \boxtimes \mathcal{B}$-modules, that is, $\mathcal{V}$-functors $\mathcal{A}^{\mathrm{op}} \boxtimes \mathcal{B} \to \underline{\mathcal{V}}$. The $\mathcal{V}$-category of $\mathcal{A}$-$\mathcal{B}$-bimodules $\underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{A}^{\mathrm{op}} \boxtimes \mathcal{B}, \underline{\mathcal{V}})$ is denoted also $\mathcal{A}$-Mod-$\mathcal{B}$:

$$\mathcal{A}\text{-Mod-}\mathcal{B}(M, N) = \int_{(A,B) \in \mathcal{A}^{\mathrm{op}} \boxtimes \mathcal{B}} \underline{\mathcal{V}}(M(B, C), N(B, C)).$$

By general properties of closed symmetric monoidal categories $\mathcal{A}$-Mod-$\mathcal{B}$ is isomorphic to $\underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{B}, \underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{A}^{\mathrm{op}}, \underline{\mathcal{V}}))$ and to $\underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{A}^{\mathrm{op}}, \underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{B}, \underline{\mathcal{V}})) = \underline{\mathcal{V}\text{-}\mathcal{C}at}(\mathcal{A}^{\mathrm{op}}, \text{Mod-}\mathcal{B})$. Thus, an $\mathcal{A}$-$\mathcal{B}$-bimodule M induces for any object A of $\mathcal{A}$ a $\mathcal{B}$-module $M(A, \text{-})$.

Example

Let S be a set. Let $\mathcal{D}$ be a *discrete $\mathcal{V}$-category* = the $\mathcal{V}$-category with $\operatorname{Ob}\mathcal{D} = S$ and objects of morphisms

$$\mathcal{D}(X, Y) = \begin{cases} \mathbb{1}, & \text{if } X = Y, \\ \emptyset, & \text{if } X \neq Y, \end{cases}$$

where $\emptyset \in \operatorname{Ob}\mathcal{V}$ is the initial object. For any $X \in \operatorname{Ob}\mathcal{V}$ we have $\emptyset \otimes X \cong X \otimes \emptyset \cong \emptyset$. The composition in $\mathcal{D}$ reduces to $l = r : \mathbb{1} \otimes \mathbb{1} \xrightarrow{\cong} \mathbb{1}$. The unit morphisms are $1_X = \mathrm{id}_{\mathbb{1}}$.
A $\mathcal{D}$-bimodule $\mathcal{A} : \mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{D} \to \underline{\mathcal{V}}$ consists of a function $\operatorname{Ob}\mathcal{A} : S \times S \to \operatorname{Ob}\mathcal{V}$, $(X, Y) \mapsto \mathcal{A}(X, Y)$ and action morphisms $\mathcal{A}(X, Y) \otimes \mathbb{1} \to \mathcal{A}(X, Y)$, $\mathbb{1} \otimes \mathcal{A}(X, Y) \to \mathcal{A}(X, Y)$. However, the action morphisms coincide with r (resp. l) (the unitors of $\mathcal{V}$) due to unitality of the action. Therefore, $\mathcal{D}$-bimodules $\mathcal{A}$ are identified with functions $\operatorname{Ob}\mathcal{A} : S \times S \to \operatorname{Ob}\mathcal{V}$.

Composition of bimodules

Bimod can be viewed as a category weakly enriched in the closed symmetric monoidal category $\mathcal{V}$-$\mathcal{C}at$. Objects of $\mathcal{V}$-bicategory Bimod are small $\mathcal{V}$-categories, denoted $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, etc. $\mathcal{V}$-categories of morphisms from $\mathcal{A}$ to $\mathcal{B}$ are $\mathrm{Bimod}(\mathcal{A},\mathcal{B}) = \mathcal{A}\text{-Mod-}\mathcal{B}$. The composition $\mathcal{V}$-functor is

$$\otimes_{\mathcal{B}} : \mathcal{A}\text{-Mod-}\mathcal{B} \boxtimes \mathcal{B}\text{-Mod-}\mathcal{C} \to \mathcal{A}\text{-Mod-}\mathcal{C},$$

$$(M, N) \mapsto M \otimes_{\mathcal{B}} N = \int^{B\in\mathcal{B}} M(\text{-}, B) \otimes N(B, \text{-}),$$

where the coend is taken in enriched sense. In other terms, $(M \otimes_{\mathcal{B}} N)(A, C)$ is the coequalizer (α is the action)

$$\coprod_{B,B'\in\mathcal{B}} M(A, B) \otimes \mathcal{B}(B, B') \otimes N(B', C) \underset{(1_{M(A,B)}\otimes^{N}_{\mathcal{B}}\alpha.\mathrm{in}_{B})_{B,B'}}{\overset{(\alpha^{M}_{\mathcal{B}}\otimes 1_{N(B',C)}.\mathrm{in}_{B'})_{B,B'}}{\rightrightarrows}}$$

$$\coprod_{B\in\mathcal{B}} M(A, B) \otimes N(B, C) \to \int^{B\in\mathcal{B}} M(A, B) \otimes N(B, C).$$

Example

Let $\mathcal{D}$ be a discrete $\mathcal{V}$-category with $\mathrm{Ob}\,\mathcal{D} = S$. The composition of $\mathcal{D}$-bimodules $\mathcal{A}$ and $\mathcal{B}$ is

$$(\mathcal{A} \otimes_{\mathcal{D}} \mathcal{B})(X, Y) = \int^{D \in \mathcal{D}} \mathcal{A}(X, D) \otimes \mathcal{B}(D, Y)$$
$$= \coprod_{D \in S} \mathcal{A}(X, D) \otimes \mathcal{B}(D, Y).$$

The object of $\mathcal{V}$-transformations from $\mathcal{A}$ to $\mathcal{B}$ is

$$\mathcal{D}\text{-Mod-}\mathcal{D}(\mathcal{A}, \mathcal{B}) = \int_{(X,Y) \in \mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{D}} \underline{\mathcal{V}}(\mathcal{A}(X, Y), \mathcal{B}(X, Y))$$
$$= \prod_{(X,Y) \in S^2} \underline{\mathcal{V}}(\mathcal{A}(X, Y), \mathcal{B}(X, Y))$$

As for general $\mathcal{D}$ the $\mathcal{V}$-category $\mathcal{D}$-Mod-$\mathcal{D}$ is monoidal.

Example

Let $\mathcal{V}$ be $\mathcal{S}$et, then a $\mathcal{V}$-category is a usual category. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be small discrete categories, identified with sets $\operatorname{Ob}\mathcal{A} = A$, $\operatorname{Ob}\mathcal{B} = B$, $\operatorname{Ob}\mathcal{C} = C$. An $\mathcal{A}$-$\mathcal{B}$-bimodule F is identified with a function $F : A \times B \to \operatorname{Ob}\mathcal{S}\mathrm{et}$, $(a, b) \mapsto F(a, b)$. The set of morphisms is

$$(\mathcal{A}\text{-Mod-}\mathcal{B})(F, G) = \prod_{a\in A, b\in B} \mathcal{S}\mathrm{et}(F(a, b), G(a, b)).$$

The composition of bimodules $F : \mathcal{A} \to \mathcal{B}$ and $H : \mathcal{B} \to \mathcal{C}$ is $(F \otimes_{\mathcal{B}} H) : \mathcal{A} \to \mathcal{C}$,

$$(F \otimes_{\mathcal{B}} H)(a, c) = \bigsqcup_{b\in B} F(a, b) \times H(b, c).$$

The horizontal composition of 2-morphisms described by $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B} \underset{K}{\overset{H}{\rightrightarrows}} \mathcal{C}$ is

$$(\mathcal{A}\text{-Mod-}\mathcal{B})(F, G) \times (\mathcal{B}\text{-Mod-}\mathcal{C})(H, K) \longrightarrow (\mathcal{A}\text{-Mod-}\mathcal{C})(F \otimes_{\mathcal{B}} H, G \otimes_{\mathcal{B}} K)$$

$$\big((f_{ab} : F(a, b) \to G(a, b))_{a.b}, (h_{bc} : H(b, c) \to K(b, c))_{b,c}\big) \longmapsto \Big(\bigsqcup_{b \in B} f_{ab} \times h_{bc}\Big)_{a,c}.$$

The horizontal composition of 2-morphisms described by $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B} \underset{K}{\overset{H}{\rightrightarrows}} \mathcal{C}$ is

$$(\mathcal{A}\text{-Mod-}\mathcal{B})(F, G) \times (\mathcal{B}\text{-Mod-}\mathcal{C})(H, K) \longrightarrow (\mathcal{A}\text{-Mod-}\mathcal{C})(F \otimes_{\mathcal{B}} H, G \otimes_{\mathcal{B}} K)$$

$$\Big((f_{ab} : F(a, b) \to G(a, b))_{a.b}, (h_{bc} : H(b, c) \to K(b, c))_{b,c}\Big) \longmapsto \Big(\bigsqcup_{b \in B} f_{ab} \times h_{bc}\Big)_{a,c}.$$

Proposition

For any complete and cocomplete symmetric monoidal closed category $\mathcal{V}$ the multiplication $\otimes_{\mathcal{B}}$ is a $\mathcal{V}$-functor.

Tensor cocategory

Since the set-like category $\mathcal{V}$ is additive, the $\mathcal{V}$-categories $\mathcal{A}$-Mod-$\mathcal{B}$ are additive as well. For an arbitrary $\mathcal{V}$-category $\mathcal{E}$ we have the diagonal $\mathcal{V}$-functor $\Delta : \mathcal{E} \to \mathcal{E}^I$, where I is a set, $\mathcal{E}^I\big((M_i)_{i\in I},(N_j)_{j\in I}\big) = \prod_{i\in I}\mathcal{E}(M_i,N_i)$. For $\mathcal{E} = \mathcal{A}$-Mod-$\mathcal{B}$ we have a left and a right adjoint to the diagonal $\mathcal{V}$-functor: the coproduct $\coprod_{i\in I} : \mathcal{E}^I \to \mathcal{E}$ and the product $\prod_{i\in I} : \mathcal{E}^I \to \mathcal{E}$. They are defined pointwise:

$$\big(\coprod_{i\in I} M_i\big)(A,B) = \coprod_{i\in I}\big[M_i(A,B)\big],$$

$$\big(\prod_{i\in I} N_i\big)(A,B) = \prod_{i\in I}\big[N_i(A,B)\big].$$

Let $\mathcal{D}$ be a small $\mathcal{V}$-category. Let $\mathcal{A}$ be a $\mathcal{D}$-bimodule. Denote by

$$T^n\mathcal{A} = \mathcal{A}^{\otimes_{\mathcal{D}} n} = \underbrace{\mathcal{A} \otimes_{\mathcal{D}} \mathcal{A} \cdots \otimes_{\mathcal{D}} \mathcal{A}}_{n}, \qquad n \geqslant 0.$$

the n-th power of $\mathcal{A}$. By definition, $T^0\mathcal{A} = \mathcal{A}^{\otimes_{\mathcal{D}} 0} = y_{\mathcal{D}}$ is the regular $\mathcal{D}$-bimodule. Denote by $T\mathcal{A}$ the $\mathcal{D}$-bimodule $T\mathcal{A} = \coprod_{n=0}^{\infty} T^n\mathcal{A} = \coprod_{n=0}^{\infty} \mathcal{A}^{\otimes_{\mathcal{D}} n}$. This is a coalgebra (comonoid) in the monoidal $\mathcal{V}$-category $\mathcal{D}$-Mod-$\mathcal{D}$. In fact,

$$T\mathcal{A} \otimes_{\mathcal{D}} T\mathcal{A} \cong \coprod_{n=0}^{\infty} \coprod_{m=0}^{\infty} \mathcal{A}^{\otimes_{\mathcal{D}} m} \otimes_{\mathcal{D}} \mathcal{A}^{\otimes_{\mathcal{D}} n}$$

$$= \coprod_{k=0}^{\infty} \coprod_{m+n=k}^{m,n \in \mathbb{N}} \mathcal{A}^{\otimes_{\mathcal{D}} m} \otimes_{\mathcal{D}} \mathcal{A}^{\otimes_{\mathcal{D}} n} \cong \coprod_{k=0}^{\infty} \prod_{m+n=k}^{m,n \in \mathbb{N}} \mathcal{A}^{\otimes_{\mathcal{D}} m} \otimes_{\mathcal{D}} \mathcal{A}^{\otimes_{\mathcal{D}} n}.$$

Deconcatenation comultiplication,

$$\Delta : T\mathcal{A} = \coprod_{k=0}^{\infty} T^k\mathcal{A} \to \coprod_{k=0}^{\infty} \prod_{m+n=k}^{m,n\in\mathbb{N}} T^m\mathcal{A} \otimes_{\mathcal{D}} T^n\mathcal{A} \cong T\mathcal{A} \otimes_{\mathcal{D}} T\mathcal{A},$$

is defined as $\coprod_{k=0}^{\infty} \Delta_{(k)}$, where components of $\Delta_{(k)} : T^k\mathcal{A} \to \prod_{m+n=k}^{m,n\in\mathbb{N}} T^m\mathcal{A} \otimes_{\mathcal{D}} T^n\mathcal{A}$ are the obvious isomorphisms $T^k\mathcal{A} \to T^m\mathcal{A} \otimes_{\mathcal{D}} T^n\mathcal{A}$, $m + n = k$.

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Let $\mathcal{A} \in \operatorname{Ob} \mathcal{D}\text{-Mod-}\mathcal{D}$, $\mathcal{B} \in \operatorname{Ob} \mathcal{E}\text{-Mod-}\mathcal{E}$. Given a $\mathcal{V}$-functor $\tilde{f} : \mathcal{D} \to \mathcal{E}$ we can consider the restriction $\mathcal{D}$-bimodule

$${}_{\tilde{f}}\mathcal{B}_{\tilde{f}} = \big(\mathcal{D}^{\mathsf{op}} \boxtimes \mathcal{D} \xrightarrow{\tilde{f}^{\mathsf{op}}\boxtimes\tilde{f}} \mathcal{E}^{\mathsf{op}} \boxtimes \mathcal{E} \xrightarrow{\mathcal{B}} \underline{\mathcal{V}}\big) : (X, Y) \mapsto \mathcal{B}(\tilde{f}X, \tilde{f}Y).$$

Definition

An augmented coalgebra morphism $f : T\mathcal{A} \to T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}})$ consists of

- $\mathcal{V}$-functor $\tilde{f} : \mathcal{D} \to \mathcal{E}$;
- Morphism of $\mathcal{D}$-bimodules $f : T\mathcal{A} \to T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}})$

such that

$$
\begin{array}{ccc}
T\mathcal{A} & \xrightarrow{\quad f \quad} & T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}}) \\
\Delta \downarrow & = & \downarrow \Delta \\
T\mathcal{A} \otimes_{\mathcal{D}} T\mathcal{A} & \xrightarrow{f \otimes_{\mathcal{D}} f} & T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}}) \otimes_{\mathcal{D}} T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}})
\end{array}
$$

$$
\begin{array}{ccccc}
T\mathcal{A} & \xrightarrow{\quad \varepsilon \quad} & y_{\mathcal{D}} & \xrightarrow{\quad \eta \quad} & T\mathcal{A} \\
f \downarrow & = & \| & = & \downarrow f \\
T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}}) & \xrightarrow{\quad \varepsilon \quad} & y_{\mathcal{D}} & \xrightarrow{\quad \eta \quad} & T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}})
\end{array}
$$

Define $\mathrm{pr}_l : T\mathcal{C} \to T^l\mathcal{C}$ as the map with components $0 : T^n\mathcal{C} \to T^l\mathcal{C}$ for $n \neq l$ and $\mathrm{id} : T^l\mathcal{C} \to T^l\mathcal{C}$.

Proposition

Augmented coalgebra morphisms $f : T\mathcal{A} \to T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}})$ are in bijection with $\mathcal{D}$-bimodule morphisms $\check{f} : T^{>0}\mathcal{A} \to {}_{\tilde{f}}\mathcal{B}_{\tilde{f}}$.

Coderivations

Let $\mathcal{A} \in \operatorname{Ob} \mathcal{D}\text{-Mod-}\mathcal{D}$, $\mathcal{B} \in \operatorname{Ob} \mathcal{E}\text{-Mod-}\mathcal{E}$, let $f : T\mathcal{A} \to T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}})$, $g : T\mathcal{A} \to T({}_{\tilde{g}}\mathcal{B}_{\tilde{g}})$ be augmented coalgebra morphisms with the corresponding $\mathcal{V}$-functors $\tilde{f}, \tilde{g} : \mathcal{D} \to \mathcal{E}$. Define a $\mathcal{D}$-bimodule

$$ {}_{\tilde{f}}\mathcal{B}_{\tilde{g}} = \bigl(\mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{D} \xrightarrow{\tilde{f}^{\mathrm{op}} \boxtimes \tilde{g}} \mathcal{E}^{\mathrm{op}} \boxtimes \mathcal{E} \xrightarrow{\mathcal{B}} \underline{\mathcal{V}}\bigr) : (X, Y) \mapsto \mathcal{B}(\tilde{f}X, \tilde{g}Y). $$

Define another $\mathcal{D}$-bimodule $Q\mathcal{B} = T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}}) \otimes_{\mathcal{D}} {}_{\tilde{f}}\mathcal{B}_{\tilde{g}} \otimes_{\mathcal{D}} T({}_{\tilde{g}}\mathcal{B}_{\tilde{g}})$. It is equipped with projections

$$ \mathrm{pr}_l : Q\mathcal{B} \to \bigoplus_{q+1+t=l} ({}_{\tilde{f}}\mathcal{B}_{\tilde{f}})^{\otimes_{\mathcal{D}} q} \otimes_{\mathcal{D}} ({}_{\tilde{f}}\mathcal{B}_{\tilde{g}}) \otimes_{\mathcal{D}} ({}_{\tilde{g}}\mathcal{B}_{\tilde{g}})^{\otimes_{\mathcal{D}} t} $$

and a kind of deconcatenation comultiplication

$$ \Delta = (\Delta \otimes_{\mathcal{D}} 1 \otimes_{\mathcal{D}} 1, 1 \otimes_{\mathcal{D}} 1 \otimes_{\mathcal{D}} \Delta) : Q\mathcal{B} = T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}}) \otimes_{\mathcal{D}} {}_{\tilde{f}}\mathcal{B}_{\tilde{g}} \otimes_{\mathcal{D}} T({}_{\tilde{g}}\mathcal{B}_{\tilde{g}}) \\ \to T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}}) \otimes_{\mathcal{D}} Q\mathcal{B} \oplus Q\mathcal{B} \otimes_{\mathcal{D}} T({}_{\tilde{g}}\mathcal{B}_{\tilde{g}}). $$

This is a particular case of the following construction.

Let $\mathcal{M}(e.g. = \mathcal{V}(\mathbb{1}, \mathcal{D}\text{-Mod-}\mathcal{D}))$ be an additive monoidal category. Let C, D be coalgebras (comonoids) in $\mathcal{M}$ and let V be an object of $\mathcal{M}$. Then $Q = C \otimes V \otimes D$ has an associative comultiplication of the following kind

$$\Delta : Q = C \otimes V \otimes D \xrightarrow{(\Delta_C \otimes 1_V \otimes 1_D, 1_C \otimes 1_V \otimes \Delta_D)} C \otimes C \otimes V \otimes D \oplus C \otimes V \otimes D \otimes D = C \otimes Q \oplus Q \otimes D.$$

Associativity means the following equation

$$\begin{array}{ccc} Q & \xrightarrow{\quad\Delta\quad} & C \otimes Q \oplus Q \otimes D \\ {\scriptstyle\Delta}\downarrow & = & \downarrow{\scriptstyle 1\otimes\Delta} \\ C \otimes Q \oplus Q \otimes D & \xrightarrow{\Delta\otimes 1} & C \otimes C \otimes Q \oplus C \otimes Q \otimes D \oplus Q \otimes D \otimes D \end{array}$$

which is verified. For any $l \geqslant 1$ the iterated comultiplication

$$\Delta^{(l)} : Q \to \bigoplus_{q+1+t=l} C^{\otimes q} \otimes Q \otimes D^{\otimes t}$$

with values in l factors is well-defined. Apply to $C = T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}})$, $D = T({}_{\tilde{g}}\mathcal{B}_{\tilde{g}})$.

Definition

The object $\mathrm{Coder}(f,g)$ of (f,g)-coderivations is the equalizer of a pair of morphisms of $\mathcal{V}$

$$\mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A},Q\mathcal{B}) \underset{\beta}{\overset{\mathcal{D}\text{-Mod-}\mathcal{D}(1,\Delta)}{\rightrightarrows}} \mathcal{D}\text{-Mod-}\mathcal{D}\big(T\mathcal{A}, T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}})\otimes_{\mathcal{D}} Q\mathcal{B} \oplus Q\mathcal{B}\otimes_{\mathcal{D}} T({}_{\tilde{g}}\mathcal{B}_{\tilde{g}})\big),$$

where β denotes the composition

$$\begin{aligned}
&\mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A},Q\mathcal{B}) \xrightarrow{(\dot{f}\otimes 1,1\otimes\dot{g})} \\
&\mathcal{D}\text{-Mod-}\mathcal{D}\big(T\mathcal{A}, T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}})\big)\otimes \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A},Q\mathcal{B})\oplus \\
&\qquad \oplus \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A},Q\mathcal{B})\otimes \mathcal{D}\text{-Mod-}\mathcal{D}\big(T\mathcal{A}, T({}_{\tilde{g}}\mathcal{B}_{\tilde{g}})\big) \\
&\xrightarrow{\otimes_{\mathcal{D}}} \mathcal{D}\text{-Mod-}\mathcal{D}\big(T\mathcal{A}\otimes_{\mathcal{D}} T\mathcal{A}, T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}})\otimes_{\mathcal{D}} Q\mathcal{B}\big)\oplus \\
&\qquad \oplus \mathcal{D}\text{-Mod-}\mathcal{D}\big(T\mathcal{A}\otimes_{\mathcal{D}} T\mathcal{A}, Q\mathcal{B}\otimes_{\mathcal{D}} T({}_{\tilde{g}}\mathcal{B}_{\tilde{g}})\big) \\
&\xrightarrow{\mathcal{D}\text{-Mod-}\mathcal{D}(\Delta,1)} \mathcal{D}\text{-Mod-}\mathcal{D}\big(T\mathcal{A}, T({}_{\tilde{f}}\mathcal{B}_{\tilde{f}})\otimes_{\mathcal{D}} Q\mathcal{B}\oplus Q\mathcal{B}\otimes_{\mathcal{D}} T({}_{\tilde{g}}\mathcal{B}_{\tilde{g}})\big).
\end{aligned}$$

The abelian group $\mathrm{Coder}(f,g)^n$ equals the abelian group

$$\big\{ r \in \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, Q\mathcal{B})^n \mid r \centerdot \Delta = \Delta \centerdot (f \otimes_{\mathcal{D}} r \oplus r \otimes_{\mathcal{D}} g) \\ : T\mathcal{A} \to T({}_{\tilde f}\mathcal{B}_{\tilde f}) \otimes_{\mathcal{D}} Q\mathcal{B} \oplus Q\mathcal{B} \otimes_{\mathcal{D}} T({}_{\tilde g}\mathcal{B}_{\tilde g}) \big\}.$$

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Proposition

The abelian group $\mathrm{Coder}(f,g)^n$ *is isomorphic to* $\mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, {}_{\tilde{f}}\mathcal{B}_{\tilde{g}})^n$*: for an element* $r \in \mathrm{Coder}(f,g)^n$ *we have*

$$r = \Delta^{(3)} \centerdot (f \otimes_{\mathcal{D}} \check{r} \otimes_{\mathcal{D}} g) : T\mathcal{A} \to Q\mathcal{B}.$$

On the other hand, for any $\check{r} \in \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, {}_{\tilde{f}}\mathcal{B}_{\tilde{g}})^n$ *this equation determines an element* $r \in \mathrm{Coder}(f,g)^n$*.*

Theorem

The morphism of $\mathcal{V}$

$$\chi = \Big[\mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, {}_{\tilde f}\mathcal{B}_{\tilde g}) \cong \mathbb{1} \otimes \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, {}_{\tilde f}\mathcal{B}_{\tilde g}) \otimes \mathbb{1}$$

$$\xrightarrow{\dot f \otimes 1 \otimes \dot g} \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, T({}_{\tilde f}\mathcal{B}_{\tilde f})) \otimes \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, {}_{\tilde f}\mathcal{B}_{\tilde g}) \otimes$$

$$\otimes \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, T({}_{\tilde g}\mathcal{B}_{\tilde g}))$$

$$\xrightarrow{\otimes_{\mathcal{D}}^{(3)}} \mathcal{D}\text{-Mod-}\mathcal{D}((T\mathcal{A})^{\otimes_{\mathcal{D}} 3}, T({}_{\tilde f}\mathcal{B}_{\tilde f}) \otimes_{\mathcal{D}} {}_{\tilde f}\mathcal{B}_{\tilde g} \otimes_{\mathcal{D}} T({}_{\tilde g}\mathcal{B}_{\tilde g}))$$

$$\xrightarrow{\mathcal{D}\text{-Mod-}\mathcal{D}(\Delta^{(3)}, 1)} \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, Q\mathcal{B})\Big]$$

induces an isomorphism for an arbitrary set-like category $\mathcal{V}$:

$$\mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, {}_{\tilde f}\mathcal{B}_{\tilde g}) \cong \mathrm{Coder}(f, g).$$

There is a morphism

$$\mathrm{Coder}(f, g) \to \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, T\mathcal{B}).$$

A_∞-categories

Let $\mathcal{A}$ be a $\mathcal{D}$-bimodule equipped with a degree 1 morphism $\check{b} : T^{>0}\mathcal{A} \to \mathcal{A}$, which we identify with $T\mathcal{A} \xrightarrow{\mathrm{pr}} T^{>0}\mathcal{A} \xrightarrow{\check{b}} \mathcal{A}$, that is, an element $\check{b} \in \mathcal{V}(\mathbb{1}[-1], \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, \mathcal{A}))$ such that $\check{b}|_{T^0\mathcal{A}} = 0$. There is a $(1,1)$-coderivation of degree 1, $b : 1 \to 1 : T\mathcal{A} \to Q\mathcal{A}$, given informally as $b = \Delta^{(3)} \centerdot (1 \otimes_{\mathcal{D}} \check{b} \otimes_{\mathcal{D}} 1) : T\mathcal{A} \to Q\mathcal{A}$. Formally, $b = \check{b}.\mathcal{V}(1, \chi)$, where

$$\mathcal{V}(1,\chi) : \mathcal{V}(\mathbb{1}[-1], \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, \mathcal{A})) \\ \to \mathcal{V}(\mathbb{1}[-1], \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, Q\mathcal{A})).$$

The morphism $\chi : \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, \mathcal{A}) \to \mathcal{D}\text{-Mod-}\mathcal{D}(T\mathcal{A}, Q\mathcal{A})$ is given in Theorem. Acting with the multiplication map $\mu^{(3)} : Q\mathcal{A} \to T\mathcal{A}$ we produce a degree 1 morphism $\bar{b} = b.\mathcal{V}(1, \mu^{(3)}) : T\mathcal{A} \to T\mathcal{A}$.

Its value is computed due to expression of b:

$$\bar{b} \centerdot \mathrm{pr}_I = b \centerdot \mu^{(3)} \centerdot \mathrm{pr}_I$$
$$= \bigl(T\mathcal{A} \xrightarrow{\Delta^{(I)}} (T\mathcal{A})^{\otimes_{\mathcal{D}} I} \xrightarrow{\sum_{q+1+t=I} 1^{\otimes_{\mathcal{D}} q} \otimes_{\mathcal{D}} \check{b} \otimes_{\mathcal{D}} 1^{\otimes_{\mathcal{D}} t}} T^I \mathcal{A} \bigr).$$

Its value is computed due to expression of b:

$$\bar{b} \centerdot \mathrm{pr}_l = b \centerdot \mu^{(3)} \centerdot \mathrm{pr}_l$$
$$= \left(T\mathcal{A} \xrightarrow{\Delta^{(l)}} (T\mathcal{A})^{\otimes_{\mathcal{D}} l} \xrightarrow{\sum_{q+1+t=l} 1^{\otimes_{\mathcal{D}} q} \otimes_{\mathcal{D}} \check{b} \otimes_{\mathcal{D}} 1^{\otimes_{\mathcal{D}} t}} T^l\mathcal{A} \right).$$

Remark

The expression $\bar{b}^2 : T\mathcal{A} \to T\mathcal{A}$ equals $r \centerdot \mu^{(3)}$ for a degree 2 (1,1)-coderivation $r : T\mathcal{A} \to Q\mathcal{A}$ with

$$r_k = \sum_{y+p+z=k} (1^{\otimes_{\mathcal{D}} y} \otimes_{\mathcal{D}} b_p \otimes_{\mathcal{D}} 1^{\otimes_{\mathcal{D}} z}) \centerdot b_{y+1+z}$$
$$= b \centerdot \mu^{(3)} \centerdot \check{b}|_{T^k\mathcal{A}} : T^k\mathcal{A} \to \mathcal{A}.$$

The $\mathcal{V}$-category $\mathcal{B}$-Mod-$\mathcal{C}$ is tensored over $\mathcal{V}$ on the left and on the right. *E.g.* the right action

$$\mathcal{B}\text{-Mod-}\mathcal{C} \boxtimes \mathcal{V} \to \mathcal{B}\text{-Mod-}\mathcal{C}, \quad (M, X) \mapsto M \otimes X,$$
$$(M \otimes X)(A, B) = M(A, B) \otimes X.$$

In a particular case we consider the right action of $\mathbb{1}[1] \in \operatorname{Ob} \mathcal{V}$ on $\mathcal{A} \in \operatorname{Ob} \mathcal{D}\text{-Mod-}\mathcal{D}$:

$$s\mathcal{A} \overset{\text{def}}{=} \mathcal{A}[1] \overset{\text{def}}{=} \mathcal{A} \otimes (\mathbb{1}[1]).$$

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Definition

The pair $(\mathcal{A}, \check{b} : Ts\mathcal{A} \to s\mathcal{A})$, $\deg \check{b} = 1$, $b_0 = 0$, is called an enriched A_∞-category if $\bar{b}^2 = 0$. Equivalently, if $b \centerdot \mu^{(3)} \centerdot \check{b} = 0$, see the previous slide.

Example

Let $\mathcal{D}$ be a small $\mathcal{V}$-category and let $\mathcal{A}$ be a differential monoid in $\mathcal{D}$-Mod-$\mathcal{D}$. That is, degree 0 morphism $m_2 : \mathcal{A} \otimes_{\mathcal{D}} \mathcal{A} \to \mathcal{A}$ is an associative multiplication, degree 0 morphism $\eta : y_{\mathcal{D}} \to \mathcal{A}$ is its unit (where $y_{\mathcal{D}}$ is a regular bimodule), degree 1 differential $d : \mathcal{A} \to \mathcal{A}$ (an element of $\mathcal{V}(\mathbb{1}[-1], \mathcal{D}\text{-Mod-}\mathcal{D}(\mathcal{A},\mathcal{A})) \cong \overline{\mathcal{D}\text{-Mod-}\mathcal{D}}(\mathcal{A}[-1],\mathcal{A}))$ is a derivation and $d^2 = 0$. We convert $\mathcal{A}$ into an enriched A_∞-category by setting

$$b_2 = -\big(\mathcal{A}[1] \otimes_{\mathcal{D}} \mathcal{A}[1] \xrightarrow{\sigma^{-1}\otimes_{\mathcal{D}}\sigma^{-1}} \mathcal{A} \otimes_{\mathcal{D}} \mathcal{A} \xrightarrow{m_2} \mathcal{A} \xrightarrow{\sigma} \mathcal{A}[1]\big),$$

$$b_1 = \big(\mathcal{A}[1] \xrightarrow{\sigma^{-1}} \mathcal{A} \xrightarrow{d} \mathcal{A} \xrightarrow{\sigma} \mathcal{A}[1]\big).$$

Other components b_n vanish. One checks that $\bar{b}^2 = 0$.

Thank you for your attention!