Open r-spin Theories with Multiple Boundary States

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Joint with
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- In 91', motivated by 2D quantum gravity, Witten conjectured that the intersection theory on the moduli of curves $\overline{\mathcal{M}}_{g,n}$ should give rise to a tau function of the KdV integrable hierarchy.
- This conjecture was proven a year later by Kontsevich, and was one of the earliest results in Gromov-Witten theory.

- The KdV hierarchy is the first ($r = 2$ case) in a family of integrable hierarchies called the Gelfand-Dickey hierarchies.
- It was natural to ask if these also correspond to intersection theories.
- And indeed, it turned out (conjectured by Witten, proved in a joint effort of many authors) that the intersection theory on the moduli of $r-$spin curves ($r \geq 2$) gives rise to a $r-$GD tau function.
- This theory was later vastly generalized by Fan, Jarvis and Ruan to the FJRW theory of quantum singularity, and was seen to satisfy interesting mirror symmetry.

Recently open analogs of the above theories were constructed:

- Pandharipande-Solomon-T, Solomon-T,T,Buryak-T constructed intersection theory on the moduli of surfaces with boundaries. It was shown that the open partition function is a KdV wave function.
- Later Buryak-Clader-T found an open $r-$spin construction in $g=0$. It was shown to be related the GD hierarchy and to be mirror dual Saito's theory of A_{r-1} singularity.
- The open $r-$spin theory was defined with a single boundary state (out of $\lfloor \frac{r}{2} \rfloor$ expected by physicists) and only in $g=0$.
- I will report on a progress, with Yizhen Zhao, on defining open $r-$spin theories with multiple boundary twists, and $g=1$.

- A (stable) twisted $r-$spin curve $(\Sigma, \mathcal{S}, z_1, \ldots, z_n)$ is a (stable) curve $(\Sigma, z_1, \ldots, z_n)$ together with a line bundle $\mathcal{S}$ which satisfies

$$\mathcal{S}^{\otimes r} \simeq \omega(-\sum_{i=1}^{n} a_i z_i).$$

- The integers $a_i \in \{0, \ldots, r-1\}$ are called the *twists*.
- A point with twist $r-1$ is called *Ramond* and otherwise *Neveu-Schwarz*.
- Half nodes inherit twists. The twists at two branches of the same node sum to $r-2$ $(mod\ r)$.

- Denote by $\overline{\mathcal{M}}^{\frac{1}{r}}_{g,a_1,\ldots,a_n}$ the *moduli space* of stable $r-$spin curves of genus g and twists a_i.
- For $a_i \geq 0$, this space is non empty exactly when $\sum a_i = (1-g)(r-2)\ (mod\ r)$.
- When non empty, the space $\mathcal{W} \to \overline{\mathcal{M}}^{\frac{1}{r}}_{0,a_1,\ldots,a_n}$, whose fiber over $(\Sigma, \mathcal{S}, z_1, \ldots, z_n)$ is $H^1(\mathcal{S})^*$ is a vector bundle. One can define a virtual analog in high genus.

- Let $\mathbb{L}_i$ be the line bundles over $\overline{\mathcal{M}}^{\frac{1}{r}}_{g,a_1,\ldots,a_n}$ whose fiber over $(\Sigma,\mathcal{S},z_1,\ldots,z_n)$ is $T^*_{z_i}\Sigma$.
- One may consider integrals

$$\left\langle\tau^{a_1}_{d_1}\tau^{a_2}_{d_2}\cdots\tau^{a_l}_{d_l}\right\rangle^{c,1/r}_g = r^{1-g}\int_{\overline{\mathcal{M}}^{\frac{1}{r}}_{g,a_1,\ldots,a_n}} c_W \cup e\Big(\bigoplus_{i=1}^n \mathbb{L}_i^{\oplus d_i}\Big).$$

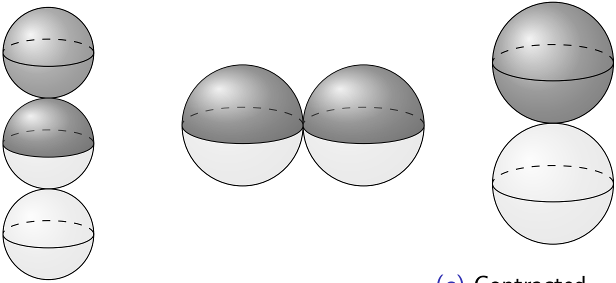
  where c_W is the virtual generalization of $e(\mathcal{W})$ to high genus.
- Let $F^{c,1/r}$ be the generating function of those integrals with all $a_i \in \{0,\ldots,r-1\},\ d_i \geq 0$.

- In 93' Witten conjectured that some simple variant of $F^{c,1/r}$, the $r-$spin potential (generating function), is a $r^{th}-$Gelfand-Dickey tau-function.
- In particular there is a Ramond vanishing phenomenon: All intersection numbers with Ramond insertions vanish.
- The motivation came from $2-$dimensional quantum gravity models coupled to matter.
- Mathematically this is an intersection theory, or curve counting theory, on the moduli space of solutions to Witten's equation.

- The rigorous consturction of the $r-$spin theory was involved, and was a joint effort of many mathematicians, Abramovich, Jarvis, Kimura, Polishchuk, Vaintrob, Vistoly...
- The conjecture was proven by Faber, Shadrin and Zvonkine.
- The $r-$spin intersection theory has initiated the development of FJRW theory, by Fan, Jarvis and Ruan, which defines and analyzes intersection theories on moduli spaces of curves with generalized spin structures.
- Those theories were found to be related to other curve counting theories and to have interesting mirror duals.

- A smooth marked disk is a tuple $(\Sigma, z_1, \ldots, z_l, x_1, \ldots, x_k)$ where
  - Σ is a disk with complex structure holomorphically equivalent to the standard unit disk,
  - $\{z_i\}_{i=1}^{l}$ are distinct internal marked points,
  - $\{x_i\}_{i=1}^{k}$ are distinct boundary marked points.
- A stable marked disk is a collection of smooth marked spheres and disks connected by simple nodes such that after smoothing the nodes a disk is obtained, and subject to some stability constraint.

- There are three types of nodes: internal and boundary nodes and contracted boundaries.
- The doubling map:



(a) Internal node

(b) Boundary node

(c) Contracted boundary

- Consider now a smooth $r-$spin disk, i.e., a marked disk $(\Sigma, z_1, \ldots, z_l, x_1, \ldots, x_k)$ together with an $r-$spin bundle $\mathcal{S}$ (on the doubled surface) with
  1. $\mathcal{S}^{\otimes r} = \omega(-\sum_{i=1}^{l} a_i(z_i + \bar{z}_i) - \sum_{i=1}^{k} b_i x_i)$,
  2. An involution $\tilde{\phi} : \mathcal{S} \to \mathcal{S}$ lifting $d\phi$ ($\tilde{\phi}^r = d\phi$).
- This structure exists when

$$e := \frac{2\sum_{i=1}^{l} a_i + \sum_{i=1}^{k} b_i + (2-r)}{r} \in \mathbb{Z}.$$

It turns out that in order to define the open intersection theory, one has to work with *graded spin disks*, meaning spin disks with

1. A choice of a positive real direction of the spin line for any boundary interval between marked points, that does not extend to the markings ("alternates" at each marking).
2. all $b_i = r-2$ (BCT theory), or $b_i = r-2, r-4, \ldots, r-2-2h$ (TZ theory for $h = 0, \ldots, \lfloor \frac{r-2}{2} \rfloor$).

The first item is non trivial only for even r, and implies (for $g=0$)

$$e = \frac{2\sum_{i=1}^{l} a_i + \sum_{i=1}^{k} b_i + (2-r)}{r} \equiv k+1 \ (mod\ 2).$$

- The above notions extend to the nodal case, to give stable $r-$spin disks.
- The twists at two opposite half nodes sum to $r-2$ $(mod\ r)$.
- The twist at a contracted component is always $r-1$.

- Let $\overline{\mathcal{M}}^{1/r}_{0,b_1,\ldots,b_k;a_1,\ldots,a_l}$ be the moduli space of graded $r-$spin disks with *twists* $b_1,\ldots,b_k,a_1,\ldots,a_l$.
- A compact smooth orbifold with corners of real dimension

$$k+2l-3.$$

- Orientable, though not canonically oriented in general.
- Topologically this moduli is identified with its spinless version $\overline{\mathcal{M}}_{0,k,l}$, (the moduli of disks with k boundary markings and l internal markings) but not as an orbifold.

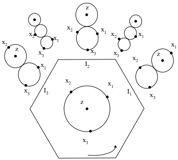


Figure: One component of $\overline{\mathcal{M}}^{1/2}_{0,3,\{a_1=0\}}$.

- The *open Witten bundle* $\mathcal{W} \to \overline{\mathcal{M}}^{1/r}_{0,b_1\ldots,b_k;a_1,\ldots,a_l}$ is the *real* bundle whose fiber is

$$\mathcal{W}_{(\Sigma,\vec{z},\vec{x},\mathcal{S},\tilde{\phi})} = (H^1(\mathcal{S})^*)^{\tilde{\phi}}.$$

- By Serre's duality one can canonically identify $\mathcal{W}$ with the bundle whose fibers are $H^0(\mathcal{J})^{\tilde{\phi}}$ where $\mathcal{J} = \omega \otimes \mathcal{S}^\vee$.
- It is useful to think of an element $v \in \mathcal{W}_{(\Sigma,\vec{z},\vec{x},\mathcal{S},\tilde{\phi})}$ as a *real* global section of $\mathcal{J}$.
- With this interpretation, for $p \in \Sigma$ one can define the evaluation map

$$ev_p : \mathcal{W}_\Sigma \to \mathcal{J}_p \quad \text{(the image is in } \mathcal{J}_p^{\tilde{\phi}} \text{ if } p \in \partial\Sigma).$$

Theorem (Buryak-Clader-T,T-Zhao)

$\mathcal{W} \to \overline{\mathcal{M}}^{1/r}_{0,b_1\ldots,b_k;a_1,\ldots,a_l}$ *is canonically relatively oriented.*

- The relative cotangent line bundle $\mathbb{L}_i \to \overline{\mathcal{M}}^{1/r}_{0,k,a_1,\ldots,a_l}$ is the complex line whose fiber at $(\Sigma, \vec{z}, \vec{x}, \mathcal{S}, \tilde{\phi})$ is $T^*_{z_i}\Sigma$.
- Alternatively, the bundle can be defined by pull back along the doubling map.
- $\mathbb{L}_i$ is endowed with a canonical complex orientation.
- Both the Witten bundle and the tautological lines satisfy nice functorial properties.

- A natural interpretation of the Euler class of a vector bundle is as the Poincare dual for the zero locus of a transverse section or multisection of the bundle.
- For example, in the closed setting, one can write

$$\left\langle \tau^{a_1}_{d_1}\tau^{a_2}_{d_2}\cdots\tau^{a_l}_{d_l}\right\rangle^{c,1/r}_0 = r\#Z(s\oplus\bigoplus_{i=1}^{l}\bigoplus_{j=1}^{d_i}s_{ij}),$$

where s is a multisection of $\mathcal{W}$ and each s_{ij} is a multisection of $\mathbb{L}_i$, and some mutual transversality is assumed.

- One could have tried to define open intersection numbers this way, as a zero count.
- The problem is that such intersection numbers **will depend on the choice of boundary conditions**.
- This problem appears because the moduli space has real codimension 1 boundaries. In a sense, that's the key problem in Open Gromov Witten theory in general.
- Are there natural, geometrical meaningful, boundary conditions for this problem?

- To illustrate the problem, and the effect of boundary conditions, let's consider the case $\underline{C} \to \overline{\Delta}$, where $\overline{\Delta}$ is the closed unit disk and $\underline{\mathbb{C}}$ is the trivial complex line.
- The weighted number of internal zeroes of a section s depends only on $s|_{\partial\overline{\Delta}}$, assuming the latter is nowhere vanishing, and equals the winding number.
- One can compare the number of zeroes of two sections s_1, s_2 by performing a transverse homotopy between $s_1|_{\partial\overline{\Delta}}, s_2|_{\partial\overline{\Delta}}$ and count its zeroes.

Examples:

1. First, if $s|_{\partial\overline{\Delta}}$ is any nonzero constant, the number of zeroes is fixed (0, actually).
2. Another way to give a family of sections with the same zero count, is to choose continuously a half-plane $\mathbb{H}_p$ for each boundary point p, and require that $s_p \in \mathbb{H}_p$.

- Buryak-Clader-T imposed boundary conditions of two types: forgetful b.c and positivity b.c.
- The first is analogous to the above example of $s|_{\partial\overline{\Delta}}$ is constant, the second to the half planes example.
- A multisection **s** of

$$\mathcal{W} \oplus \bigoplus_{i=1}^{l} \mathbb{L}_i^{\oplus d_i} \to \overline{\mathcal{M}}^{1/r}_{0,b_1\ldots,b_k;a_1,\ldots,a_l}$$

is *canonical* if it satisfies the two boundary conditions.

Theorem (Buryak-Clader-T)

When

$$rk\left(\mathcal{W}\oplus\bigoplus_{i=1}^{l}\mathbb{L}_i^{\oplus d_i}\right)=dim\overline{\mathcal{M}}^{1/r}_{0,k,a_1,\ldots,a_l}$$

one can find transverse canonical multisection **s**, *and hence define*

$$\left\langle\tau^{a_1}_{d_1}\tau^{a_2}_{d_2}\cdots\tau^{a_l}_{d_l}\sigma^k\right\rangle_0^{o,1/r}=\int_{\overline{\mathcal{M}}^{1/r}_{0,k,a_1,\ldots,a_l}}e(\mathcal{W}\oplus\bigoplus_i\mathbb{L}_i^{\oplus d_i},\mathbf{s})=\#Z(\mathbf{s}),$$

The result is independent of the canonical transverse **s**.

Theorem (Buryak-Clader-T)

- $$\left\langle \tau_0^{a_1}\tau_0^{a_2}\cdots\tau_0^{a_l}\sigma^k\right\rangle_0^{o,1/r} = \frac{(k+l-2)!}{(-r)^{l-1}}.$$
- *The open $r-$spin free energy $F_0^{o,1/r}$ (defined analogously to the spinless case) satisfies the open string equation.*
- $\exp(F_0^{c,1/r}+F_0^{o,1/r})$ *is (the genus* 0 *part of) a $r^{th}-$Gelfand-Dickey wave function, after a simple change of variables.*

Theorem (Gross-Kelly-T)

The canonical coordinates of the Saito-Frobenius algebra of the A_{r-1} singularity, written as functions of the flat coordinates, are generating functions of the $g=0$ open $r-$spin numbers.

BCT theory allows only one type of boundary state $r-2$. Can one construct theories with other types of boundary twists?

Theorem (T-Zhao, to appear soon)

There exists a family, indexed by $h = 1, \ldots, \lfloor \frac{r}{2} \rfloor$*, of open* $r-$*spin theories. The* h^{th} *theory allows the boundary twists to range in* $r-2, r-4, \ldots, r-2h$.

- *The* $h = 1$ *theory is equivalent to BCT.*
- *The genus* 0 *open intersection numbers satisfy a TRR which allows their calculation.*

The main new idea was to replace the forgetful b.c. by a *point insertion b.c.*

- The BCT theory and the TZ family of theories satisfy different looking recursions, which allow calculating all $g = 0$ numbers.
- Although there are more boundary twists, in the h^{th} TZ theory most primaries contining *internal twists* smaller than h vanish.
- The point insertion idea can be generalized to open Gromov-Witten setting, under some assumptions on the target pair (M, L) and similar recursions are expected to hold.
- Interestingly, we were able to extend the construction to $g = 1$.

Some questions:

- Calculate the different open $r-$spin theories in $g = 0, 1$; predict high genus; find relations with integrable hierarchies and mirror symmetry.
- Do all the theories carry the same amount of information? Are there transformations between different theories?
- Study more general open FJRW theories based on the new $r-$spins.

Thank you for listening!

- **Positivity boundary conditions:** At the heart of the constructions lies the remarkable fact that *the positivity constraints in high codimensional corners can be simultaneously satisfied*.
- Consider any other codimension 1 boundary $\overline{\mathcal{M}}_\Gamma$. It will either parameterize surface with a **contracted boundary** n_1, or surfaces with a **boundary node** n **such that if** n_1 **is a half-node of it with an even twist** t, **then** $t > 0$.
- In both cases the fiber $\mathcal{J}^{\tilde{\phi}}_{n_1}$ is a real line, oriented by the grading.
- **We say that a multisection** s **of** $\mathcal{W} \to \overline{\mathcal{M}}_\Gamma$ **satisfies the positivity boundary conditions if for any** $p \in \overline{\mathcal{M}}_\Gamma$, $ev_{n_1}(s_p) \in \mathcal{J}^{\tilde{\phi}}_{n_1}$ **is positive.***
- *- I cheat here a bit.

- The consistency of the positivity b.c. is a surprising feature of the open Witten bundle, and that what makes the definition of the open $r-$spin intersection theory possible.
- Why is it surprising?

- Suppose that for a given intersection problem the real rank of the Witten bundle is d, and consider a stratum with m nodes which give rise to m positivity conditions.
- One can show that the positivity conditions are non degenerate in the sense that each of them cuts a half-space out of the fiber of the Witten bundle.
- Simultaneous satisfaction means that all those half spaces have a common intersection. When $m \leq d$ that's reasonable (m "generic" half spaces in $\mathbb{R}^d$ will always have a common intersection).
- For $m > d$ on the other hand, "generic" half spaces need not to have a common intersection. **This is where the fact that the boundary twists are $r-2$ is used.**

- **Forgetful b.c.:** Suppose the boundary stratum $\overline{\mathcal{M}}_\Gamma$ parameterizes stable disks with a **single boundary node** n**, such that one of its half nodes** n_1 **has a twist** 0.
- Let $Detach : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2$ be the map which on the surface level normalizes the node n ($\overline{\mathcal{M}}_1$ is the moduli which parameterizes the component of the normalized surfaces which contain n_1).
- Let $For_{n_1} : \overline{\mathcal{M}}_1 \to \overline{\mathcal{M}}'_1$ be the map which forgets n_1, and set

$$\pi = (For_{n_1} \times id) \circ Detach : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}'_1 \times \overline{\mathcal{M}}_2$$

- We have $\mathcal{W} \to \overline{\mathcal{M}}_\Gamma \cong \pi^* \left(\mathcal{W}'_1 \boxplus \mathcal{W}_2\right) = \pi^*(\mathcal{W} \to \overline{\mathcal{M}}'_1 \times \overline{\mathcal{M}}_2)$. The same holds for $\mathbb{L}_i \to \overline{\mathcal{M}}_\Gamma$, except for one irrelevant case.
- **A multisection** s **of** $\mathcal{W} \to \overline{\mathcal{M}}_\Gamma$ **or** $\mathbb{L}_i \to \overline{\mathcal{M}}_\Gamma$ **satisfies the forgetful boundary conditions if** $s = \pi^* s'$ **for some** s'.

- Assuming the constructions, the reason intersection numbers are well defined (independent of choices) is as follows.
- One can perform an homotopy between two different canonical choices, and count zeroes.
- Along boundaries with positivity b.c. a simple linear homotopy will not add zeroes, by positivity.
- In the BCT case there are additional boundaries with forgetful b.c., if we construct an homotopy which is itself pulled-back from the moduli after forgetting the half node, then it can be considered as a section of a bundle E of rank $k+2l-3$ over a moduli $[0,1]\times\overline{\mathcal{M}}'_1\times\overline{\mathcal{M}}_2$ of dimension $k+2l-4$. A transverse homotopy of this type will not vanish as well.