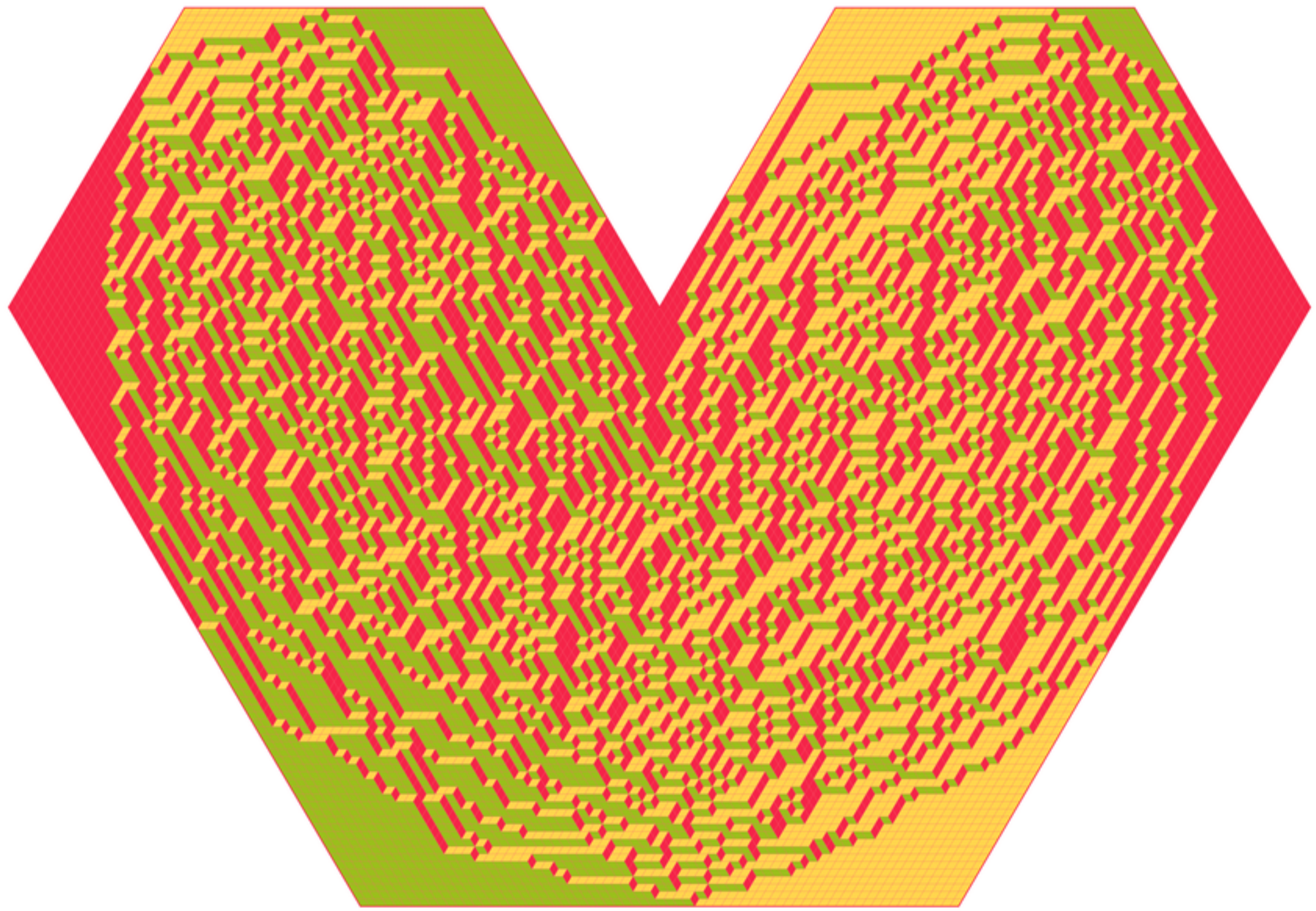
Large random matrices are everywhere



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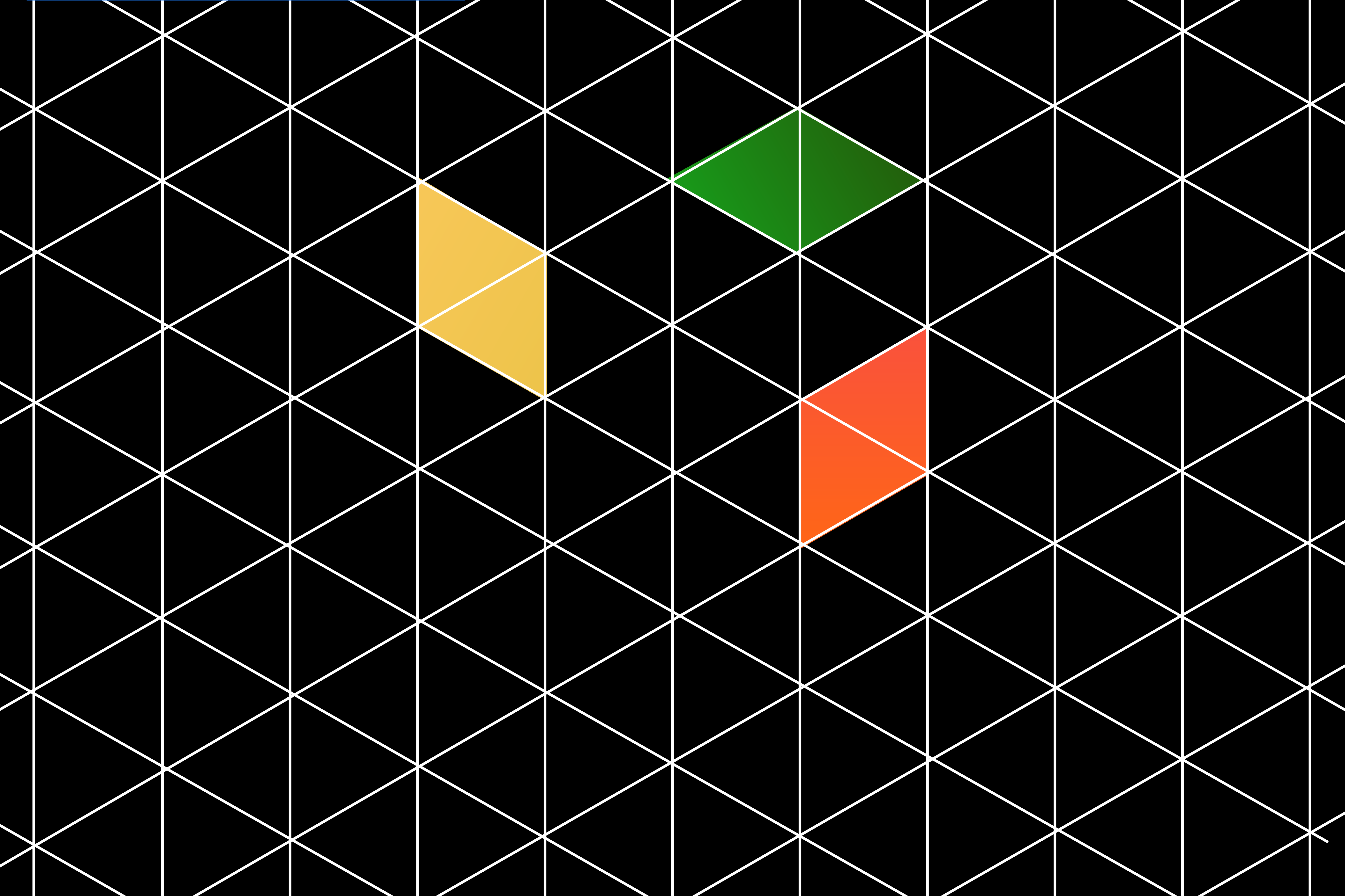
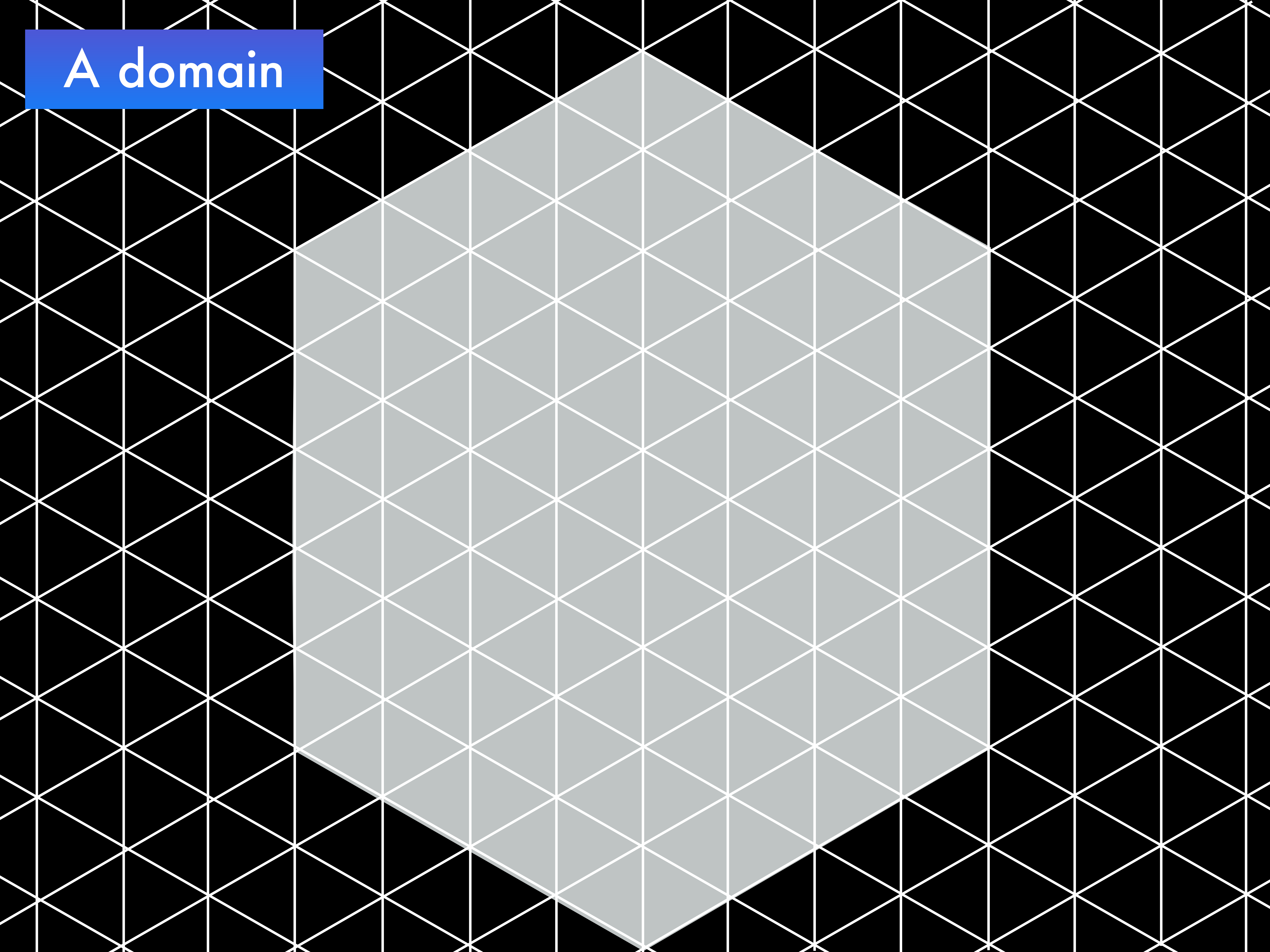
1

Tilings and sugar melting

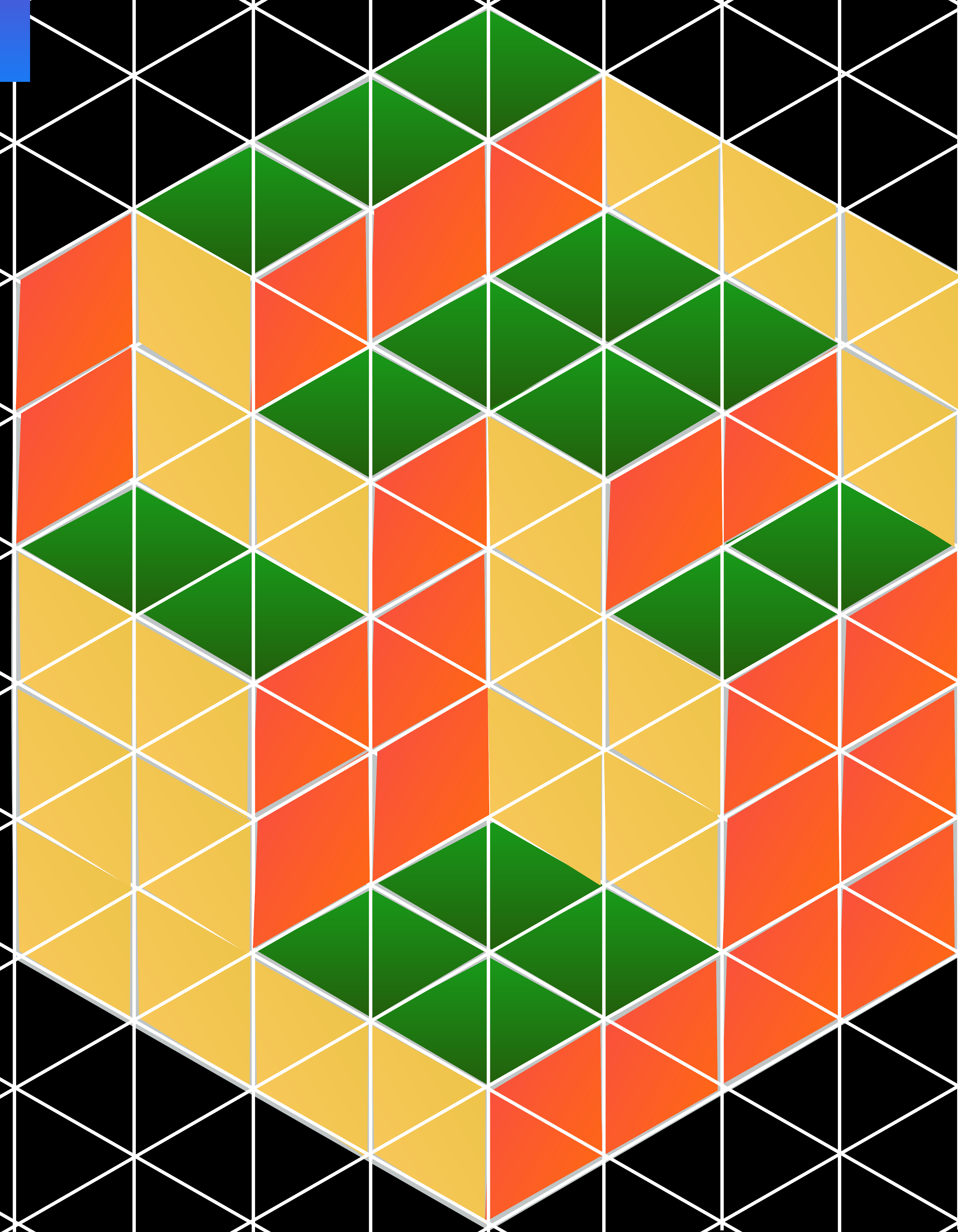
3 types of tiles



A domain

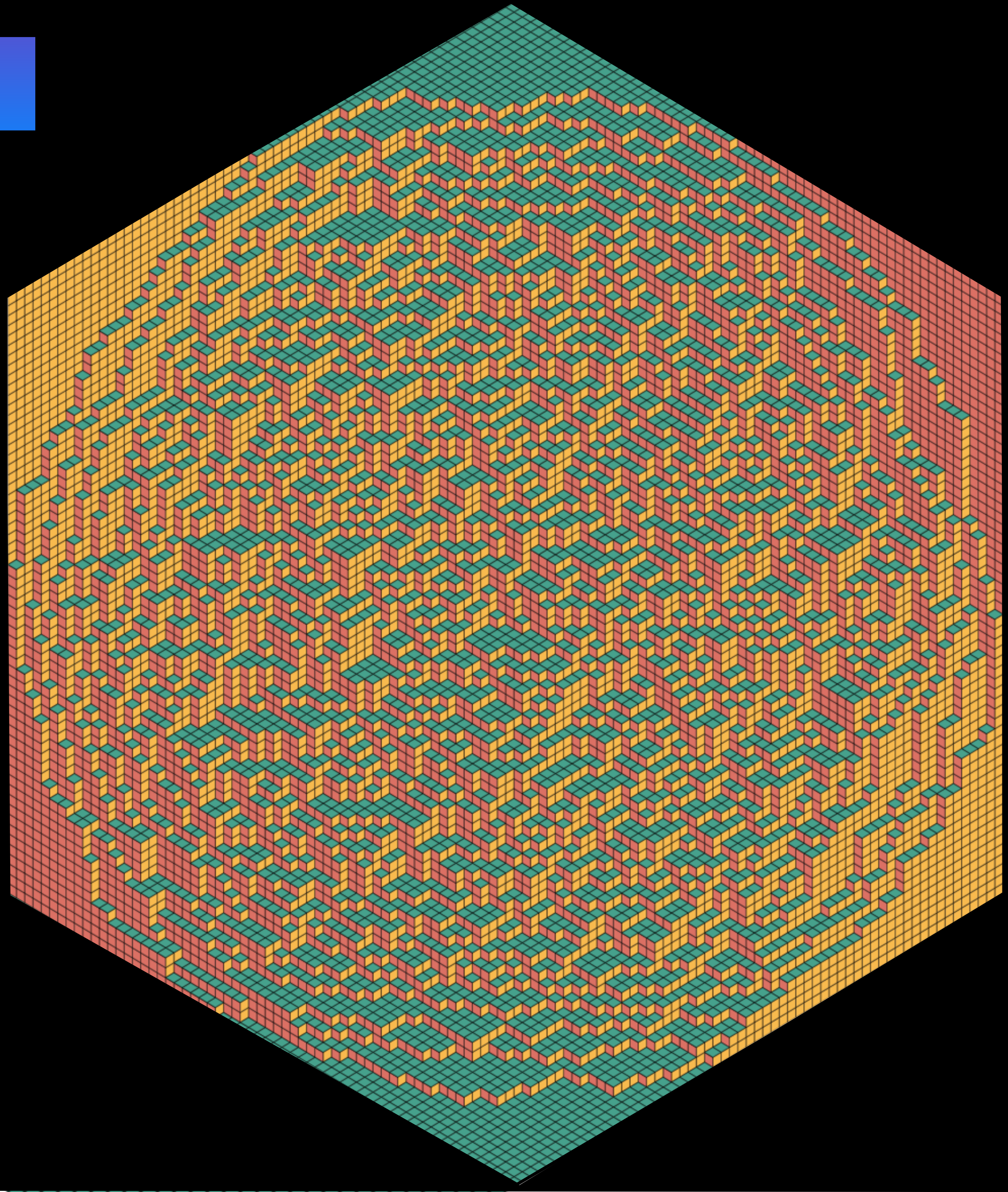


A tiling



A large tiling

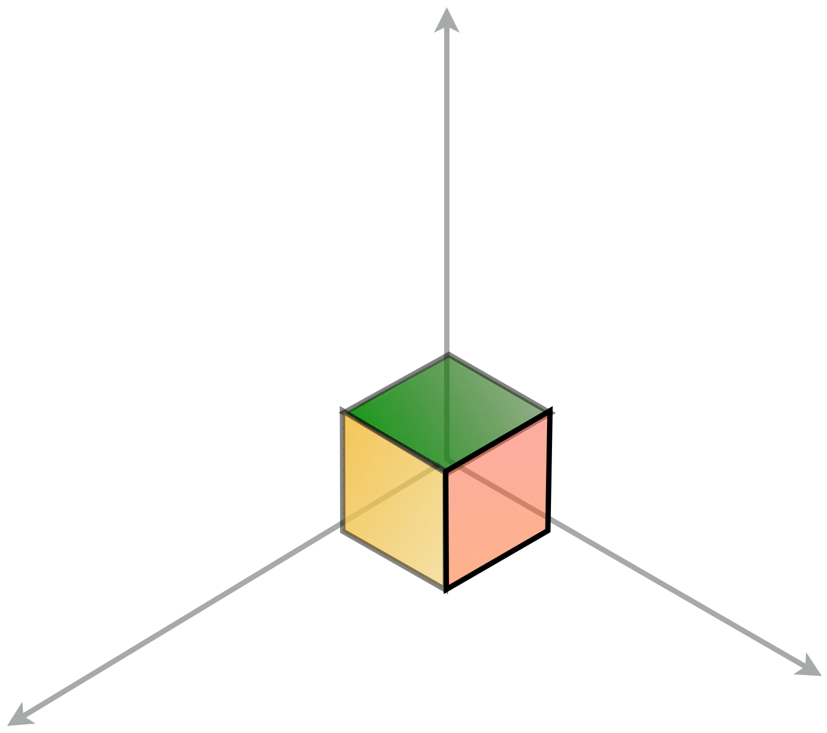
Simulation by
L. Petrov

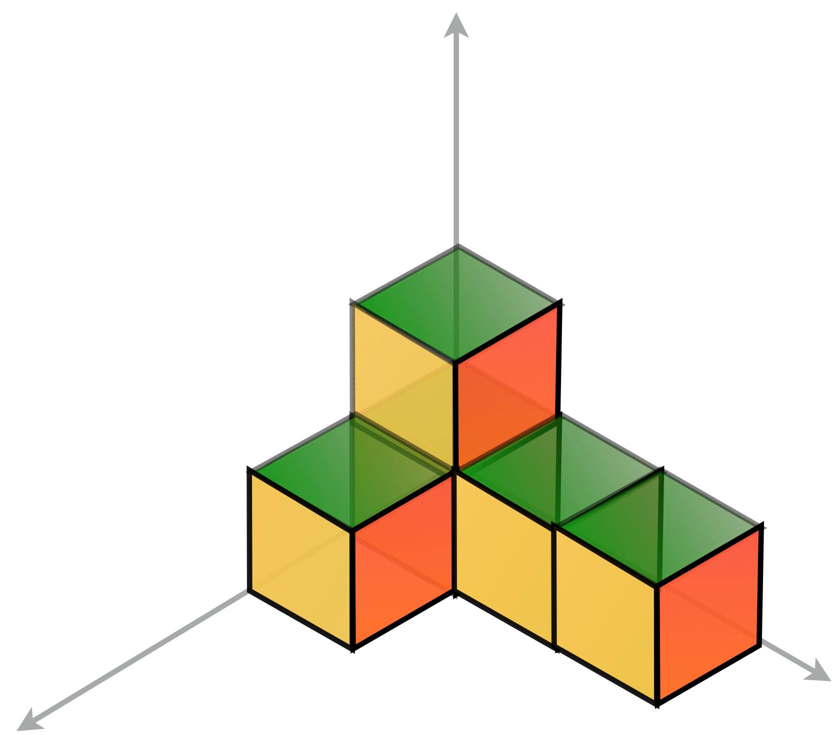


1. Random tilings

- These are 2d projections of a 3d picture

2d tiling $\longleftrightarrow$ Piling up cubes in the corner of a room





1. Random tilings

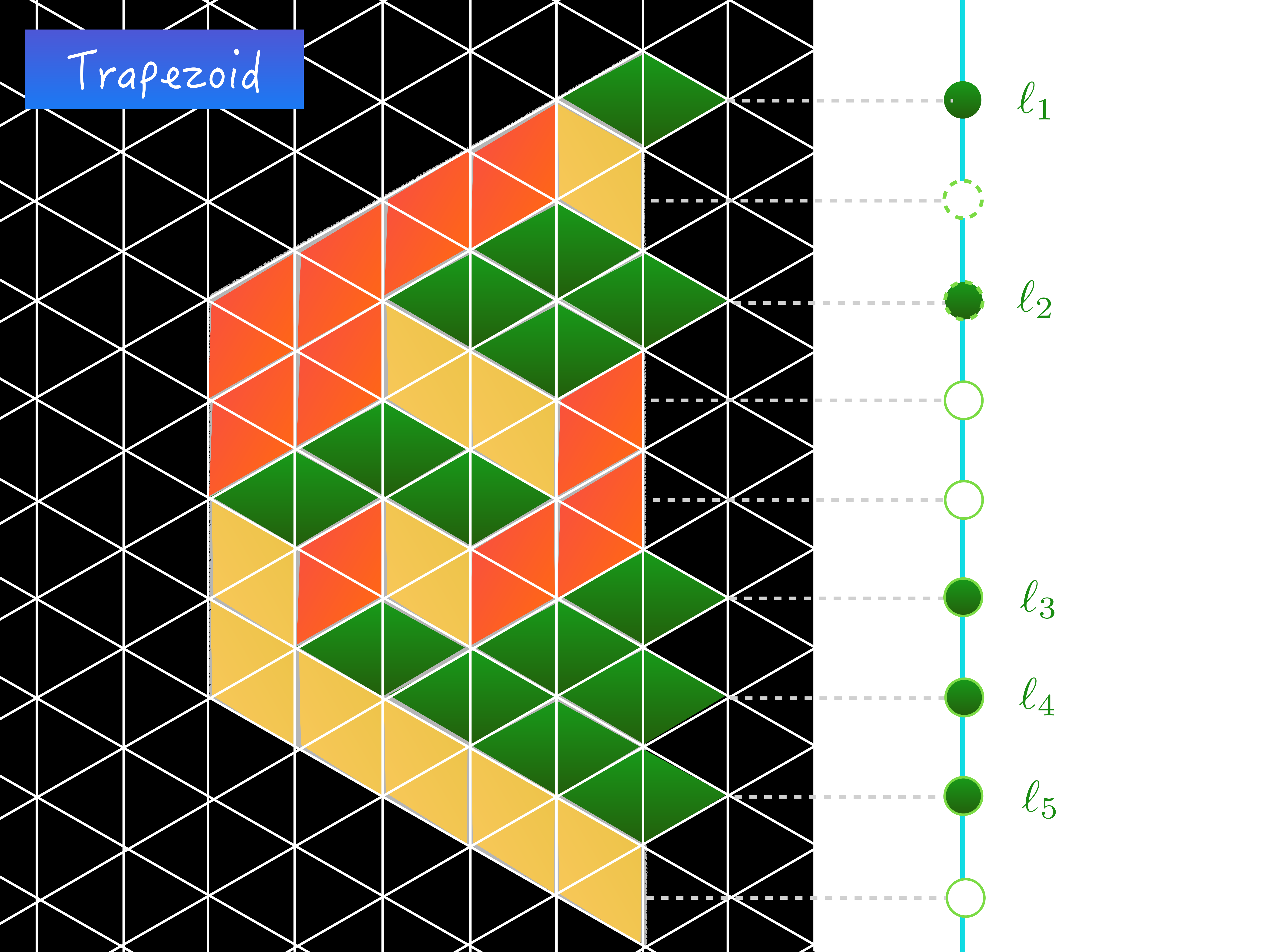
- These are 2d projections of a 3d picture

2d tiling $\longleftrightarrow$ Piling up cubes in the corner of a room

- Model for random piling/sugar melting

- Choose N randomly, with probability proportional to $\exp\left(-\frac{\epsilon N}{k_B T}\right)$
(Boltzmann law: ϵ energy per cube, T temperature)

- Choose a piling with N cubes, randomly


Trapezoid
ℓ_1
ℓ_2
ℓ_3
ℓ_4
ℓ_5

1. Random tilings

(Gelfand-Tsetlin 1950, Cohn-Larsen-Propp 1998)

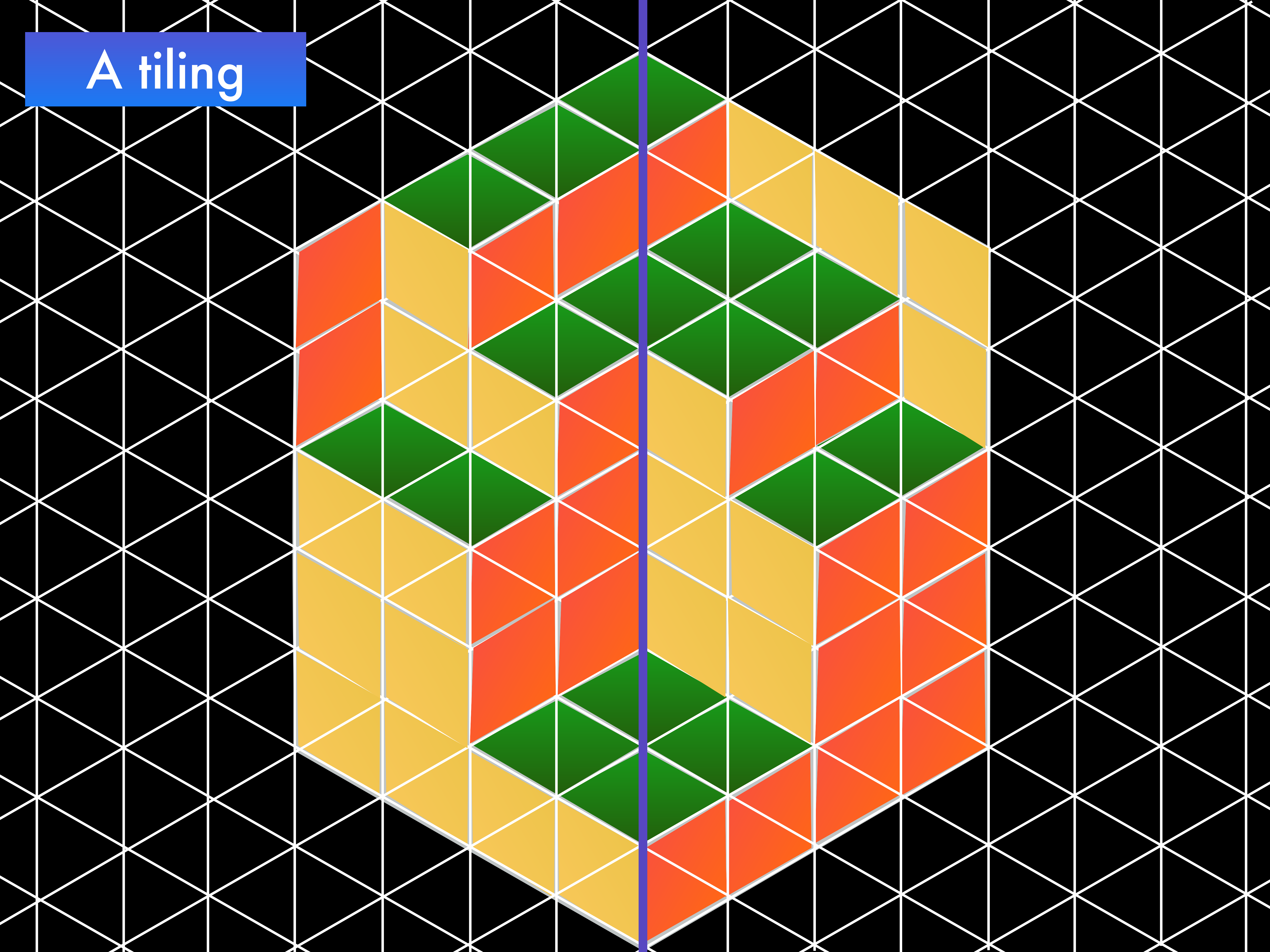
There are $\prod_{i<j} \frac{\ell_i - \ell_j}{i-j}$ tilings of the trapezoid having green tiles sticking out at positions $\ell_1 > \ell_2 > \dots$

$\Longrightarrow$ There are $P(\ell) = \left(\prod_{i<j} \frac{\ell_i - \ell_j}{i-j} \right)^2$ tilings of the hexagon having green tiles crossing the vertical section at positions $\ell_1 > \ell_2 > \dots$

- Random choice of tiling of the hexagon

$\Longrightarrow$ Probability to see $\ell_1, \ell_2, \dots$ is proportional to $P(\ell)$

A tiling



1. Random tilings

Random choice of tiling $$P(\ell) = \left(\prod_{i<j} \frac{\ell_i - \ell_j}{i-j} \right)^2$$

Probability to observe ℓ_i, ℓ_j close to each other is small

$\longrightarrow$ $\ell_1, \ell_2, \dots$ are not independent from each other

They rather tend not to be close to each other !

1. Random tilings

For very large tilings chosen at random

Observation: Arctic circle phenomenon

there is a (non-random) curve c such that
with probability ~1 when N is large

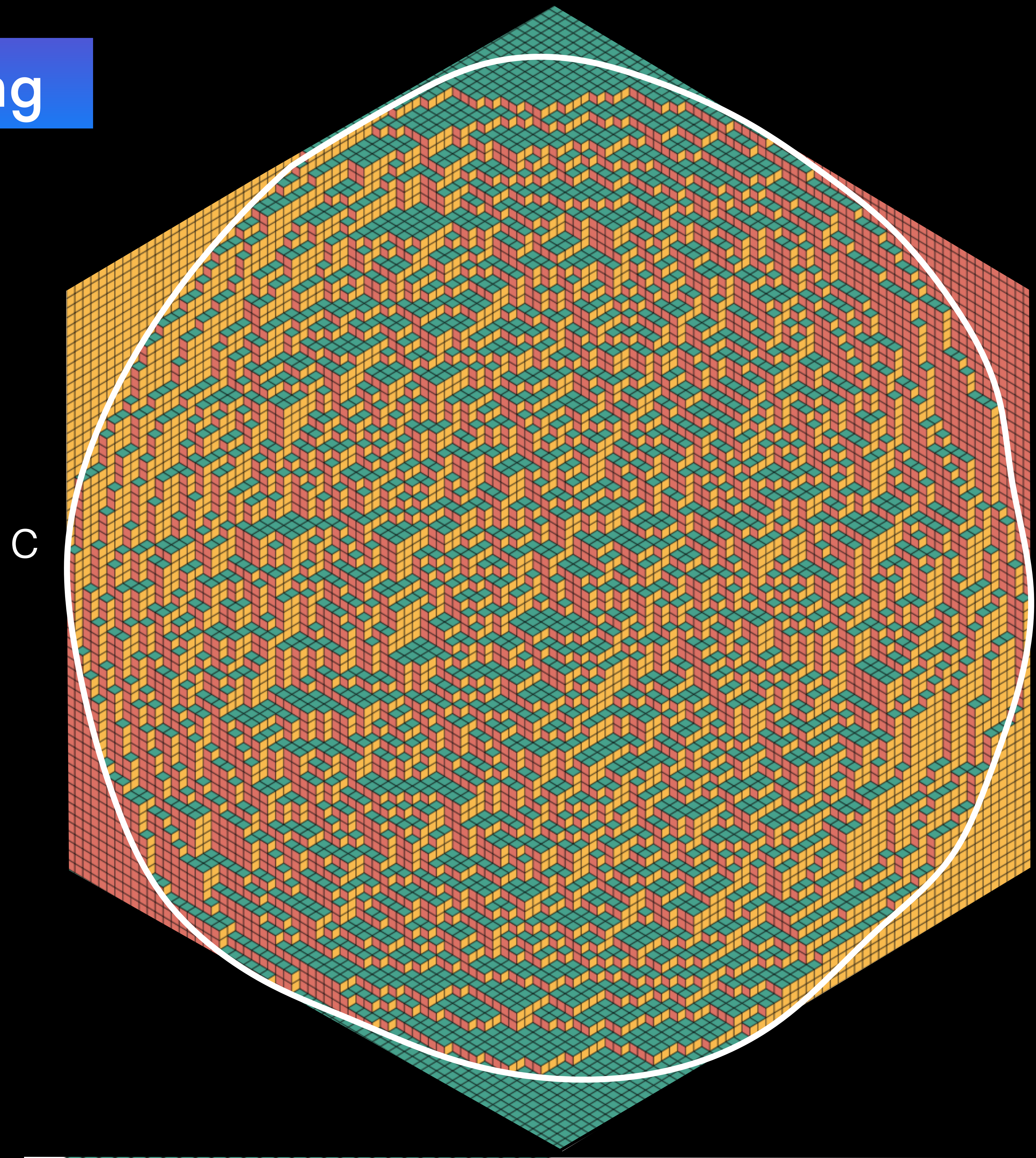
- outside c: frozen tiles
- inside c: fluctuating tiles (surface looks rough)

Questions

- Describe c / the distribution of $\frac{\ell_1}{\sqrt{N}}$
- describe the law of (microscopic) fluctuations inside c

A large tiling

Simulation by
L. Petrov



2

Spectrum of (random) matrices

2. Spectrum of (random) matrices

- A matrix is a table filled with numbers
- A matrix of size n x n represents a linear transformation

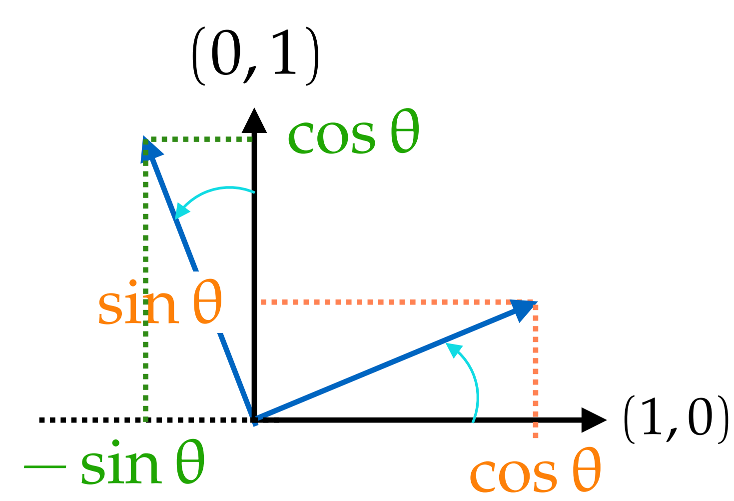
$$\begin{pmatrix} A(1,1) & A(1,2) & \cdots & A(n,1) \\ A(2,1) & A(2,2) & \cdots & A(2,n) \\ \vdots & & & \vdots \\ A(n,1) & A(n,2) & \cdots & A(n,n) \end{pmatrix}$$

$$\begin{aligned} x_1 &\longmapsto A(1,1)x_1 + A(1,2)x_2 + \cdots + A(1,n)x_n \\ x_2 &\longmapsto A(2,1)x_1 + A(2,2)x_2 + \cdots + A(2,n)x_n \\ &\vdots \\ x_n &\longmapsto A(n,1)x_1 + A(n,2)x_2 + \cdots + A(n,n)x_n \end{aligned}$$

- Example: for n = 2

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

represents the rotation of angle θ in the plane

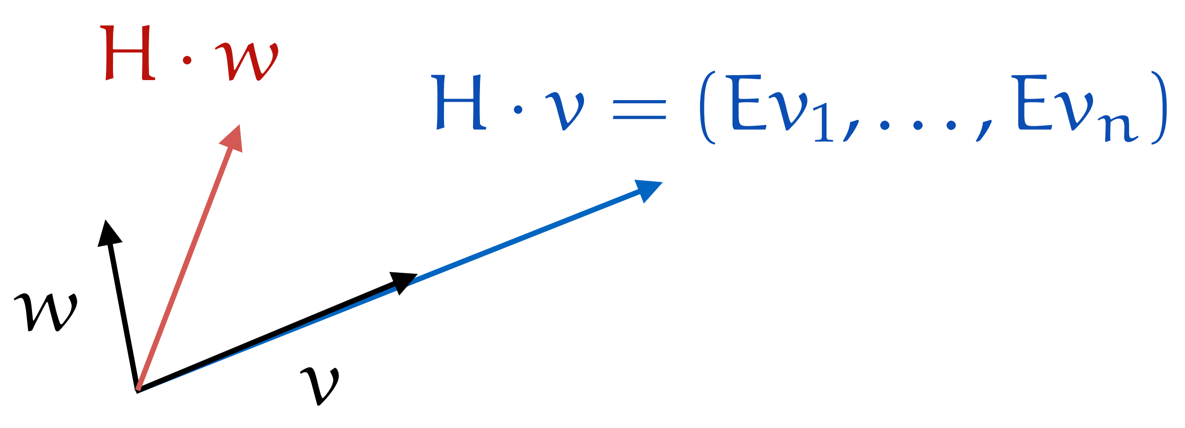


2. Spectrum of (random) matrices

- Given a matrix H of size n x n

  $v = (v_1, \ldots, v_n)$ is an eigenstate with eigenvalue E

  if H acts on v as rescaling by the factor E



- Under certain assumptions on H (hermitian)

  there are exactly n eigenstates (up to scale) and n corresponding eigenvalues $E_1 \geqslant \cdots \geqslant E_n$

= spectrum of H

2. Spectrum of (random) matrices

- **Random** (hermitian) matrix of size n x n

  $\longrightarrow$ **random** spectrum $E_1 \geqslant \cdots \geqslant E_n$

- When the random model does not have a preferred direction

(1900 … Dyson, Wigner, Mehta … 1960)

Probability to find E_i near x_i at precision $\delta \ll 1$

is proportional to $\delta^n \cdot \rho(x) \cdot \prod_{i<j} (x_i - x_j)^2$

Questions When n is large

- How do $E_1 \geqslant \cdots \geqslant E_n$ distribute ?
- How does $E_1 = \max(E_1, \ldots, E_n)$ behave ?

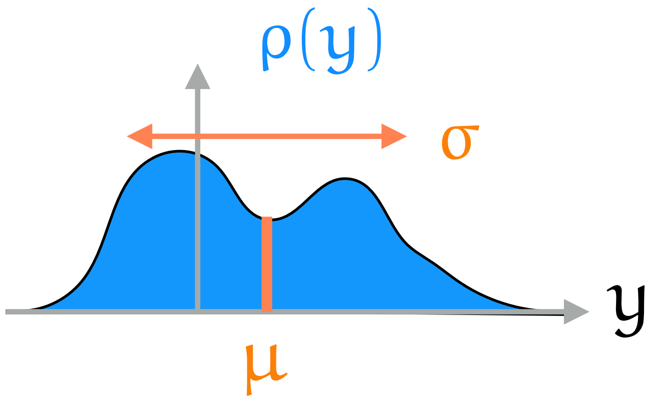
3

What happens for independent random variables ?

3. Independent random variables

Consider a random variable Y

$\mu = \mathrm{Mean}(Y) \qquad \sigma = \mathrm{Variance}(Y)$

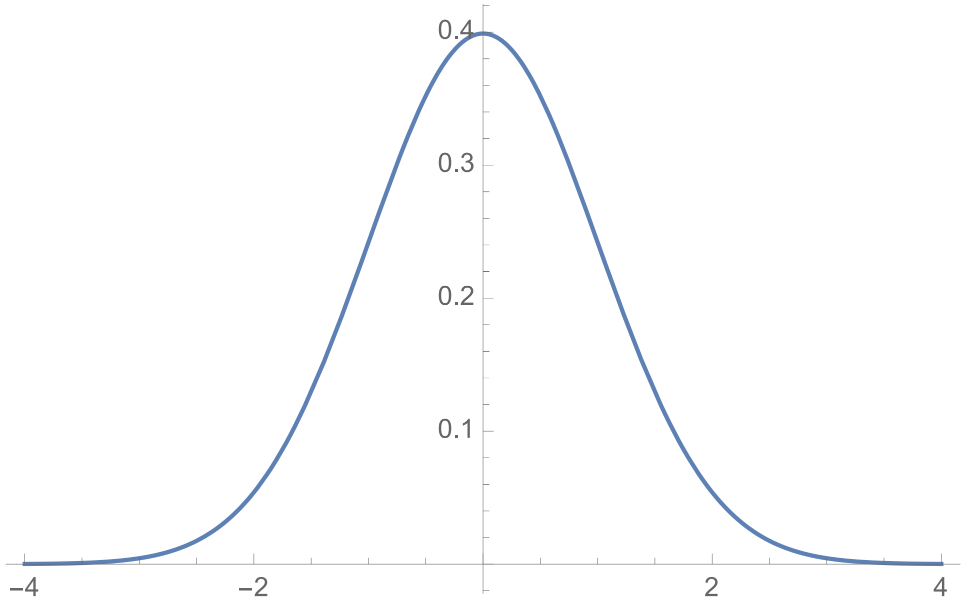


Make n independent measurements of it : $y_1, \ldots, y_n$

Central limit theorem

$$y_1 + \cdots + y_n \underset{n \text{ large}}{\approx} n\mu + \sqrt{n}\sigma\, s$$

s random, with Gaußian distribution



(Gauß law)

3. Independent random variables

Make n independent measurements : $y_1, \ldots, y_n$

$m_n = \max(y_1, \ldots, y_n)$

Fluctuations of the maximum

(Fisher-Tippett 1928, Gumbel 1935, Gnedenko 1942)

Assume $\rho(y) \underset{y \text{ large}}{\approx} y^\alpha e^{-\beta y^\gamma}$ with $\beta, \gamma > 0$

Then $m_n \underset{n \text{ large}}{\approx} a_n + b_n \xi$

Non-random $a_n = \left(\frac{\ln(n)}{\beta}\right)^{1/\gamma}$ and $b_n = (a_n)^{1-\gamma} \ll a_n$

ξ random with Gumbel distribution $\rho(\xi) = \exp(-\xi - \exp(-\xi))$

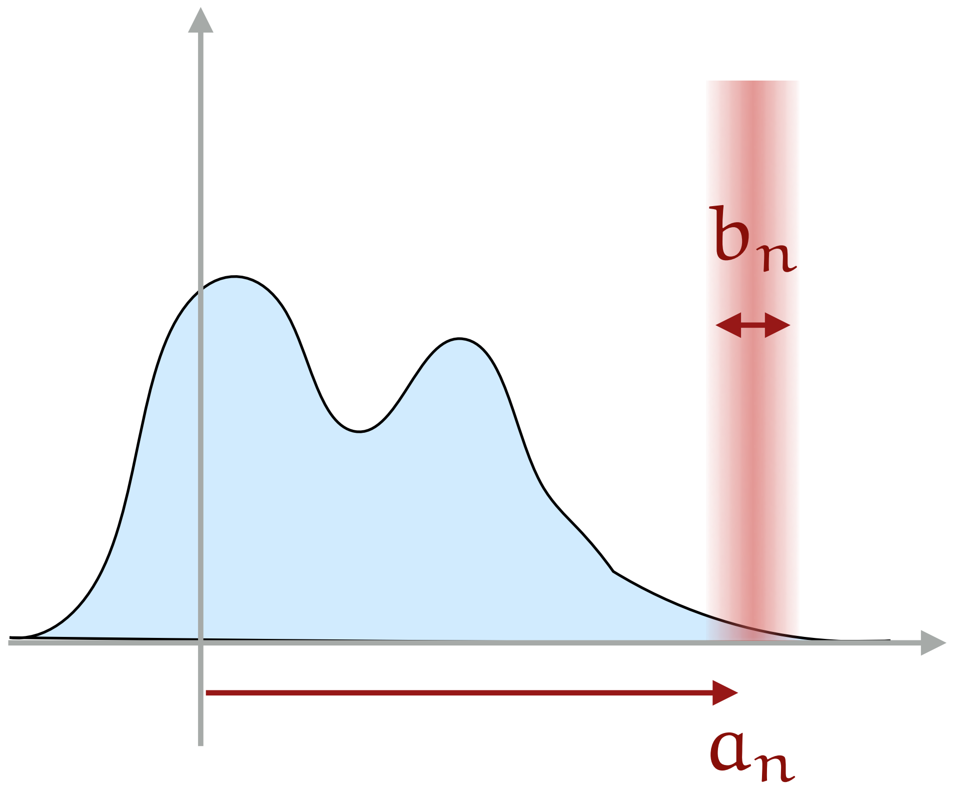
3. Independent random variables

$$m_n = \max(y_1, \ldots, y_n) = a_n + b_n \xi$$

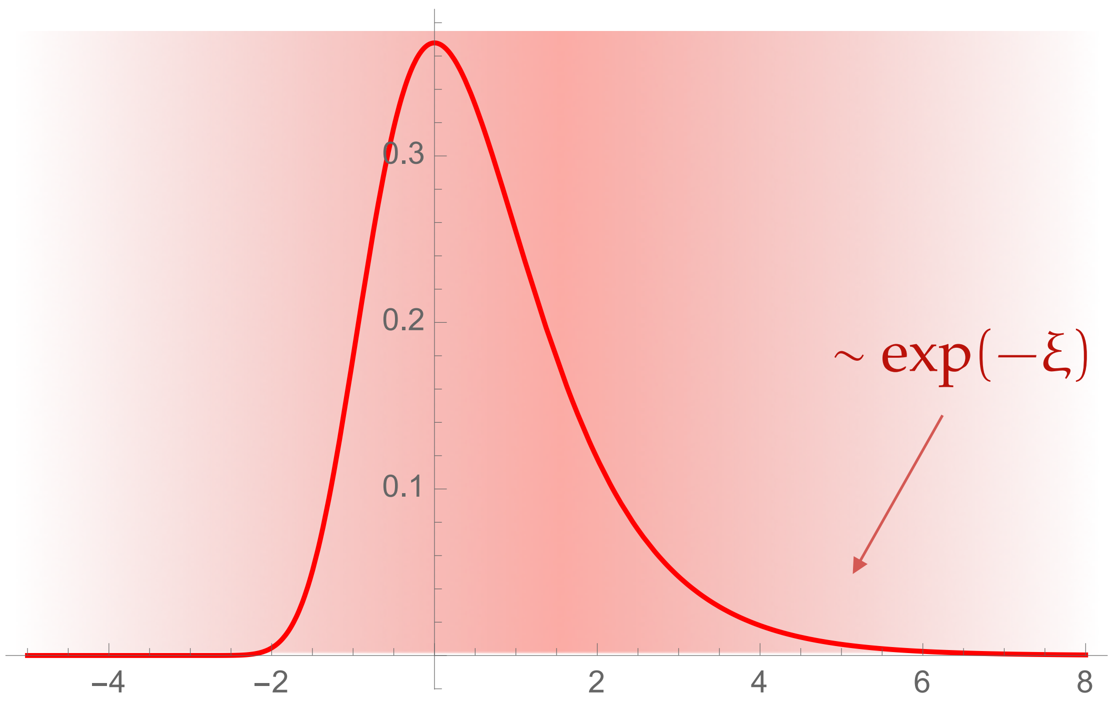
$$a_n = \left(\frac{\ln(n)}{\beta}\right)^{1/\gamma}$$

$$b_n = (a_n)^{1-\gamma} \ll a_n$$

Prob. distribution of y



Prob. distribution of ξ



(Gumbel's law)

3. Independent random variables

Summary

- The behavior of the sum and the max of independent n random variables with n large is well-understood since ~a century (Gauß's law, Gumbel's law, etc.)

- Universality phenomenon
  it depends very little on the random model considered (on the details of $\rho(y)$)

4

What happens for eigenvalues of large random matrices ?

4. Eigenvalues of large random matrices

- In random tilings $\ell_1 > \cdots > \ell_n$ are integers
- In random matrices $E_1 \geqslant \ldots \geqslant E_n$ are real numbers

In both cases, probability to find them around $x_1, \ldots, x_n$ at precision $\delta \ll 1$ is proportional to $\prod_{i<j} (x_i - x_j)^2$

$\longrightarrow$ these random variables are strongly correlated
(x_i tends to avoid being close to x_j)

The previous laws (Gauß, Gumbel, etc.) do not apply !

4. Eigenvalues of large random matrices

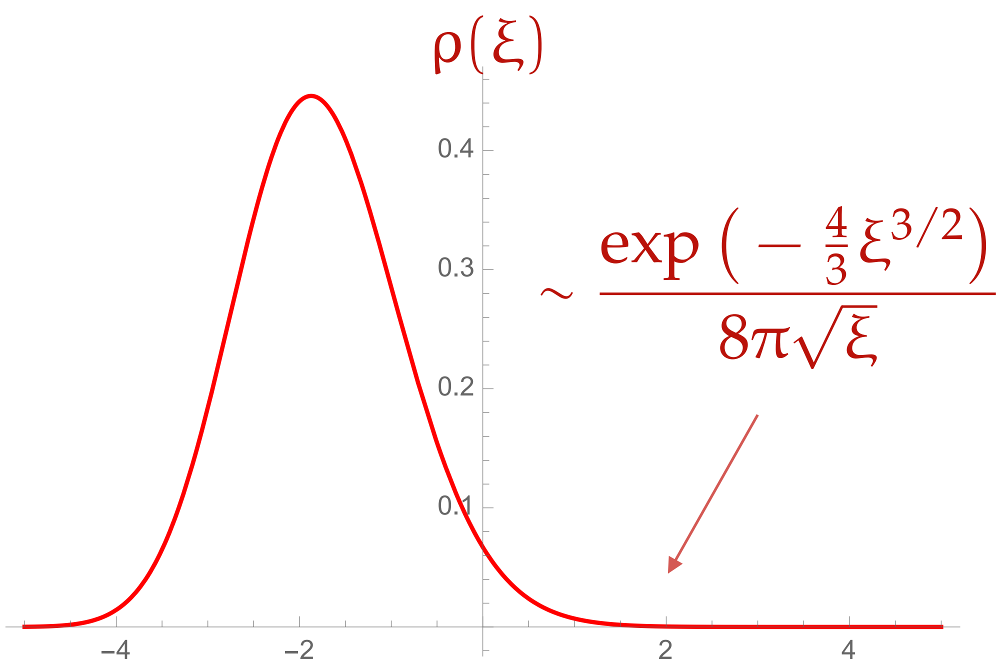
Consider a random (hermitian) matrix of size n x n
with independent Gauß-distributed entries with variance σ^2

Fluctuations of the maximum Tracy-Widom 1992

$$E_1 = \max(E_1, \dots, E_n) \underset{n \text{ large}}{\approx} 2\sigma\sqrt{n} + \sigma n^{-1/6}\,\xi$$

ξ random with Tracy-Widom distribution

- Quite different from Gumbel's law !
- Universal !
- Describes as well fluctuations of ℓ_1 on vertical section of large random tilings

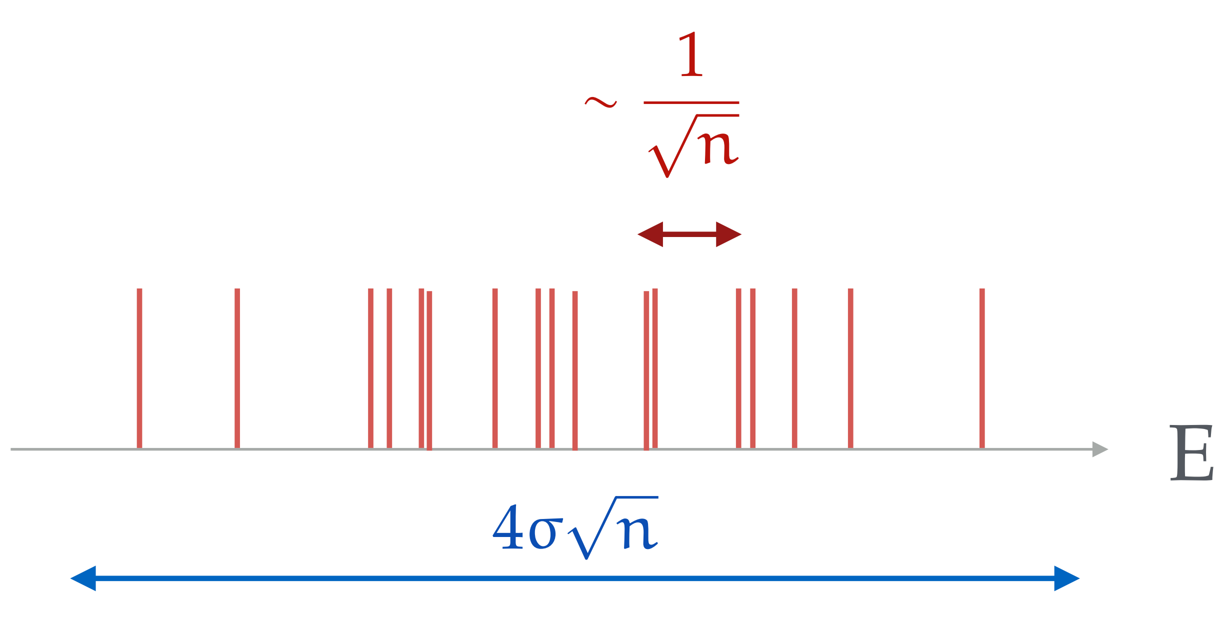


4. Eigenvalues of large random matrices

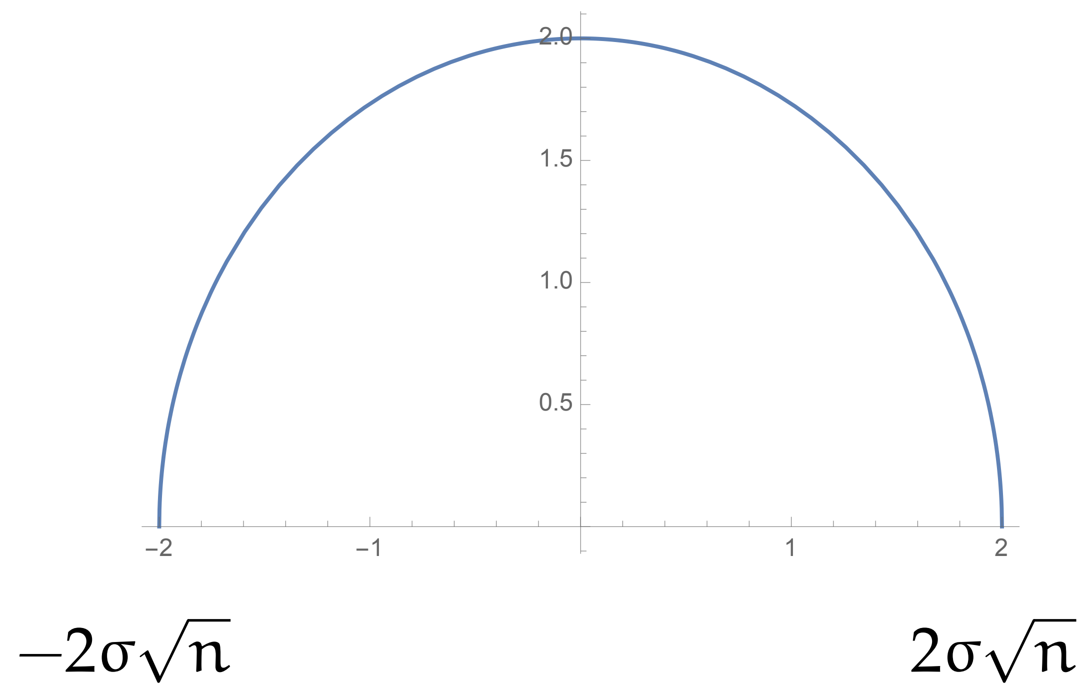
Consider a random (hermitian) matrix of size n x n
with independent Gauß-distributed entries with variance σ^2

n large : macroscopic distribution of eigenvalues becomes non-random
but its shape is not universal (Wigner 1950)

position of n eigenvalues



large n density of eigenvalues

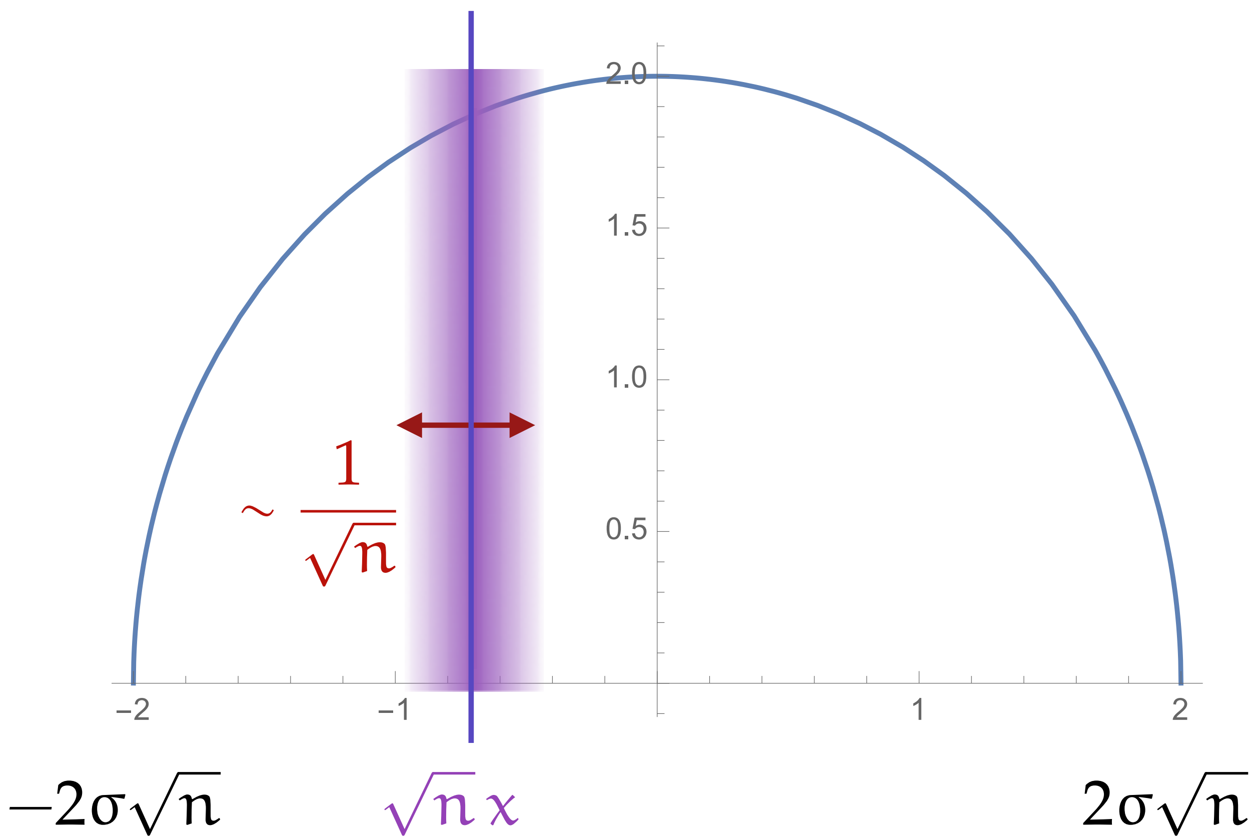


4. Eigenvalues of large random matrices

Pair correlations

The probability distribution of finding a eigenvalue at at distance $\frac{r}{\sqrt{n}}$ from an eigenvalue at $\sqrt{n}\,x$

when n becomes large, is also known (Wigner, Dyson, Mehta 50s …)



5

Energy levels of heavy atomic nuclei

5. Heavy atomic nuclei

In quantum mechanics

- the state of a system is represented by a point $X(t) = (x_1(t), \ldots, x_n(t))$ in n-dimensional space

- the time evolution is described by a matrix H

$$i\hbar \frac{X(t+\delta) - X(t)}{\delta} \underset{\delta \text{ small}}{\approx} H \cdot X(t)$$ (Schrödinger's equation)

- H is not random: it is specified by the components of the system and their interactions

5. Heavy atomic nuclei

In general, understanding the evolution of the system

= finding the states with fixed energy (eigenvectors of H)
and the value of these energies (eigenvalues of H)

For heavy atomic nuclei (between 70 and 100 protons)

H is very complicated !

Wigner's idea (1950s)

statistics of eigenvalues of H may look like the
statistics of eigenvalues of a typical large matrix **chosen at random**

5. Heavy atomic nuclei

Eigenvalues of H for atomic nuclei

= energy of particles that it can absorb (resonances)

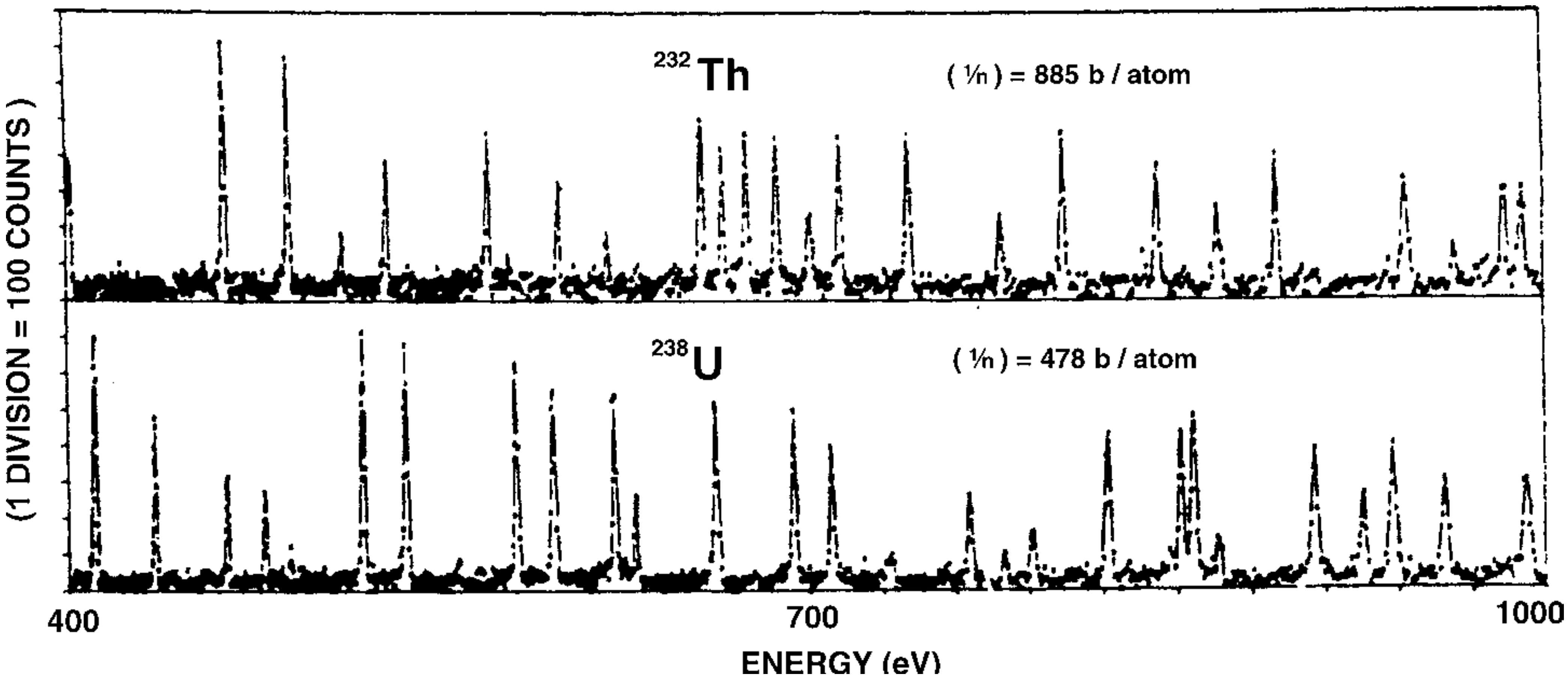
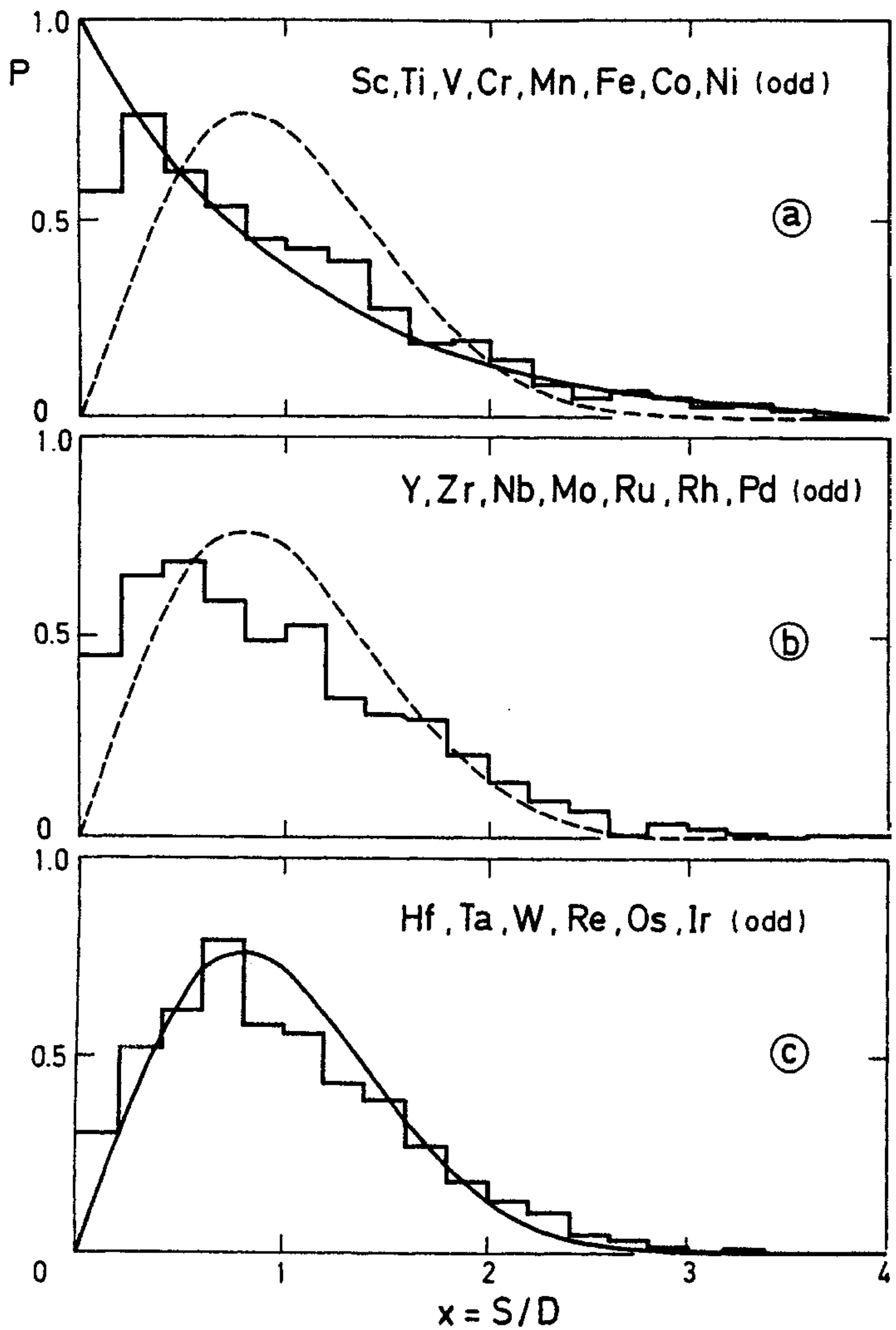


Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei. Reprinted with permission from The American Physical Society, Rahn et al., Neutron resonance spectroscopy, X, *Phys. Rev. C* 6, 1854–1869 (1972). (from Mehta's book *Random matrices*)

5. Heavy atomic nuclei



Pair correlations

- Cumulative histogram of distance between high energy resonances in certain heavy nuclei

vs.

- Predicted probability distributions for eigenvalues of a large random hermitian matrix

corresponding increase in strength of the spin dependent forces. Reprinted with permission from Annales Academiae Scientiarum Fennicae, Porter C.E. and Rosenzweig N., Statistical properties of atomic and nuclear spectra, *Annale Academiae Scientiarum Fennicae, Serie A VI, Physica* 44, 1–66 (1960).

(from Mehta's book *Random matrices*)

6

Other surprising apparitions of large random matrices

6. Other apparitions

Summary

- Universality phenomenon: repulsive random variables can exhibit the same statistics as the eigenvalues of large random matrices

- Proving universality (for large classes of models of random matrices) has occupied mathematicians from the 60s until now, is still an active topic of research

- We have encountered applications to

  - random tilings
  - statistics of high energy resonances in heavy nuclei

6. Other apparitions

Many other apparitions of the theory of (large) random matrices

- Data analysis (economics, linguistics, phylogenetics, …)
- Statistics of zeroes of the Riemann zeta function
  (related statistics of prime numbers)
  (Montgomery 1970)
- Random crystal growth/interface growth
  (Takeuchi-Sano 2010, Sasamoto-Spohn 2010)
- Statistics of distance between pine trees in Swedish forests
  (le Caer 1990)
- Statistics of bus waiting times in Cuernavaca (Mexico)
  (Krbalek-Seba 2000, Baik-Borodin-Deift-Suidan 2006)

Thank you for your attention !