Holomorphic curves, boundaries, skeins, and recursion

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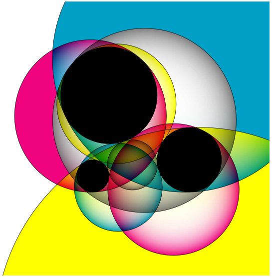
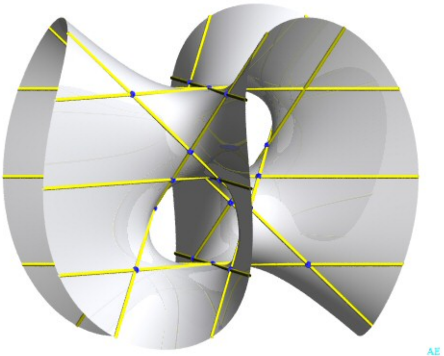
Curve counting in antiquity

- Euclid (c. 300 BC): 2 points determine a line
- Apollonius (c. 200 BC): 8 circles are tangent to 3 given ones



Curve counting in the 19th century

- (1848) Steiner: 7776 conics tangent to 5 given ones
- (1859) de Jonquières: actually, no, there are 3264
- (1849) Cayley, Salmon: 27 lines on a cubic surface



- (1874) Schubert: general methods

Curve counting: the quintic

- 2875 lines (classical)
- 609250 conics (S. Katz, 1986)
- String theorists! (Candelas, de la Ossa, Green, Parkes; 1991): 2875, 609250, 317206375, 242467530000, 229305888887625, 248249742118022000, 295091050570845659250, 375632160937476603550000, 503840510416985243645106250, ...
- Kontsevich : can mathematically define the topological A-model — and make calculations
- Givental (1995) : and the physicists prediction is correct

Since then the Gromov-Witten theory has been the source of endless mathematics.

Why Gromov-Witten invariants exist

Algebraic approach

Use derived geometry (Kontsevich) or virtual fundamental classes (Li-Tian, Behrend-Fantechi).

Analytic approach

Perturb holomorphic curve equation (Gromov, Floer, Ruan-Tian) either locally in order to construct virtual classes (Fukaya-Ono, Pardon) or globally (Hofer-Wysocki-Zehnder).

J-holomorphic curves: local theory

If (X, J) is an almost complex manifold, and (C, j) is a Riemann surface, we can ask for a map $f : C \to X$ satisfying $df \circ j = J \circ df$.

This is a nonlinear elliptic PDE. Its linearization is Fredholm with index $2((3-d)(1-g) + \int_C f^* c_1(M))$ (by Riemann-Roch).

In an ideal world, the index is the dimension of the moduli space of such maps.

In the real world, we perturb the equation to make this true.

Index vanishes when M is Calabi-Yau ($c_1(M) = 0$) of complex dimension d $= 3$.

J-holomorphic maps: global theory

When there's a symplectic form ω so that $g = \omega(\cdot\,, J\,\cdot)$ is a metric, then the quantity

$$I = \int_C f^*\omega$$

is both topological (because ω is closed) and is the g-area of f(C).

Using this control, Gromov classified how families of such maps may degenerate, and showed that after allowing certain explicit bubbling behaviors, the space of solutions becomes compact (after fixing topological data).

Gromov-Witten invariants of Calabi-Yau 3-folds

Since the moduli of holomorphic maps to a Calabi-Yau 3-fold is compact (for fixed topological data), and zero dimensional (after perturbation), one can define a number by counting the points of this set.

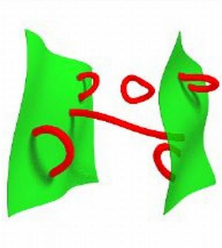
To see this did not depend on the perturbation ξ, one considers a 1-parameter family of perturbations ξ_t connecting two generic choices ξ_0 and ξ_1.

The moduli space of maps which are holomorphic for some perturbation in the path $M(\xi_t)$ gives a 1-dimensional cobordism from $M(\xi_0)$ to $M(\xi_1)$.

Degenerations have complex codimension 2, so are not encountered in this path, so

$$0 = [\partial M(\xi_t)] = [M(\xi_0)] - [M(\xi_1)]$$

Open Gromov-Witten theory?



In string theory, the strings may end on extended objects called branes.

Translation: there should be an ‘open’ Gromov-Witten theory which counts holomorphic curves ending on Lagrangian submanifolds.

Why open Gromov-Witten invariants do not exist (?)

Recall we consider moduli of perturbed solutions $M(\xi_0)$ and $M(\xi_1)$, and try to compare them by considering a 1-parameter family interpolating between them, $M(\xi_t)$

However, when studying curves with boundary, $\partial M(\xi_t)$ has other components!

This is because the space of choices to smooth a boundary degeneration is $\mathbb{R}$, whereas the space of choices to smooth an interior degeneration is $\mathbb{C}$. So for curves with boundary, degenerations appear in codimension one.

Naively, this seems to mean the corresponding count is not well defined, since $0 = [\partial M(\xi_t)] \neq [M(\xi_1)] - [M(\xi_0)]$.

But they have been calculated!

Nevertheless, in the string literature, there are various calculations of open Gromov-Witten invariants! (Ooguri-Vafa, Aganagic-Klemm-Marino-Vafa, ...)

Some are even confirmed by mathematical calculations (Katz-Liu, Graber-Zaslow, ...) — which however were made in the absence of a definition!

Important hint from Witten: the open Gromov-Witten invariants should have something to do with the Chern-Simons theory on the Lagrangian.

Ooguri-Vafa conjecture

Transport the conormal to a knot across the conifold transition. The open Gromov-Witten invariants of the resulting Lagrangian are the colored HOMFLYPT invariants of the knot.

Knots

Definition

A knot is an embedding of a circle into three dimensional space,

$$\begin{aligned} f : S^1 &\rightarrow \mathbb{R}^3 \\ \theta &\mapsto (x(\theta), y(\theta), z(\theta)) \end{aligned}$$

Two knots are the same (isotopic) if there is a 1-parameter family of such embeddings interpolating between them.

Knots in antiquity

Knots in the 19th century

Gauss

For two knots $f(\theta), g(\phi)$ consider the work done by simultaneously moving a magnetic charge along g, and an electric charge along f:

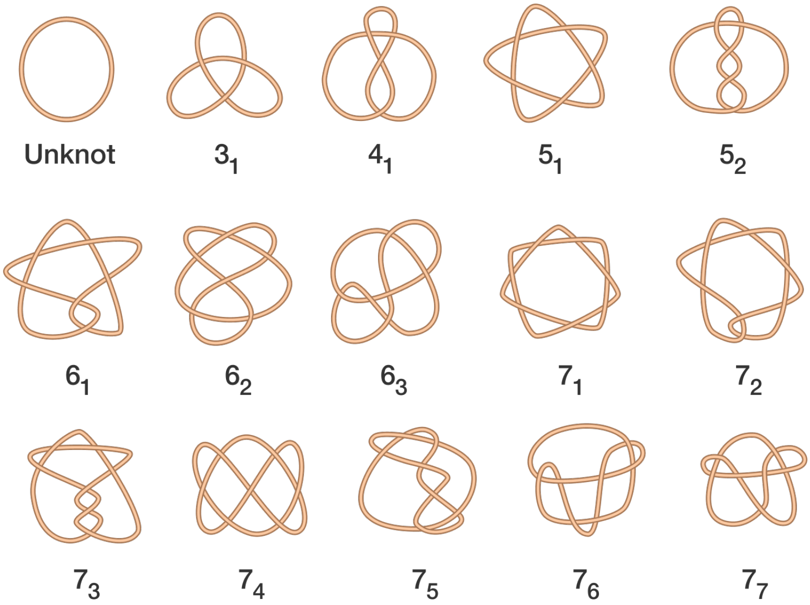
$$W \sim \frac{1}{4\pi} \int \int \frac{f(\theta) - g(\phi)}{|f(\theta) - g(\phi)|^3} \cdot (df \times dg)$$

Lord Kelvin

Maybe atoms are knots in the ether?

Tait

Well, anyway, let's classify them...



Unknot
3_1
4_1
5_1
5_2
6_1
6_2
6_3
7_1
7_2
7_3
7_4
7_5
7_6
7_7

How to tell knots apart?

Knot invariants

are rules for assigning some quantity to each knot, so that the quantity stays constant in 1-parameter families. Two knots assigned different values are thus different knots!

A knot invariant is most useful when:

It is computable from a knot presentation, has a geometric meaning, and takes different values on different knots. These desiderata are in considerable tension.

One way to construct invariants

Give some formula in terms of a 2d projection of the knot, and check it doesn't change under the Reidemeister moves. This method *does not* usually clarify the geometric meaning of the invariant.

Knot polynomials

Alexander polynomial

1-variable Laurent polynomial; up to normalization the generator of the ideal in $\mathbb{Z}[\pi_1/[\pi_1, \pi_1]]$ which annihilates the module $\mathbb{Z}[\pi_1, \pi_1]/[[\pi_1, \pi_1], [\pi_1, \pi_1]]$. Easy to compute; relatively weak.

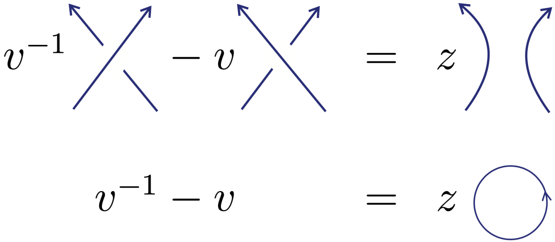
Jones polynomial

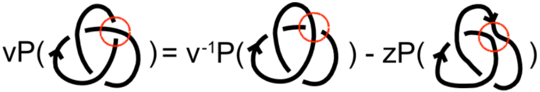
1-variable Laurent polynomial; comes from braid representations of operator algebras from statistical mechanics; defined in terms of 2d projection; satisfies skein relation. Harder to compute, but stronger.

HOMFLYPT polynomial

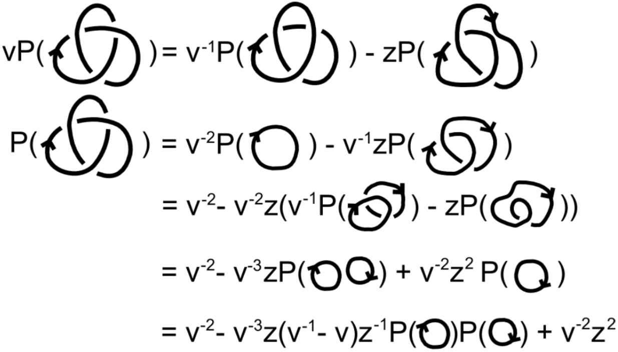
2-variable Laurent polynomial; generalizes both Jones and Alexander; defined in terms of 2d projection; satisfies skein relation.

Skein relations





Skein relations



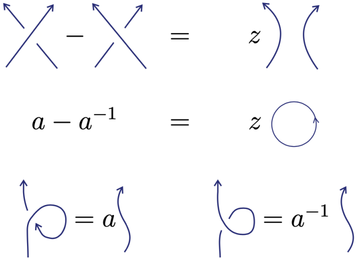
$$= v^{-2} - v^{-4} + v^{-2} + v^{-2}z^{2}$$

$$= 2v^{-2} - v^{-4} + z^{2}v^{-2}$$

taken from https://www.maths.dur.ac.uk/Ug/projects/highlights/PR4/Goulding_Knot_Theory_report.pdf

Skein module

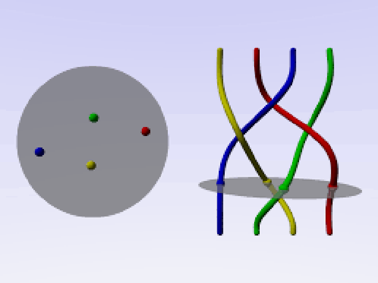
For an oriented 3-manifold M, the skein module $Sk(M)$ is the module freely generated over $\mathbb{Z}[a^{\pm}, z^{\pm}]$ by framed links; modulo the skein relations.



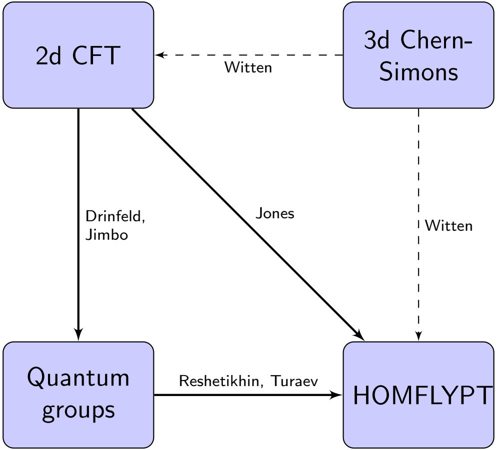
Existence of the HOMFLYPT invariant is equivalent to the statement that $Sk(S^3)$ is generated by the class of the empty link. Indeed, $\langle K \rangle = \mathrm{HOMFLYPT}(K) \cdot \langle \emptyset \rangle$.

HOMFLYPT from 2d

$$\begin{tikzcd}
\text{2d CFT} \arrow[d, "\text{Drinfeld, Jimbo}"] \arrow[dr, "\text{Jones}"] & \\
\text{Quantum groups} \arrow[r, "\text{Reshetikhin, Turaev}"] & \text{HOMFLYPT}
\end{tikzcd}$$



HOMFLYPT from 3d



$$\langle K \rangle = \int \mathrm{Hol}_K(A)\, e^{\frac{ik}{4\pi}\int \mathrm{tr}\, A\wedge dA + \frac{2}{3} A\wedge A\wedge A}\, DA$$

This integral over all connections has no existing mathematical definition

Two problems

Define open Gromov-Witten invariants

The main obstruction is dealing with the boundary degenerations which appear in codimension one in moduli.

Give a rigorous, geometric interpretation of the HOMFLYPT invariant

The existing mathematical constructions begin in one way or another from a 2d projection; the existing physical interpretation uses the nonexistent Feynman integral.

These problems solve each other!

Open Gromov-Witten invariants

Theorem (Ekholm-Shende)

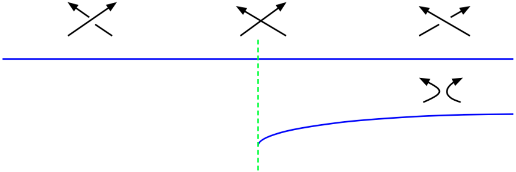
For CY3 X and appropriate Lagrangian L, there exists a space of parameters $\mathcal{P}$, and for $\lambda \in \mathcal{P}$, compact zero dimensional moduli spaces $M_\chi(X, L, \lambda)$ of bare (no symplectic area zero components) maps $u : (\Sigma, \partial\Sigma) \to (X, L)$ with domain of Euler characteristic χ, and an invariant

$$\Psi_{X,L,\lambda} = 1 + \sum_{\chi} \sum_{u \in M_\chi(X,L,\lambda)} wt(u) \cdot z^{-\chi} \cdot Q^{u_*[\Sigma]} \cdot a^{u \circ L} \cdot \langle u(\partial\Sigma) \rangle \in Sk(L)[[Q]]$$

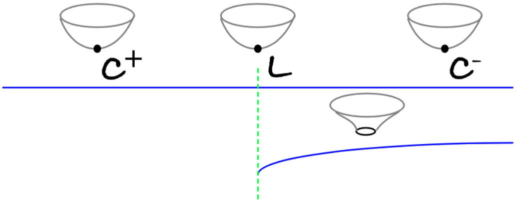
which is independent of λ, and invariant under deformations of X, L.

Here, $wt(u) \in \mathbb{Q}$ is for dealing with orbifold issues; $u_*[\Sigma]$ is the fundamental class of the image, and $u \circ L$ is a linking number.

Proof



$$P(\times) - P(\times) = z\,P(\,)\,(\,)$$



$$a - a^{-1} = z\,P(\bigcirc)$$

Geometric meaning of the HOMFLYPT polynomial

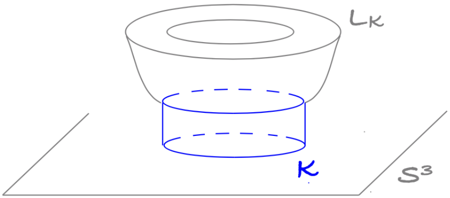
Theorem (Ekholm-Shende)

Let $K \subset S^3$ be a knot, and $L \subset T^*S^3 \setminus S^3$ a pushoff of the conormal bundle of K. Then the degree 1 term of $\Psi_{T^*S^3 \setminus S^3, L}$ is the HOMFLYPT polynomial of K.

One can also fill $T^*S^3 \setminus S^3$ with the resolved conifold and have the same result. This establishes the prediction of Ooguri and Vafa.

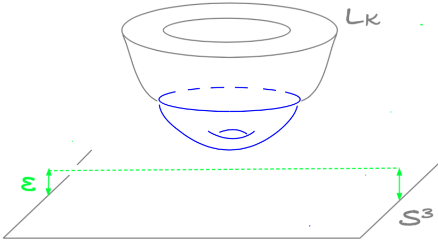
Proof

Consider in T^*S^3 the quantity $\Psi_{T^*S^3, S^3 \cup L}$. By the degree 1 term, we mean the contribution of curves which wrap once around the generator in $H_1(L)$. For appropriate almost complex structure there is exactly one such curve: the annulus tracing the path of the knot K under the pushing off of L. This annulus contributes as exactly $\langle K \rangle \in Sk(S^3)$.



Proof

Now degenerate the almost complex structure along a cosphere bundle which separates L from the zero section S^3, in sense of symplectic field theory. In the limit we can have only broken curves with one part in $T^*S^3 \setminus S^3$ with boundary on L, and another part in T^*S^3 with boundary on the zero section S^3. These curves may be asymptotic to index zero Reeb orbits — but there are no such orbits, so the curves must be compact. This means there are no curves whatsoever in the S^3 component.



Proof

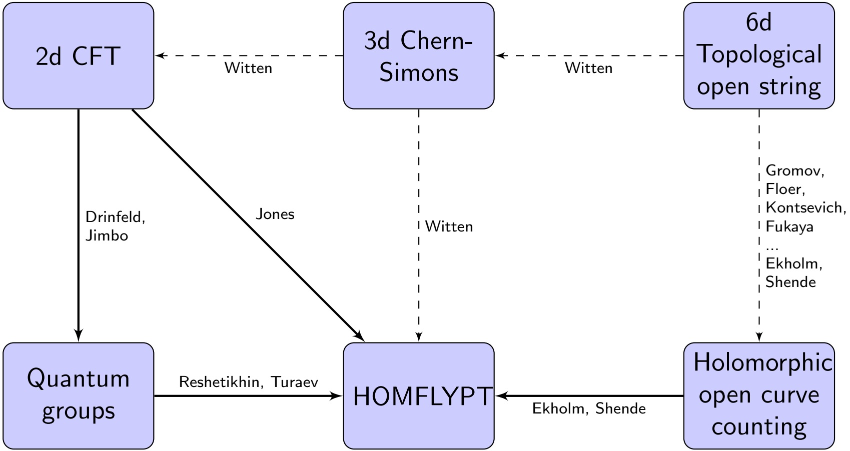
By invariance we have the following equation in $Sk(S^3)$:

$$\langle K \rangle = \Psi_{before} = \Psi_{after} = \Psi_{T^*S^3, S^3 \cup L} \otimes \langle \emptyset \rangle$$

But since $\langle K \rangle = \mathrm{HOMFLYPT}(K) \otimes \langle \emptyset \rangle$, we conclude

$$\Psi_{T^*S^3, S^3 \cup L} = \mathrm{HOMFLYPT}(K). \quad \blacksquare$$

HOMFLYPT from 6d



Colored HOMFLYPT

More generally, Ooguri and Vafa predicted that the full series $\Psi_{T^*S^3 \setminus S^3, L}$ is determined by the colored HOMFLYPT invariants

They argued as follows: if going once around the cylinder introduces the Wilson line along K given by the trace of the holonomy in the standard representation, then going n times around the cylinder must introduce the Wilson line given by tracing the n'th power of the holonomy.

Colored HOMFLYPT

In our setup, this translates to the prediction

$$\Psi_{T^*S^3 \setminus S^3, L} = \sum_{\pi} W_\pi \cdot \mathrm{HOMFLYPT}_\pi(K) \in Sk(L)$$

where the sum is over partitions, the $W_\pi \in Sk(L)$ are certain distinguished elements corresponding to the irreps of $\mathrm{GL}(n)$, and the $\mathrm{HOMFLYPT}_\pi(K)$ are the corresponding 'colored' HOMFLYPT invariants.

Theorem (Ekholm-Shende): This is true.

The HOMFLYPT skein of the solid torus

Carries a commutative ring structure since a pile of two donuts can be smushed into one donut.

Freely by variables indexed by $\mathbb{Z} \setminus 0$, corresponding to certain knots winding that many times around. (Turaev)

The HOMFLYPT skein of the solid torus

Isomorphic to the tensor square of the ring of symmetric functions.

Carries an action by the skein of the torus cylinder $\mathbb{T}^2 \times [0,1]$ (put it on the outside). This action is a specialization of the action of the Elliptic Hall algebra on symmetric functions (Morton-Samuelson).

$$\begin{aligned} P_{1,0} W_\lambda &= (\bigcirc + a(q^{1/2} - q^{-1/2}) c_\lambda(q)) \cdot W_\lambda \\ P_{0,1} W_\lambda &= \sum_{\lambda + \square = \mu} W_\mu \end{aligned}$$

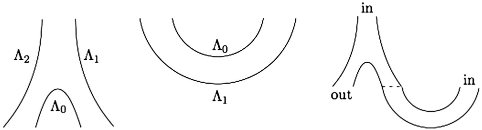
In particular, the Schur basis W_λ is identified as the eigenskeins of encirclement ($P_{1,0}$).

Relations from infinity

When we study curves possibly asymptotic to Reeb chords (or orbits) at infinity, the index receives contributions from these asymptotes.

In particular, we can find positive dimensional moduli spaces.

The boundaries of these moduli spaces come from boundary breaking as before, and also from SFT degenerations along chords.



Relations from infinity

Taking a skein valued count cancels the boundary breaking; what is left is the SFT degenerations. By compactness, the total contribution of these is zero.

The simplest scenario is when L has only Reeb chords of index ≥ 1. In this case, all SFT degenerations will consist of some curves in the symplectization union some compact curves entirely in the interior.

In this case, writing A for the count of curves in the symplectization, and $\Psi = \Psi_{T^*S^3 \setminus S^3, L}$ for the count of curves in the interior as before, we have $A\Psi = 0$.

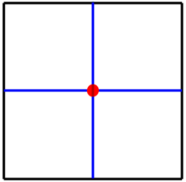
Unknot conormal

When L is the conormal to the unknot, there are indeed only Reeb chords of index ≥ 1.

A natural map $S^*S^3 \to \mathbb{P}^1 \times \mathbb{P}^1$ sends the unknot conormal torus to the standard 'Clifford' torus.

This can be used to count the holomorphic curves in the symplectization, yielding:

$$A = \bigcirc - P_{1,0} - P_{0,1} + a^2 a_L^{-1} P_{1,1}$$



Unknot conormal

Finally, the equation $(\bigcirc - P_{1,0} - P_{0,1} + a^2 a_L^{-1} P_{1,1})\Psi = 0$ has the (unique up to scalar) solution

$$\Psi = \sum_{\pi} W_\pi \prod_{\square \in \pi} \frac{a q^{c(\square)/2} - a^{-1} q^{-c(\square)/2}}{q^{h(\square)/2} - q^{-h(\square)/2}}$$

where c is the content and h is the hook-length.

Conormal to longitude in solid torus

To pin down the contribution of the multiply covered cylinder, we study finally the cotangent bundle to the solid torus, and consider the zero section and a pushoff of the conormal to the longitude.

This geometry can be obtained from $T^*_0\mathbb{R}^2 \cup T^*_{\mathbb{R}^2}\mathbb{R}^2 \subset T^*\mathbb{R}^2$ multiplication with T^*S^1.

In particular, its holomorphic curve theory at infinity is determined by that of the (well studied) Legendrian Hopf link.

Conormal to longitude in solid torus

This time we obtain

$$A = (P_{1,0} - \bigcirc) \otimes a_2 - a_1 \otimes (P_{1,0} - \bigcirc) \in Sk(\mathbb{T}^2 \times [0,1]) \otimes Sk(\mathbb{T}^2 \times [0,1])$$

Because the W_π are the eigenskeins of $P_{1,0}$, this means that $\Psi = \sum_\pi c_\pi W_\pi \otimes W_\pi$.

It follows from the unknot calculation that $c_\pi = 1$.

Hence, the full multiple cover contribution of the annulus is $\sum_\pi W_\pi \otimes W_\pi$, as predicted by Ooguri and Vafa.