Topological recursion in Hurwitz theory

(on a joint work with B.Bychkov, P.Dunin-Barkowsky, S.Shadrin)

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Hypergeometric-type family of solutions to KP hierarchy

Variables: $p = (p_1, p_2, \dots)$, parameters: $q = (q_1, q_2, \dots)$, $c = (c_1, c_2, \dots)$.

$$
\begin{aligned}
F = & p_1 q_1 + \frac{p_1^2}{4}\left(c_1^2 q_1^2 + c_2 q_1^2 + 2c_1 q_2\right) + \frac{p_2}{2}\left(c_1 q_1^2 + q_2\right) + \frac{p_1^3}{36}\left(6c_1^4 q_1^3 + 24 c_1^3 q_1 q_2 + 18 c_2 c_1^2 q_1^3\right.\\
&\quad + 18 c_1^2 q_3 + 6 c_3 c_1 q_1^3 + 36 c_2 c_1 q_1 q_2 + 6 c_2^2 q_1^3 + c_4 q_1^3 + 6 c_3 q_1 q_2 + \left.6 c_2 q_3\right) + \frac{p_2 p_1}{6}\left(4 c_1^3 q_1^3\right.\\
&\quad + 6 c_1 c_2 q_1^3 + c_3 q_1^3 + 12 c_1^2 q_2 q_1 + 6 c_2 q_2 q_1 + \left.6 c_1 q_3\right) + \frac{p_3}{6}\left(3 c_1^2 q_1^3 + c_2 q_1^3 + 6 c_1 q_2 q_1 + 2 q_3\right) + \dots\\
& + \hbar^2\Bigg(\frac{p_1^2}{48}\left(c_1^4 q_1^2 + 4 c_1^3 q_2 + 6 c_2 c_1^2 q_1^2 + 4 c_3 c_1 q_1^2 + 12 c_2 c_1 q_2 + 3 c_2^2 q_1^2 + c_4 q_1^2 + 4 c_3 q_2\right)\\
&\qquad + \frac{p_2}{12}\left(c_1^3 q_1^2 + 3 c_1^2 q_2 + 3 c_2 c_1 q_1^2 + c_3 q_1^2 + 3 c_2 q_2\right) + \dots\Bigg)\\
& + \hbar^4\Bigg(\frac{p_2}{240}\left(c_1^5 q_1^2 + 5 c_1^4 q_2 + 10 c_2 c_1^3 q_1^2 + 10 c_3 c_1^2 q_1^2 + 30 c_2 c_1^2 q_2 + 15 c_2^2 c_1 q_1^2 + 5 c_4 c_1 q_1^2 + 20 c_3 c_1 q_2\right.\\
&\quad + 10 c_2 c_3 q_1^2 + c_5 q_1^2 + 15 c_2^2 q_2 + \left.5 c_4 q_2\right) + \frac{p_1^2}{1440}\left(c_1^6 q_1^2 + 6 c_1^5 q_2 + 15 c_2 c_1^4 q_1^2 + 20 c_3 c_1^3 q_1^2 + 60 c_2 c_1^3 q_2\right.\\
&\quad + 45 c_2^2 c_1^2 q_1^2 + 15 c_4 c_1^2 q_1^2 + 60 c_3 c_1^2 q_2 + 60 c_2 c_3 c_1 q_1^2 + 6 c_5 c_1 q_1^2 + 90 c_2^2 c_1 q_2 + 30 c_4 c_1 q_2 + 15 c_2^3 q_1^2\\
&\quad + 10 c_3^2 q_1^2 + 15 c_2 c_4 q_1^2 + c_6 q_1^2 + 60 c_2 c_3 q_2 + \left.6 c_5 q_2\right)\Bigg) + \dots\\
& + \dots
\end{aligned}
$$

Hypergeometric-type family of solutions to KP hierarchy

$$\psi(y) = \sum_{k=1}^{\infty} c_k \frac{y^k}{k!}, \qquad Y(z) = \sum_{k=1}^{\infty} q_k z^k.$$

Definition

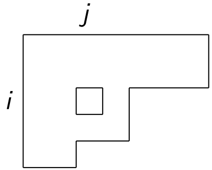
$$e^{\frac{1}{\hbar^2}F} = \sum_{\lambda} e^{\sum_{(i,j)\in\lambda} \psi(\hbar(j-i))} s_\lambda(p/\hbar) s_\lambda(q/\hbar)$$

s_λ Schur function,

$p/\hbar = (p_1/\hbar, p_2/\hbar, \dots)$,

$q/\hbar = (q_1/\hbar, q_2/\hbar, \dots)$.

Young diagram λ



$c(i,j) = j - i$
content of a cell

Genus expansion: $F = \sum_{g\geq 0, n\geq 1} \frac{\hbar^{2g}}{n!} \sum_{k_1,\dots,k_n} f_{g,(k_1,\dots,k_n)} p_{k_1} \dots p_{k_n}$

$f_{g,(k_1,\dots,k_n)}$ ‘correlators’ or (generalized) Hurwitz numbers

Specializaion: enumeration of maps (ribbon graphs)

$$\psi(y)=\sum_{k=1}^{\infty} c_k \frac{y^k}{k!}, \qquad Y(z)=\sum_{k=1}^{\infty} q_k z^k.$$

$$F=\sum_{g\ge 0, n\ge 1} \frac{\hbar^{2g}}{n!} \sum_{k_1,\dots,k_n} f_{g,(k_1,\dots,k_n)} p_{k_1}\dots p_{k_n} = \hbar^2 \log\Big(\sum_{\lambda} e^{\sum_{(i,j)\in\lambda} \psi(\hbar(j-i))} s_\lambda(p/\hbar) s_\lambda(q/\hbar)\Big)$$

Example

$$e^{\psi(y)}=1+y,\ Y(z)=z^2 \qquad \Longrightarrow \qquad f_{g,(k_1,\dots,k_n)} = \sum_{\substack{\gamma:\ \text{maps of given} \\ \text{combinatorial type} \\ (g,(k_1,\dots,k_n))}} \frac{1}{|\mathrm{Aut}(\gamma)|}$$

A *map* is a graph embedded to a closed surface such that each connected component of the complement is homeomorphic to a disk.

$$g = \mathrm{genus}(C),$$
$$n = \text{ the number of vertices}$$
$$(k_1,\dots,k_n) = \text{ valencies of the vertices}$$
$$2-2g = n - \tfrac{1}{2}\sum k_i + \#(faces)$$

Specializaion: Hurwitz numbers

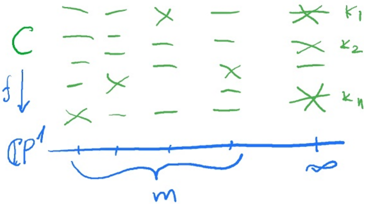
$$\psi(y)=\sum_{k=1}^{\infty} c_k \frac{y^k}{k!}, \qquad Y(z)=\sum_{k=1}^{\infty} q_k z^k.$$

$$F=\sum_{g\ge 0,n\ge 1}\frac{\hbar^{2g}}{n!}\sum_{k_1,\dots,k_n} f_{g,(k_1,\dots,k_n)} p_{k_1}\dots p_{k_n} = \hbar^2 \log\Big(\sum_{\lambda} e^{\sum_{(i,j)\in\lambda}\psi(\hbar(j-i))} s_\lambda(p/\hbar)s_\lambda(q/\hbar)\Big)$$

Example

$$\psi(y)=y,\ Y(z)=z \qquad \Longrightarrow \qquad f_{g,(k_1,\dots,k_n)}=\sum_{\substack{\gamma:C\to\mathbb{C}P^1 \text{ with} \\ \text{the ramification data} \\ (g,(k_1,\dots,k_n))}} \frac{1}{|\mathrm{Aut}(\gamma)|}$$

Hurwitz numbers enumerate ramified coverings of $\mathbb{C}P^1$.



$$g=\mathrm{genus}(C),$$
$$(k_1,\dots,k_n)= \text{ ramification type over } \infty$$
$$m=2g-2+n+\sum k_i$$

Specializations: types of Hurwitz numbers

Hurwitz numbers	$\phi(y)$
usual	e^y
r-spin (atlantes)	e^{y^r}
monotone	$\dfrac{1}{1-y}$
strictly monotone (maps)	$1+y$
hypermaps	$(1+u\,y)(1+v\,y)$
BM-Sch numbers	$(1+y)^m$
weighted	$1+\sum_{k=1}^{\infty} c_k \frac{y^k}{k!}$

Variations	$Y(z)$
simple	z
orbifold	z^m
double	$\sum_{m=1}^{\infty} q_m z^m$

Bibliography I

- S. Kharchev, A. Marshakov, A. Mironov, and A. Morozov. Generalized Kazakov-Migdal-Kontsevich model: group theory aspects. Internat. J. Modern Phys. A, 10(14):2015–2051, 1995.
- A. Yu. Orlov and D. M. Shcherbin. Hypergeometric solutions of soliton equations. Teoret. Mat. Fiz., 128(1):84–108, 2001.
- L. Chekhov and B. Eynard. "Hermitian matrix model free energy: Feynman graph technique for all genera". In: J. High Energy Phys. 3 (2006), pp. 014, 18.
- L. Chekhov, B. Eynard, and N. Orantin. "Free energy topological expansion for the 2-matrix model". In: J. High Energy Phys. 12 (2006), pp. 053, 31.
- I. P. Goulden and D. M. Jackson. "The KP hierarchy, branched covers, and triangulations". In: Adv. Math. 219.3 (2008), pp. 932–951.
- V Bouchard and M Mariño. Hurwitz numbers, matrix models and enumerative geometry. In From Hodge theory to integrability and TQFT: tt^*-geometry, volume 78 of Proc. Sympos. Pure Math., pages 263–283. Amer. Math. Soc., Providence, RI, 2008.
- Gaëtan Borot, Bertrand Eynard, Motohico Mulase, and Brad Safnuk. A matrix model for simple Hurwitz numbers, and topological recursion. J. Geom. Phys., 61(2):522–540, 2011.
- S. Shadrin, L. Spitz, and D. Zvonkine. On double Hurwitz numbers with completed cycles. J. Lond. Math. Soc. (2), 86(2):407–432, 2012.

Bibliography II

- Norman Do, Alastair Dyer, Daniel V. Mathews. Topological recursion and a quantum curve for monotone Hurwitz numbers. 2014. arXiv:1408.3992
- Maxim Kazarian, Peter Zograf. Virasoro constraints and topological recursion for Grothendieck's dessin counting. Letters in Mathematical Physics. 2015. Vol. 105. No. 8. P. 1057-1084. arXiv:1406.5976
- Norman Do, Maksim Karev. Monotone orbifold Hurwitz numbers. 2015. arXiv:1505.06503
- P. Dunin-Barkowski, D. Lewanski, A. Popolitov, and S. Shadrin. Polynomiality of orbifold Hurwitz numbers, spectral curve, and a new proof of the Johnson-Pandharipande-Tseng formula. J. Lond. Math. Soc. (2), 92(3):547–565, 2015.
- M. E. Kazaryan and S. K. Lando. Combinatorial solutions to integrable hierarchies. Uspekhi Mat. Nauk, 70(3(423)):77–106, 2015.
- B. Eynard. Counting surfaces. Vol. 70. Progress in Mathematical Physics. CRM Aisen- stadt chair lectures. Birkhäuser/Springer, [Cham], 2016.
- A. Alexandrov, D. Lewanski, and S. Shadrin. Ramifications of Hurwitz theory, KP integrability and quantum curves. J. High Energy Phys., (5):124, front matter+30, 2016.
- P. Dunin-Barkowski, N. Orantin, A. Popolitov, and S. Shadrin. "Combinatorics of loop equations for branched covers of sphere". In: Int. Math. Res. Not. IMRN 18 (2018), pp. 5638–5662.
- Reinier Kramer, Danilo Lewanski, and Sergey Shadrin. Quasi-polynomiality of monotone orbifold Hurwitz numbers and Grothendieck's dessins d'enfants. Doc. Math., 24:857–898, 2019.

Bibliography III

A. Alexandrov, G. Chapuy, B. Eynard, and J. Harnad. "Weighted Hurwitz numbers and topological recursion". In: Comm. Math. Phys. 375.1 (2020), pp. 237–305.

G. Borot and E. Garcia-Failde. "Simple maps, Hurwitz numbers, and topological recursion". In: Comm. Math. Phys. 380.2 (2020), pp. 581–654.

B. Bychkov, P. Dunin-Barkowski, and S. Shadrin. "Combinatorics of Bousquet-Mélou–Schaeffer numbers in the light of topological recursion". In: Europ ean J. Combin. 90 (2020), p. 103184.

Gaëtan Borot, Norman Do, Maxim Karev, Danilo Lewanski, and Ellena Moskovsky. Double Hurwitz numbers: polynomiality, topological recursion and intersection theory. arXiv e-prints, Feb 2020. arXiv:2002.00900.

Petr Dunin-Barkowski, Maxim Kazarian, Aleksandr Popolitov, Sergey Shadrin, and Alexey Sleptsov. Topological recursion for the extended Ooguri–Vafa partition function of colored HOMFLY–PT polynomials of torus knots. arXiv e-prints, Oct 2020. arXiv:2010.11021.

B. Bychkov, P. Dunin-Barkowski, M. Kazarian, and S. Shadrin. Explicit closed algebraic formulas for Orlov-Scherbin n-point functions. 2020. arXiv: 2008.13123

B. Bychkov, P. Dunin-Barkowski, M. Kazarian, and S. Shadrin. Topological recursion for Kadomtsev-Petviashvili tau functions of hypergeometric type. 2020. arXiv: 2012.14723

B. Bychkov, P. Dunin-Barkowski, M. Kazarian, and S. Shadrin. Generalized ordinary vs fully simple duality for n-point functions and a proof of the Borot–Garcia-Failde conjecture. 2021. arXiv:2106.08368

Rationality and z-coordinates

n-point functions

$$H_{g,n}(X_1,\dots,X_n)=\sum_{k_1,\dots,k_n} f_{g,(k_1,\dots,k_n)}X_1^{k_1}\dots X_n^{k_n}$$

Rationality principle

There exists a suitable local change of coordinates $X=X(z)$ such that with certain assumptions on ψ and Y, $H_{g,n}$ becomes a rational function in $z_1,\dots,z_n$ after substitution $X_i=X(z_i)$.

It follows that $H_{g,n}$ can be written in a closed form for any particular (g,n).
Topological recursion is an inductive procedure to compute $H_{g,n}$ explicitly

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Example

Maps: $X(z)=\dfrac{z}{1+z^2} \Leftrightarrow z=\frac{1-\sqrt{1-4X^2}}{2X}$.

$$H_{0,3}=\frac{z_1z_2z_3(z_1+z_2+z_3+z_1z_2z_3)}{(1-z_1^2)(1-z_2^2)(1-z_3^2)},\qquad z_i=\frac{1-\sqrt{1-4X_i^2}}{2X_i}$$

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Example

Hurwitz numbers: $X(z)=z\,e^{-z}$.

$$H_{0,3}=\frac{z_1z_2z_3}{(1-z_1)(1-z_2)(1-z_3)}$$

Rationality and z-coordinates

n-point functions

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Rationality principle

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Example ([ACEH])

If $Y(z)$ and $e^{\psi(y)}$ are polynomials, then $H_{g,n}$ becomes rational after the change

$$X_i=X(z_i),\qquad X(z)=z\,e^{-\psi(Y(z))}.$$

Main Theorem

$$X_i = X(z_i), \qquad X(z) = z\, e^{-\psi(Y(z))}.$$

Theorem ([BDKS])

For each (g,n) with $2g-2+n>0$ there is a universal closed expression for $H_{g,n}$ as a finite sum of the form

$$H_{g,n} = \sum_{j_1,\dots,j_n} D_1^{j_1}\dots D_n^{j_n} \frac{P_{g,n}^{(j_1,\dots,j_n)}}{Q_1\dots Q_n} + \mathrm{const}.$$

$$D_i = X_i \frac{d}{dX_i} = \frac{z_i}{Q_i}\frac{d}{dz_i}, \qquad Q_i = \frac{z_i}{X(z_i)}\frac{dX(z_i)}{dz_i} = 1 - z_i Y'(z_i)\psi'(Y(z_i)),$$

and $P_{g,n}^{(j_1,\dots,j_n)}$ is a polynomial combination of the functions of the form

$$\frac{z_j}{z_i - z_j}, \quad \psi^{(k)}(Y(z_i)), \quad \left(z_i\frac{d}{dz_i}\right)^k Y(z_i), \quad k \geq 1, \quad i,j = 1,\dots,n.$$

Main Theorem

$$X_i = X(z_i), \qquad X(z) = z\, e^{-\psi(Y(z))}.$$

Example

$$H_{0,3} = \frac{\psi_1'}{Q_1}\frac{z_2}{z_1-z_2}\frac{z_3}{z_1-z_3} + \frac{\psi_2'}{Q_2}\frac{z_3}{z_2-z_3}\frac{z_1}{z_2-z_1} + \frac{\psi_3'}{Q_3}\frac{z_1}{z_3-z_1}\frac{z_2}{z_3-z_2} - \psi'(0),$$

$$H_{1,1} = \frac{(\psi_1')^2 Y_1^{[1]} - 2\psi_1' + \psi_1'' Y_1^{[2]}}{24 Q_1} + D_1\left(\frac{(\psi_1')^2 Y_1^{[2]} + \psi_1'' Y_1^{[1]}}{24 Q_1}\right) + \frac{\psi'(0)}{12}.$$

$$\psi_k^{(i)} = \psi^{(i)}(Y(z_k)), \quad i \geq 1, \quad k = 1, \dots, n,$$
$$Y_k^{[i]} = \left(z_k \frac{\partial}{\partial z_k}\right)^i Y(z_k), \quad i \geq 1, \quad k = 1, \dots, n.$$

Main Theorem

$$X_i = X(z_i), \qquad X(z) = z\, e^{-\psi(Y(z))}.$$

Example

$g = 0$, $n \geq 3$:

$$D_1 \dots D_n H_{0,n} = \sum_{j_1 \dots j_n} D_1^{j_1} \dots D_n^{j_n} [v_1^{j_1} \dots v_n^{j_n}] \sum_{\substack{\gamma \in \text{ trees with} \\ n \text{ numbered vertices}}}$$

$$\prod_{i=1}^{n} \frac{e^{-v_i \psi(y)} \partial_y^{\mathrm{val}_\gamma(i)-1} e^{v_i \psi(y)} \big|_{y=Y(z_i)}}{Q_i} \prod_{(i,j) \in E(\gamma)} \frac{z_i z_j}{(z_i - z_j)^2}$$

The equality implies that the poles on the diagonals $z_i = z_j$ cancel out on the right hand side

Main Theorem

$$X_i = X(z_i), \qquad X(z) = z\, e^{-\psi(Y(z))}.$$

Corollary

If $\psi'(y)$ and $Y'(z)$ are rational functions then $H_{g,n}$ is rational in the case $2g-2+n>0$.

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Corollary

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Remark

The unstable *n*-point functions are given explicitly by

$$D_1 H_{0,1} = Y(z_1),$$

$$H_{0,2} = \log\left(\frac{z_1^{-1}-z_2^{-1}}{X_1^{-1}-X_2^{-1}}\right).$$

Concept of topological recursion (CEO-recursion): an overview

- $H_{g,n}$ is a rational function in $z_1, \dots, z_n$.
- Its possible poles are at $z_i = a_j$ where $\{a_1, a_2, \dots, a_N\}$ are the critical points of $X(z)$.
- Consider $H_{g,n}$ as a rational function in z_1. The *loop equations* determine the principal parts of its pole at $z_1 = a_j$ for $j = 1, 2, \dots, N$ inductively in (g, n).
- A rational function is uniquely determined by the principle parts of its poles (up to an additive constant which is fixed by the condition of vanishing at $z_1 = 0$). Hence, $H_{g,n}$ is determined by the loop equations uniquely.

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$H_{g,n}$ is determined by its behaviour at the points beyond the convergency domain of its power expansion at the origin!

Remarks:

- Rationality is an essential ingredient of the topological recursion and there is no way to apply (and even to formulate) it in the case when $\psi(y)$ and $Y(z)$ are general power series.

Topological Recursion

Remarks:

- Rationality is an essential ingredient of the topological recursion and there is no way to apply (and even to formulate) it in the case when $\psi(y)$ and $Y(z)$ are general power series.
- An application of the topological recursion requires explicit knowledge of the critical points a_j of the function $X(z)$. Their positions are determined by a high order algebraic equation. So that even in the cases when the topological recursion holds true it is not always useful for practical computations.

Topological Recursion

The actual relations of topological recursion are formulated in terms of *correlator differentials*:

$$\omega_{g,n} = d_1 \dots d_n H_{g,n} = \sum_{k_1 \dots k_n} f_{g,(k_1,\dots,k_n)} \prod_{i=1}^{n} k_i X_i^{k_i - 1} dX_i, \quad (g,n) \neq (0,2)$$

$$\omega_{0,2} = d_1 d_2 H_{0,2} + \frac{dX_1 dX_2}{(X_1 - X_2)^2}$$

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$$\omega_{0,2} = d_1 d_2 H_{0,2} + \tfrac{dX_1 dX_2}{(X_1 - X_2)^2}$$

Proposition

$$\omega_{0,1} = Y(z_1) \frac{dX(z_1)}{X(z_1)},$$

$$\omega_{0,2} = \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$$

Topological Recursion: initial data

Spectral curve data $(\Sigma, X, \omega_{0,1}, \omega_{0,2})$:

- $\Sigma = \mathbb{C}P^1$ with affine coordinate z;
- X is a local coordinate on Σ near the point $z = 0$;
- $\omega_{0,1} = Y(z_1)\frac{dX(z_1)}{X(z_1)}$ a differential on Σ;
- $\omega_{0,2} = \dfrac{dz_1\,dz_2}{(z_1 - z_2)^2}$ a bidifferential on $\Sigma \times \Sigma$.

Analyticity assumptions:

- $\frac{dX}{X}$ extends as a global meromorphic (i.e. rational) 1-form on Σ such that all its zeroes $a_1, \dots, a_N$ are simple;
- $Y(z)$ extends as a global rational function such that dY is nonvanishing at $a_1, \dots, a_N$.

Topological Recursion

Definition

The sequence of symmetric differentials $\omega_{g,n}$, $g \geq 0$, $n \geq 1$ on $\Sigma = \mathbb{C}P^1$ satisfy topological recursion with the spectral curve data $(\Sigma, X, \omega_{0,1}, \omega_{0,2})$, if for each (g,n) with $2g - 2 + n > 0$ the form $\omega_{g,n}$ viewed as a 1-form in the first argument satisfies:

- Rationality: $\omega_{g,n}$ extends as a global rational differential form on Σ^n.
- Projection property: $\omega_{g,n}$ has no poles other that $a_1, \dots, a_N$.
- The principal parts of its poles at $a_1, \dots, a_N$ satisfy the Linear and Quadratic Loop Equations.

LLE: $\omega_{g,n} \in d\widetilde{\Xi}$

QLE: $$\Big(\omega_{g-1,n+1}(z,z,z_{2,\dots,n}) + \sum_{\substack{g_1+g_2=g\\ J_1\sqcup J_2=\{2,\dots,n\}}} \omega_{g_1,|J_1|+1}(z,z_{J_1})\omega_{g_2,|J_2|+1}(z,z_{J_2})\Big)\frac{X}{dX} \in d\widetilde{\Xi}$$

where $\widetilde{\Xi}$ is the space of functions on Σ defined in a vicinity of the points $a_1, \dots, a_N$ and spanned by the functions of the form $D^k f$, $k \geq 0$, where f might have a pole at a_j of order at most one, and $D = X\frac{d}{dX} = \frac{1}{Q} z \frac{d}{dz}$.

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Poposition

If $(\Sigma, X, \omega_{0,1}, \omega_{0,2})$ *satisfy the natural analyticity assumptions, then the Rationality, Projection property, LLE, and QLE determine the forms* $\omega_{g,n}$ *uniquely*

Blobbed Topological Recursion

Theorem

Let $\psi(y)$ and $Y(z)$ satisfy the natural analyticity conditions. Then the corresponding correlator differentials $\omega_{g,n}$ encoding generalized Hurwitz numbers correspond to the spectral curve data $(\Sigma, X, \omega_{0,1}, \omega_{0,2})$ with

$$\Sigma = \mathbb{C}P^1, \qquad X = z\, e^{-\psi(Y(z))}, \qquad \omega_{0,1} = Y(z)\frac{dX(z)}{X(z)}, \qquad \omega_{0,2} = \frac{dz_1 dz_2}{(z_1 - z_2)^2},$$

and satisfy the Rationality property, LLE, QLE

The only (possibly) missing ingredient of TR is the projection property.

Proof: it is an obvious corollary of the very structure of the closed formula for $\omega_{g,n}$ (and a similar formula for the expression entering the QLE).

$\hbar^2$-deformations

Extend the definition of the potential F and the corresponding correlator functions $H_{g,n}$ and correlator differentials $\omega_{g,n}$ by allowing the initial functions ψ and Y depend in $\hbar^2$ as power series, i.e. replace ψ and Y in the definition of F by

$$\widehat{\psi}(y,\hbar) = \sum_{i=1}^{\infty} \hat{c}_i(\hbar^2) y^i = \psi(y) + O(\hbar^2),$$

$$\widehat{Y}(z,\hbar) = \sum_{i=1}^{\infty} \hat{q}_i(\hbar^2) z^i = Y(z) + O(\hbar^2).$$

Then:

- a formula for $\omega_{g,n}$ is valid in (almost) unchanged form;
- the spectral curve equation is determined by the $\hbar = 0$ components of $\widehat{\psi}$ and $\widehat{Y}$.

Corollary

Assume that ψ and Y satisfy the natural analyticity assumptions. Then, the rationality, LLE, and QLE hold true for any $\hbar^2$-deformations of the initial data such that the coefficients of $\hbar^{2k}$ in $\widehat{\psi}$ and $\widehat{Y}$ are rational

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General principle

For given ψ and Y there exist suitable their $\hbar^2$-deformations $\widehat{\psi}$, $\widehat{Y}$ such that the Projection property holds true and $\omega_{g,n}$'s satisfy Topological Recursion

This principle is not proved in full generality but there are many cases where it does work.

$\hbar^2$-deformations

Some cases for which the projection property (and thus the TR) is proved

$e^{\psi(y)}$, $Y(z)$ are polynomials (or rational functions)	$\widehat{\psi}=\psi$, $\widehat{Y}=Y$
$\psi(y)=y$, $Y(z)$ is a polynomials (or a rational function)	$\widehat{\psi}=\psi$, $\widehat{Y}=Y$
$\psi(y)$ polynomial, $Y(z)$ rational	$\widehat{\psi}(y)=\mathcal{S}(\hbar\partial_y)\psi(y)$, $\widehat{Y}=Y$
$\psi(y)=y$, $e^{Y(z)}$ is rational	$\widehat{\psi}=\psi$, $\widehat{Y}(z)=\frac{1}{\mathcal{S}(\hbar z\partial_z)}Y(z)$

$$\mathcal{S}(u)=\frac{e^{u/2}-e^{-u/2}}{u}$$

Operator formalism

$\mathcal{F} = \mathbb{C}[[p_1, p_2, \dots]]$ (bosonic) Fock space
$|0\rangle = 1$, $\langle 0| : \mathcal{F} \to \mathbb{C}$, $f \mapsto f(0)$
$J_k = k\partial_{p_k}$, $J_{-k} = p_k$, $k > 0$, $J_0 = 0$
$\mathcal{D}_\psi : s_\lambda \mapsto \sum_{(i,j)\in\lambda} \psi(\hbar(j-i))\, s_\lambda$
Then,

$$e^{\frac{1}{\hbar^2}F} = e^{\mathcal{D}_\psi} e^{\sum_{k=1}^{\infty} \frac{q_k J_{-k}}{k\hbar}} |0\rangle$$

All operators involved belong to the Lie algebra $\widehat{\mathfrak{gl}}_\infty$ of infinitesimal symmetries of KP hierarchy. This implies that $e^{\frac{1}{\hbar^2}F}$ is a KP tau-function

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Set

$$H_n = \sum_{g=0}^{\infty} \hbar^{2g-2+n} H_{g,n}, \qquad H_n^\bullet = \sum_{\{1,\dots,n\}=\bigsqcup_\alpha I_\alpha} \prod_\alpha H_{|I_\alpha|}(X_{I_\alpha})$$

Then we can represent $H_n^\bullet$ as a *vaccuum expectation value*

$$H_n^\bullet = \langle 0| \prod_{i=1}^{n} \left(\sum_{m=1}^{\infty} \frac{X_i^m}{m} J_m \right) e^{\mathcal{D}_\psi} e^{\sum_{k=1}^\infty \frac{q_k J_{-k}}{k\hbar}} |0\rangle$$

Computation of $H_{g,n}$, basic steps

1. Represent $H_n^\bullet$ as a Vacuum Expectation Value.
2. Apply the known commutation relations for the action of $\widehat{\mathfrak{gl}}_\infty$ in order to compute explicitly these VEV's as *power series*.
3. Apply inclusion/exclusion arguments to extract connected correlators
4. Apply Lagrange inversion formula in order to represent the result in a closed form

$$\sum_m X^m [z^m] e^{m\psi(Y(z))} G(z) = \frac{1}{Q(z)} G(z), \qquad X = z\, e^{-\psi(Y(z))}$$

5. Take care about the details: special cases, the contribution of singular terms etc.
6. Check the obtained relations by numerical computer experiments.