Summary of results in topological recursion

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This is an unofficial, bibliographical guide for the theory of the topological recursion and its applications[1]*. If you find that some results have been omitted or are incorrectly referred to or presented, you are welcome to contact me at gborot@mpim-bonn.mpg.de*

Contents

1 General properties

The topological recursion (TR) is an axiomatic construction of a family of correlation functions $\omega_{g,n}$ indexed by two integers $g, n \geq 0$, from the initial data of a spectral curve $\mathcal{C}$ equipped with a differential $\omega_{0,1}$ and a fundamental bidifferential of the second kind $\omega_{0,2}$. This construction has many properties which suggests that it provides non-trivial, interesting geometric invariants of a new type, and makes it suited for computations. This geometric nature has not been unveiled yet precisely enough, but is the subject of current work. Here is a list of those properties:

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1.a TR is defined as a normalized solution of loop equations [68, 25] – which are 'local' Virasoro constraints (more precisely constraints forming the positive part of a Witt Lie algebra). The set of all solutions to the loop equations is described by a generalization of TR, called blobbed TR [32]. [34] sets up a global version of topological recursion, useful in formulating the theory in case of spectral curve with higher order ramification points [36].

1.b TR fits in a more general (involving other Lie algebras) procedure of quantization of quadratic Lagrangians in symplectic vector spaces [83, 7].

1.c The recursive structure in TR is encoded in a Hopf algebra of graphs à la Ronco-Loday [54, 47].

2. For compact spectral curves $\mathcal{C}$, symplectic invariance of $\omega_{g,0}$ [69, 70] when $\mathcal{C}$ is embedded as a Lagrangian subvariety of $T^*\mathbb{C}$. This is one of the deepest and still mysterious property of TR.

3.a Variational formula – with respect to the parameters of the initial data $\omega_{0,1}$. $\omega_{g,n}$ encode the n-th order derivative of $\omega_{g,0}$ [68]. One advantage of TR, compared to other constructions say in integrable systems, is that it is a pointwise construction in the space of spectral curves.

3.b Holomorphic anomaly equation – obtained by studying variations of $\omega_{0,2}$, and specializing to $\omega_{0,2}$ which is modular, but non-holomorphic in the periods of the spectral curve [66].

3.c Singular limits: if a family of spectral curve degenerates, the $\omega_{g,n}$ diverge and one understands how [68].

4. Non-perturbative TR: when the spectral curve is compact and has non trivial topology, one can add corrections to the TR wave function so as to make it a section of a spinor line bundle on the spectral curve. These corrections involve theta functions and their derivatives, and offer a window into non-perturbative effects in matrix models and topological strings [65].

2 Existing open problems

1. **Quantum curve conjecture**: the TR wave function is annihilated by a differential (or difference) equation whose coefficients have controlled singularities. This has been proved for compact spectral curves of genus 0 [35], and for many other special cases *e.g.* [33, 105, 4, 102]. Quantum variety conjecture: for any $n \geq 1$ the TR n-point wave function is annihilated by a n-body quantum Hamiltonian whose coefficients have controlled singularities. This is known at least for the spectral curves coming from the 1-hermitian matrix model. A strong version of the conjecture is that loop equations are equivalent to this property.

2. **Integrability conjecture**: for compact spectral curve, the partition function is a formal tau function of the multi-KP hierarchy [20]. In particular, this would imply the quantum variety conjecture for compact spectral curves.

3. Proving (and understanding the geometric nature of) **symplectic invariance** in the most general setting possible. In particular, symplectic invariance for spectral curves of the form $\mathrm{Polynomial}(e^x, e^y) = 0$ is not covered by the current results, but is expected to be true as it is related to framing invariance in topological strings [37].

4. The **$\hat{A}$-TR conjecture** [45, 46, 21]: the non-perturbative TR wave function on the $\mathrm{SL}_2(\mathbb{C})$-character variety of hyperbolic knot in $\mathbb{S}^3$ is annihilated by the $\hat{A}$-difference equation – which is the difference equation satisfied by the colored Jones polynomial. A related (and stronger, in a sense) version of this conjecture is that the non-perturbative TR wave function computes the asymptotic expansion of the colored Jones polynomial.

5. The **mirror symmetry-TR conjecture**: show that global TR can compute (when there is no LG model available, the initial data should be determined) Gromov-Witten theory ancestor potential of symplectic varieties with non semi-simple quantum cohomology.

6. The **TR-CFT conjecture**. It is known that the loop equations can be identified with Ward identities in a CFT. More precisely, if the spectral curve is chosen as that of a Fuchsian (connection with simple poles only) Hitchin system with Lie group G on a Riemann surface $\mathcal{C}$, the TR amplitudes should give the asymptotic expansion (in the heavy limit) of the currents in the $\mathcal{W}_G$-algebra conformal field theory obtained by the Sugawara construction. This should work for both the classical case ($\beta = 2 \Rightarrow Q = 0$) and non-classical case ($\beta \neq 2 \Rightarrow Q \neq 0$). Unpublished work of Eynard and Ribault explores this problem. This may be used to obtain a TR proof of the already proved cases of the AGT conjecture (relating conformal blocks to Nekrasov partition functions of supersymmetric gauge theories), as well as its generalizations.

7. Prove the remaining unknown cases of TR in Hurwitz theory. In particular, the $r \geq 4$-spin q-orbifold Hurwitz numbers in genus > 0 which remain unsolved. In the latter case, the conjectural TR is equivalent to the conjectural r-spin (and q-orbifold generalization of) the ELSV-like formula of Zvonkine [101, 86].

8. What is the meaning in enumerative geometry (CohFT and their K-theoretic generalization, mirror symmetry, gauge theory, etc.) of the β-deformation (aka non-commutative) TR ?

3 Large N expansion of matrix models

1. The computation of the all-order large N expansion of $U(N)$-invariant correlation functions of a wide class of unitarily invariant hermitian matrix models, in computed by TR in a universal way from algebraic geometry on the spectral curve: 1-hermitian matrix model [5, 6, 56], chain of matrices and matrix model in external field [55, 72], multi-trace matrix model [25, 16] These expansions are formal in application to combinatorics and quantization, and have the status of Poincaré asymptotic series in random matrix theory. In the latter case, a growing corpus of works in analysis [1, 77, 28, 27, 78, 29, 15, 26] shows that such asymptotic expansions do exist.

2. Understanding the geometric nature of the first terms in this large N expansion. The leading covariance is related to the fundamental bidifferential of the second kind $\omega_{0,2}$, therefore to a (1d restriction of) the Gaussian free field on the half-spectral curve. The order 1 correction to the free energy $\omega_{g=1,n=0}$ is related to the Bergman tau function on Hurwitz space and the determinant of the Laplacian on the spectral curve [81, 63, 64].

3. Hermitian matrix models provide particular examples of KP and Toda integrable systems [76], generating series of enumerative geometry of surfaces (*e.g.* Kontsevich matrix model [82]), conformal field theories on the spectral curve [84, 85], partition function of gauge theories and topological string theory [90, 61, 62]. Therefore, the intuition developed in matrix models was important in reaching new applications of the topological recursion to geometry, although these applications do not enter the realm of matrix models.

4. β matrix models provide a 1-parameter deformation of hermitian matrix models, and their large N expansion is governed by the β-deformation of the topological recursion [41, 42, 43, 13]. β governs the transition between commutative and non-commutative curves. Namely, when $\beta N = O(1)$, the initial data for topological recursion becomes a D-module instead of a curve. Ongoing work of Eynard brings the theory to a axiomatic form and extends most properties of the TR to this non-commutative TR.

Here are two original applications to random matrix theory.

5. The all-order finite size corrections to the large deviations of the maximum eigenvalue in β-matrix ensembles is computed by β-TR [23, 31]. As a consequence, modulo a proof that two limits commute (which has not been obtained yet), the all-order left and right tails of Tracy-Widom β distribution (describing the fluctuations of the maximum of the eigenvalue of the random β-matrix model in a generic situation) are computed by TR. Only the first two terms are rigorously known in the literature, and this conjecture predicts in particular an expression for the transcendental

constant giving the order 1 term, which matches the (difficult) known cases for $\beta = 1, 2, 4$.

6. TR gives the all-order asymptotics of Töplitz determinants – which are related to integrals over $U(N)$ – for a class of symbols supported on several arcs [88]. This case does not belong to the Fisher-Hartwig class, which could be handled by Riemann-Hilbert steepest descent analysis, and therefore was not accessible before.

4 Flat connections on surfaces, WKB, Higgs moduli space

1 For any flat connection with meromorphic singularities on a compact curve $\mathcal{C}$, and any basis of flat sections, one can construct correlators which are solutions of loop equations [12, 11, 60, 10]. In the case of $\hbar$-connections, if the flat sections are of topological type (this constrains the form of their formal $\hbar \to 0$ expansion), their formal WKB expansion is computed by the TR wave function. This provides a conditional converse to the quantum curve conjecture. By analysis of the Knizhnik-Zamolodchikov(-Bernard) system, this also gives a perspective to study conformal blocks, as well as the quantization of the Higgs moduli space, whose implications are still under investigation. One may hope that it will shed light and give a topological recursion derivation of the AGT conjecture.

2.a The above theory has been applied to show that the formal $\hbar \to 0$ expansion of tau functions associated with all the six Painlevé equations coincide by the TR free energies [79]. In particular, various asymptotics of the universal local distributions in hermitian random matrix theory are computed by TR – *e.g.* left-tail asymptotics of Tracy-Widom distribution related to Painlevé II [18], large gap asymptotics of the sine-gap distribution related to Painlevé V [89].

2.b [74] relates WKB expansion of the GKZ hypergeometric system to topological recursion.

3. The theory of 1. has been extended to handle difference equations on $\mathbb{C}$ in [87] – see [Section 7, 3.] for an application.

4. [96, 51] proposed that the wave function of TR for Hitchin spectral curve satisfies a quantum curve equation – in close relation with quantization of the Higgs moduli space - although there is consensus that the proof is not complete as such for curves of genus > 0.

5. The Taylor expansion of the special Kähler metric on the base of Hitchin integrable systems is computed by the genus 0 sector of TR [9].

6. For elliptic spectral curves, [33] proved the quantum curve conjecture, and showed that it implies new identities between modular forms.

5 Combinatorics

1. TR solves the problem, on surfaces of arbitrary topology, of enumerating maps (aka fatgraphs, discretized surfaces, etc.) [59], maps carrying statistical physics models (like Ising model, loop model, Potts model, etc.) [19, 25], Grothendieck dessins d'enfants and hypermaps [80, 52]. As a consequence of the singular limit property, it provides a proof of critical exponents on surfaces of any topology, and formula for the asymptotics number of maps in the limit of a large number of vertices.

2. Colored tensor models have a combinatorial interpretation as generating series of discretized spaces of dimension D – and matrix models are retrieved for $D = 2$. Certain colored tensor models can be represented as multi-trace matrix models, and as such, their formal large N expansion is computed by blobbed TR [98, 14].

2.a TR solves the problem of enumerating branched covers (Hurwitz theory) of $\mathbb{P}^1$ with one arbitrary, sorted by genus of the covering – this was Bouchard-Mariño conjecture [38] proved in [67] (the earlier paper [24] proposed a proof which was later found to be incomplete, due to ill-defined manipulations of formal series). As a result, it gives a new proof – which is purely combinatorial proof if one is allowed to use Kontsevich theorem (former Witten conjecture) as an input – of the ELSV formula. TR also solves a large class of variants of this enumeration problem [2], as well as a q-orbifold [49] version and an r-spin version proved for $r = 3$, and for arbitrary r but genus 0 in [30]. This also results in various ELSV-type formulas for these different kind of Hurwitz numbers.

2.b A large class of Hurwitz problems were solved by TR in the series of papers [2, 3]. Other interesting cases which are not covered by these papers but are solved by TR are simple Hurwitz numbers (see 2.a), weakly monotone Hurwitz numbers [48],

3. As TR computes sums over fatgraphs, it has applications to combinatorics of the moduli space [40, 97, 8].

6 Chern-Simons theory

1. The large rank expansion of the LMO invariant of the Seifert 3-manifolds $\mathbb{S}^3/\Gamma$ [75, 91], where Γ is a finite isometry group, and colored HOMFLY invariant of the knots going along the fibers of the Seifert fibration, are computed by a specialization of the spectral curve of the relativistic ADE Toda chain [39, 22, 17]. When Γ is a finite cyclic group, this includes the case of HOMFLY invariants of torus knots [39]. When Γ is non abelian, it proves that the large rank expansion of the colored HOMFLY of knots along the fibers have singularities in the $u = N\hbar$-complex plane [22]. This

is contrast with colored HOMFLY invariants of knots in $\mathbb{S}^3$, which are Laurent polynomials in $e^{u/d}$ and $e^{\hbar/d}$ for some integer d.

2. See Section 2, 4.

7 Enumerative geometry of surfaces and mirror symmetry

0.a Kontsevich proved in [82] Witten's conjecture [104], stating that ψ-classes intersection numbers on $\overline{\mathcal{M}}_{g,n}$ satisfy Virasoro constraints. These constraints are equivalent to TR for generating series of ψ-classes intersections for the spectral curve $x = y^2/2$.

0.b The topological recursion for the Bessel curve $xy^2 = 1$ gives [50] the topological expansion of the Brézin-Gross–Witten matrix model – and this matrix integral is known since [94] to be a tau function of the KdV hierarchy. [99] constructs a sequence $(\Theta_{g,n})_{g,n}$ of cohomology classes on $\overline{\mathcal{M}}_{g,n}$, satisfying axioms different from that of a cohomological field theory, and such that TR amplitudes for the Bessel curve are the intersection of these classes with ψ-classes.

0.c Via hyperbolic geometry, Mirzakhani obtained in [95] a famous recursion on $2g - 2 + n > 0$ for the Weil-Petersson volume of the moduli space of bordered surfaces of genus g with n boundaries. After Laplace transform, this recursion is equivalent to TR for the spectral curve $y = \sin(2\pi\sqrt{2x})$.

1. These two examples led to far reaching generalizations. For any initial data, $\omega_{g,n}$ can be expressed in terms of tautological intersection theory on Deligne-Mumford moduli space of curves [57, 58, 44]. An important result of [92, 53] gives a partial converse: TR computes the correlation function of semi-simple cohomological field theories, and the action of the Givental group on CohFT can be explicitly carried to the initial data of TR. This result uses the difficult result of Teleman [103] that Givental group action is transitive on CohFT on a given semi-simple Frobenius algebra. The identification of this action on initial data of TR was done by study of graphs. It is also understood now in a spirit closer to Givental's original formula via [Section 1, 1.b].

2 Based on these results, the most remarkable achievement of TR is the proof [71, 73] of Bouchard-Klemm-Mariño-Pasquetti conjecture [37] for toric Calabi-Yau 3-folds X: TR for their mirror curve of X computes the closed and open Gromov-Witten invariants of X. The initial data of TR in this case is provided by toric mirror symmetry (identification of Frobenius manifolds: quantum cohomology on A, Landau-Ginzburg model B side). It should be noted that all those cases have semi-simple Frobenius manifolds.

3 In the case of the A_N-singularity, the loop equations are equivalent to W-constraints for the ancestor potential. [93]

4 0.a admits a generalization to the moduli space of open Riemann surfaces, and a modification of TR computes the relevant intersection numbers in this case [100]. This theory is still under investigation.

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