Introduction to the AdS/CFT correspondence
HW set 2

April 27, 2012

1 Reading

Read pages 1-30 from Martin, pages 1-14 from Lykken and pages 17-34 from Quevedo.
MAGOO pages 31-36 and Nastase pages 58-65
Review of AdS/CFT Integrability, Chapter VI.1: Superconformal Symmetry, Niklas Beisert 1012.3982

2 $\mathcal{N} = 4$ Lagrangian

Repeat the derivation of the $\mathcal{N} = 4$ SYM Lagrangian that we did in class.

- Use Martin (pages 21-27) to write down the most general $\mathcal{N} = 1$ theory with one (off-shell) vector multiplet and three (off-shell) chiral multiplets, all in the adjoint representation of the gauge group.
- $SU(3)$ symmetry rotates the three off-shell chiral multiplets.
- Convince yourself that

$$W = \alpha \frac{i\,g}{3!} \epsilon_{ijk} \operatorname{Tr} \phi^i \left[\phi^j, \, \phi^k\right] = \alpha \frac{g}{3!} \epsilon_{ijk} \, f^{abc} \, \phi^i_a \, \phi^j_b \, \phi^k_c \tag{2.1}$$

  is the only superpotential that leads to a renormalizable (by power-counting) Lagrangian that has $SU(4)$ R-symmetry plus gauge invariance. ($SU(4)$ forbids mass terms)

- Fix the unknown coefficient α so that $SU(4)$ R-symmetry is visible in the Yukawa terms.
  You must use the fact that the fermions λ^A_α is a **4** of $SU(4)$ and the scalars X^{AB} are in the **6** of $SU(4)$ that is a rank 2 anti-symmetric matrix. The X^{AB} are related to the three complex scalars ϕ^i as

$$X^{AB} = \left(\begin{array}{cc|cc} 0 & \phi^1 & \phi^2 & \phi^3 \\ -\phi^1 & 0 & \phi^*_3 & -\phi^*_2 \\ \hline -\phi^2 & -\phi^*_3 & 0 & \phi^*_1 \\ -\phi^3 & \phi^*_2 & -\phi^*_1 & 0 \end{array}\right) \tag{2.2}$$

  and obey the self-duality constraint

$$(X^{AB})^\dagger = \bar{X}_{AB} \equiv \frac{1}{2} \epsilon_{ABCD} \, X^{CD} \, . \tag{2.3}$$

- Finally, integrate out the auxiliary fields.

3 Conformal transformations

Check that for a conformal transformation $x'_\mu = x_\mu + v_\mu$ that rescales the metric

$$ds^2 = dx'_\mu dx'_\mu = [\Omega(x)]^{-2} dx_\mu dx_\mu \tag{3.1}$$

by the conformal factor is $\Omega(x) = 1 - \sigma_v(x)$, the infinitesimal conformal transformation is

$$\partial_\mu v_\nu + \partial_\nu v_\mu = 2\sigma_v \delta_{\mu\nu} \tag{3.2}$$

with

$$\sigma_v = \frac{1}{d}\partial \cdot v \qquad \text{and} \qquad v_\mu = a_\mu + \omega_{\mu\nu} x_\nu + \lambda x_\mu + b_\mu x^2 - 2x_\mu b \cdot x \tag{3.3}$$

4 Conformal algebra

Derive the conformal algebra in terms of $P_\mu, J_{\mu\nu}, K_\mu, D$ from the SO(d,2) algebra, given that $J_{d+1,d+2} = D$, $J_{\mu,d+1} = \frac{1}{2}(K_\mu - P_\mu)$, $J_{\mu,d+2} = \frac{1}{2}(K_\mu + P_\mu)$.

5 Inversion

Prove that the special conformal transformation

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2} \tag{5.1}$$

can be obtained by an inversion

$$I : x'_\mu = \frac{x_\mu}{x^2}$$

followed by a translation, and another inversion.

6 Oscillators

The oscillator representation is particularly convenient for superconformal algebras. For $\mathcal{N} = 4$ SYM we define the following bosonic and fermionic creation and annihilation operators

$$[\mathbf{a}^\alpha, \bar{\mathbf{a}}_\gamma] = i\delta^\alpha_\gamma, \qquad [\mathbf{b}^{\dot\alpha}, \bar{\mathbf{b}}_{\dot\gamma}] = i\delta^{\dot\alpha}_{\dot\gamma}, \qquad \{\mathbf{c}^a, \bar{\mathbf{c}}_c\} = \delta^a_c.$$

Generators are represented through oscillator bilinears:

$$\begin{aligned}
\mathrm{L}^\alpha{}_\gamma &\simeq \bar{\mathbf{a}}_\gamma \mathbf{a}^\alpha - \tfrac{1}{2}\delta^\alpha_\gamma \bar{\mathbf{a}}_\epsilon \mathbf{a}^\epsilon, & \mathrm{R}^a{}_c &\simeq \bar{\mathbf{c}}_c \mathbf{c}^a - \tfrac{1}{4}\delta^a_c \bar{\mathbf{c}}_e \mathbf{c}^e, \\
\bar{\mathrm{L}}_{\dot\gamma}{}^{\dot\alpha} &\simeq \mathbf{b}^{\dot\alpha} \bar{\mathbf{b}}_{\dot\gamma} - \tfrac{1}{2}\delta^{\dot\alpha}_{\dot\gamma} \mathbf{b}^{\dot\epsilon} \bar{\mathbf{b}}_{\dot\epsilon}, & \mathrm{D} &\simeq \tfrac{1}{2}\bar{\mathbf{a}}_\alpha \mathbf{a}^\alpha + \tfrac{1}{2}\mathbf{b}^{\dot\alpha}\bar{\mathbf{b}}_{\dot\alpha}, \\
\mathrm{P}_{\gamma\dot\alpha} &\simeq \bar{\mathbf{a}}_\gamma \bar{\mathbf{b}}_{\dot\alpha}, & \mathrm{K}^{\gamma\dot\alpha} &\simeq \mathbf{b}^{\dot\alpha} \mathbf{a}^\gamma, \\
\mathrm{Q}^a_\gamma &\simeq \bar{\mathbf{a}}_\gamma \mathbf{c}^a, & \mathrm{S}^\gamma_a &\simeq \bar{\mathbf{c}}_a \mathbf{a}^\gamma, \\
\bar{\mathrm{Q}}_{\dot\gamma a} &\simeq \bar{\mathbf{c}}_a \bar{\mathbf{b}}_{\dot\gamma}, & \bar{\mathrm{S}}^{\dot\gamma a} &\simeq \mathbf{b}^{\dot\gamma} \mathbf{c}^a.
\end{aligned}$$

Check that they satisfy the $\mathcal{N} = 4$ superconformal algebra provided that

$$(\bar{\mathbf{a}}_\alpha)^\dagger = \bar{\mathbf{b}}_{\dot\alpha}, \qquad (\mathbf{a}^\alpha)^\dagger = \mathbf{b}^{\dot\alpha}, \qquad (\bar{\mathbf{c}}_a)^\dagger = \mathbf{c}^a.$$

Moreover, given the fact that the $\mathcal{N}=4$ fileds are represented as:

$$\begin{aligned} \mathcal{F} &= |0\rangle \\ \lambda^a_\alpha &= c^\dagger \mathbf{b}^\dagger |0\rangle \\ \Phi^{ab} &= \left(\mathbf{c}^\dagger\right)^2 |0\rangle \\ \bar{\lambda}_{a\dot{\alpha}} &= \left(\mathbf{c}^\dagger\right)^3 \mathbf{a}^\dagger |0\rangle \\ \bar{\mathcal{F}} &= \left(\mathbf{c}^\dagger\right)^4 |0\rangle \end{aligned} \tag{6.1}$$

check their supersymmetry transformations.