Introduction to the AdS/CFT correspondence
HW set 4

May 25, 2012

1 Reading

- E. D'Hoker and D. Z. Freedman, Supersymmetric gauge theories and the AdS/CFT correspondence, hep-th/0201253 (section 4 pages 25-38)
- O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Large N field theories, string theory and gravity, hep-th/9905111 (subsection 1.3.1 pages 16-19 and subsection 2.2.1 pages 36-45)
- M. J. Duff, "TASI Lectures on Branes, Black Holes and Anti-de Sitter Space" hep-th/9912164 (pages 23-32)

2 Type IIA supergravity in 10D

With the help of P. West hep-th/9811101 pages 58-66 (or Polchinski "String theory" volume 2, chapter 12, pages 84-89), derive the bosonic part of type IIA supergravity Lagrangian in 10D by dimensionally reducing the bosonic part of $\mathcal{N} = 1$ supergravity in 11D.

3 AdS Space

The anti-de Sitter AdS_{p+2} is the hyperboloid defined by embedding in $p + 3$ dimensional space

$$-(X^0)^2 - (X^{p+2})^2 + \sum_{i=1}^{p+1} (X^i)^2 = -R^2 \ . \tag{1}$$

This definition makes its $SO(p + 1, 2)$ isometries manifest.
The metric in the embedding $p + 3$ space reads

$$(ds)^2 = -(dX^0)^2 - (dX^{p+2})^2 + \sum_{i=1}^{p+1} (dX^i)^2 \tag{2}$$

3.1 Global coordinates

Substitute [1]

$$
\begin{aligned}
X^0 &= R \cosh\rho \, \cos t \\
X^{p+2} &= R \cosh\rho \, \sin t \\
X^i &= R \sinh\rho \, \Omega^i \quad i = 1, \dots, p+1
\end{aligned} \tag{3}
$$

in (2) to find

$$
(ds)^2 = R^2 \left[d\rho^2 - \cosh^2\rho dt^2 + \sinh^2\rho \, (d\Omega_{p+1})^2 \right] \tag{4}
$$

The time coordinate the way it was defined has range $0 \leq t \leq 2\pi$. The hyperboloid has closed timelike curves. Consider the covering space $-\infty \leq t \leq +\infty$ that has no closed timelike curves. This is what we will call "the AdS space".
Change variable to $\tan\theta = \sinh\rho$. Show that the new variable has range $0 \leq \theta \leq \pi/2$ and that the metric becomes

$$
(ds)^2 = \frac{R^2}{\cos^2\theta} \left[-dt^2 + d\theta^2 + \sin^2\theta \, (d\Omega_{p+1})^2 \right] \tag{5}
$$

Note that the topology of this space is $\mathbb{R} \times \mathbb{S}^{p+2}$ as $d\theta^2 + \sin^2\theta \, (d\Omega_{p+1})^2 = (d\Omega_{p+2})^2$.

3.2 Poincaré coordinates

Substitute

$$
\begin{aligned}
X^0 &= \frac{1}{2u} \left[1 + u^2 \left(R^2 + (\vec{x})^2 - t^2 \right) \right] \\
X^{p+2} &= R \, u \, t \\
X^{p+1} &= \frac{1}{2u} \left[1 - u^2 \left(R^2 + (\vec{x})^2 - t^2 \right) \right] \\
X^i &= R \, u \, x^i \qquad \text{with} \quad i = 1, \dots, p
\end{aligned} \tag{6}
$$

in (2) and get

$$
(ds)^2 = R^2 \left[\frac{(du)^2}{u^2} + u^2 \left(-dt^2 + (d\vec{x})^2 \right) \right] \tag{7}
$$

Finally, show that setting $u = \frac{1}{y}$ leads to

$$
(ds)^2 = \frac{R^2}{y^2} \left(dy^2 - dt^2 + (d\vec{x})^2 \right) \tag{8}
$$

[1] where Ω_i are coordinates on a $p+1$-sphere.

4 Reissner-Nordström black holes and AdS space

Write down Einstein's equations with gravity coupled to a $U(1)$ gauge field.
Show that they admit a solution of the form

$$\begin{aligned} ds^2 &= -\Delta_+(\rho)\Delta_-(\rho)dt^2 + \Delta_+(\rho)^{-1}\Delta_-(\rho)^{-1}d\rho^2 + \rho^2 d\Omega_2^2 \\ F_{t\rho} &= \frac{Q}{\rho^2} \\ \Delta_\pm(\rho) &= \left(1 - \frac{r_\pm}{\rho}\right) \qquad r_\pm = G\left(M \pm \sqrt{M^2 - Q^2}\right) . \end{aligned} \tag{9}$$

with two horizons located at $r = r_+$ and $r = r_-$.

Cosmic censorship forbids a naked singularity[2] and requires that the singularity at $r = 0$ is hidden behind a horizon. Show that this implies that

$$M \geq |Q| . \tag{10}$$

This inequality should remind you of the BPS bound we discovered when studied susy algebras with central charges!

Consider the extremal geometry $M = |Q|$, and let the double horizon be at r_0. Change the radial coordinate to

$$r \equiv \rho - r_0 ; \tag{11}$$

and show that

$$\Delta_\pm = 1 - \frac{r_0}{\rho} = \left(1 + \frac{r_0}{r}\right)^{-1} \equiv H(r)^{-1} \quad \text{and} \quad \rho^2 = r^2\left(1 + \frac{r_0}{r}\right)^2 , \tag{12}$$

so that

$$ds^2_{\text{ext}} = -H(r)^{-2}dt^2 + H(r)^2\left(dr^2 + r^2 d\Omega_2\right)^2 . \tag{13}$$

In these coordinates $SO(3)$ is manifest (isotropic coordinates).

Then show that the near the horizon geometry ($r = 0$) looks like

$$\begin{aligned} ds^2 = & -\left(\frac{r}{r + r_0}\right)^2 dt^2 + \left(1 + \frac{r_0}{r}\right)^2 \left(dr^2 + r^2 d\Omega_2\right)^2 \\ & \to -\frac{r^2}{{r_0}^2}dt^2 + \frac{r_0^2}{r^2}dr^2 + r_0^2 d\Omega^2 . \end{aligned} \tag{14}$$

Defining yet another new coordinate

$$z \equiv \frac{r_0^2}{r} , \tag{15}$$

so that $dz/z = dr/r$, we find a direct product of an anti-deSitter spacetime with a sphere:

$$ds^2 \quad \to \underbrace{\frac{r_0^2}{z^2}\left(-dt^2 + dz^2\right)}_{AdS_2 } + \underbrace{r_0^2 d\Omega_2^2}_{\times \quad \mathbb{S}^2 .} . \tag{16}$$

Note that the Reissner-Nordström solution is also asymptotically flat! This means that it interpolates between two maximally symmetric spacetimes!

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[2]Naked is a singularity that is not "hidden" from the observer by an event of horizon. An observer can travel there and return with no obstruction to report on what was observed.

5 The D3 Brane solution

First derive the Type IIB sugra field equations:

$$R_{\mu\nu} = \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}{}^{\rho\sigma} + e^{2\Phi} \left(F_{1\mu} F_{1\nu} + \frac{1}{4} \tilde{F}_{3\mu\sigma\rho} \tilde{F}_{3\nu}{}^{\rho\sigma} + \frac{1}{24} \tilde{F}^{+}_{5\mu\rho\sigma\tau\upsilon} \tilde{F}^{+\rho\sigma\tau\upsilon}_{5\nu} \right) \tag{17}$$

where

$$\begin{cases} F_1 = dC \\ H_3 = dB \\ F_3 = dA_2 \\ F_5 = dA_4^{+} \end{cases} \qquad \begin{cases} \tilde{F}_3 = F_3 - CH_3 \\ \tilde{F}_5 = F_5 - \frac{1}{2} A_2 \wedge H_3 + \frac{1}{2} B \wedge F_3 \end{cases} \tag{18}$$

Then consider the following Ansatz: a constant dilaton ϕ, vanishing axion $C = 0$, vanishing two-forms $A_{(2)\mu\nu} = B_{\mu\nu} = 0$, $F_{(5)\mu\nu\rho\sigma\tau} \sim \epsilon_{\mu\nu\rho\sigma\tau\upsilon} \partial^{\upsilon} H$ and a metric of the form

$$ds^2 = H^{-\frac{1}{2}}(\vec{y}) dx^{\mu} dx_{\mu} + H^{\frac{1}{2}}(\vec{y}) d\vec{y}^2 \tag{19}$$

where x^{μ}, $\mu = 0, \cdots, 3$ will be the coordinates along the brane, while $\vec{y} \in \mathbb{R}^6$ the coordinates perpendicular to the brane. Show that the Ansatz above is a solution of the the sugra equations provided H is harmonic in the transverse directions (i.e. satisfies $\Box_y H = 0$, except at the position of the brane, where a pole will occur). Then, plug in the susy transformations dilatino and gravitino susy transformations

$$\begin{aligned} \delta\lambda &= \frac{i}{\kappa} \Gamma^{\mu} \epsilon^{*} \frac{\partial_{\mu}\tau}{\mathrm{Im}\tau} - \frac{i}{24} G_{3\mu\nu\rho} \Gamma^{\mu\nu\rho} \epsilon + (\mathrm{Fermi})^2 \\ \delta\psi_{\mu} &= \frac{1}{\kappa} D_{\mu}\epsilon + \frac{i}{480} F_{5\mu_1\cdots\mu_5} \Gamma^{\mu_1\cdots\mu_5} \Gamma_{\mu} \epsilon + \frac{1}{96} (\Gamma_{\mu}{}^{\rho\sigma\tau} G_{3\rho\sigma\tau} - 9 \Gamma^{\nu\rho} G_{3\mu\nu\rho}) \epsilon^{*} + (\mathrm{Fermi})^2 \end{aligned} \tag{20}$$

Observe that most of the terms immediately go to zero apart from one from which we lern that this solution preserves 16 supersymmetries (i.e. half of the total number).