Introduction to the AdS/CFT correspondence
HW set 6

June 29, 2012

1 Reading

- Richard J. Szabo "An Introduction to String Theory and D-brane Dynamics" chapter 2 and 3 discuss the bosonic string, subsections 5.2 and 5.3 T-duality, subsection 6.4 presents a derivation of the DBI action from the ws and finally, Chapter 7 discusses properties of the DBI.
- Barton Zwiebach "A First Course in String Theory" (Get the Second Edition for more!) chapters 12 and 13 discuss open and closed strings while 15 includes Quantization of open strings on Dp-branes. T-duality of closed strings is in 17 and T-duality of open strings in 18. Finally, read the subsections 23.5, 23.6 and 23.7 for a discussion very similar with the one we had in class motivating the AdS/CFT correspondence.
- E. D'Hoker and D. Z. Freedman, Supersymmetric gauge theories and the AdS/CFT correspondence, hep-th/0201253 (section 5 pages 40-47)
- O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Large N field theories, string theory and gravity, hep-th/9905111 (subsection 3.1 pages pages 55-61)

2 Closed vs open Strings

Begin by showing that Polyakov's action $S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}g^{\alpha\beta}\,\partial_\alpha X^\mu \partial_\beta X^\nu\,\eta_{\mu\nu}$ enjoys scale invariance. This will allow you to write Polyakov's action in the conformal gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\ \partial_\alpha X \cdot \partial^\alpha X \tag{1}$$

where the e.o.m. for X^μ reduce to the free wave equation

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right) X^\mu = 0 \qquad \text{or} \qquad \partial_+\partial_- X^\mu = 0 \tag{2}$$

using lightcone coordinates $\sigma^\pm = \tau \pm \sigma$.

You must still make sure that the equation of motion for $g_{\alpha\beta}$ is still satisfied!

Show that the equation of motion for $g_{\alpha\beta}$ is $T_{\alpha\beta} = 0$ which in conformal gauge looks like

$$T_{01} = \dot{X} \cdot X' = 0 \quad \text{and} \quad T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0 \qquad \text{or} \qquad (\partial_+ X)^2 = (\partial_- X)^2 = 0 \tag{3}$$

You will now solve the **free wave equations** subject to **two constraints** arising from $T_{\alpha\beta} = 0$. Imposing the boundary condition

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau) \tag{4}$$

(this is for **closed strings**) show that the solution can be written as

$$X^\mu(\sigma, \tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \tag{5}$$

where

$$X^\mu_L(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \, \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \, \tilde{\alpha}^\mu_n \, e^{-in\sigma^+} \; , \tag{6}$$

$$X^\mu_R(\sigma^-) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \, \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \, \alpha^\mu_n \, e^{-in\sigma^-} \; . \tag{7}$$

with x^μ and p^μ being the center of mass position and momentum of the string.
Then show that

$$\alpha_n = (\alpha^\mu_{-n})^* \quad , \quad \tilde{\alpha}_n = (\tilde{\alpha}^\mu_{-n})^* \tag{8}$$

because X has to be real.
Use the definition of Virasoro's

$$L_n = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m \qquad \tilde{L}_n = \frac{1}{2} \sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m \tag{9}$$

to show that the constraints take the form

$$(\partial_- X)^2 = \alpha' \sum_n L_n \, e^{-in\sigma^-} = 0 \qquad \text{and} \qquad (\partial_+ X)^2 = \alpha' \sum_n \tilde{L}_n \, e^{-in\sigma^+} = 0 \tag{10}$$

where by definition the zero mode

$$\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}} p^\mu \tag{11}$$

Putting everything together derive

$$M^2 \; = \; -p_\mu p^\mu \; = \; \frac{4}{\alpha'} N \; = \; \frac{4}{\alpha'} \tilde{N} \tag{12}$$

where

$$N = \sum_{n > 0} \alpha_n \cdot \alpha_{-n} \quad \text{and} \quad \tilde{N} = \sum_{n > 0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n} \tag{13}$$

$N = \tilde{N}$ is called the **level matching condition**.
As we discuss in class this is why closed strings lead to a theory spacetime of gravity.

For open strings σ lives on an interval $[0, \pi]$ rather that a circle. The mode expansion is obtained from that of the closed string through the "**doubling trick**" which identifies $\sigma \sim -\sigma$ on the $\mathbb{S}^1$. This immediately implies the **Neumann boundary condition** ($\partial_\sigma X = 0$)

$$X^\mu(\tau, \sigma) = X^\mu(\tau, -\sigma) \qquad \Rightarrow \qquad \alpha_n^\mu = \tilde{\alpha}_n^\mu \tag{14}$$

For open strings there is **only one set of oscillators** and the definition of the zero mode changes to

$$\alpha_0^\mu = \sqrt{2\alpha'} p^\mu \tag{15}$$

that finally leads to

$$M^2 \; = \; -p_\mu p^\mu \; = \; \frac{1}{\alpha'} N \tag{16}$$

In contrast to closed strings, for open strings we have states created by only one oscillator that yield spacetime gauge fields.

3 T-duality and the string coupling

Problem 18.5 from Zwiebach (this numbers correspond to the second edition of the book – the problems are also there in the first edition)

4 Born-Infeld action

Show that in four dimensions the Born-Infeld Lagrangian

$$\mathcal{L}_{BI} = -b^2 \sqrt{-\det\left(\eta_{\mu\nu} + \frac{F_{\mu\nu}}{b}\right)} + b^2 \tag{17}$$

can be written as

$$\mathcal{L}_{BI} = -b^2 \sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2} - \frac{F_{\mu\nu}\tilde{F}^{\mu\nu}}{16b^4}} + b^2 = -b^2 \sqrt{1 - \frac{E^2 - B^2}{b^2} - \frac{(\vec{E} \cdot \vec{B})^2}{b^4}} + b^2 \tag{18}$$

where E and B is the electric and magnetic field respectively.

The point of this exercise is to make you see that this action is a regularized version of Maxwell theory. The Born-Infeld theory removes the divergence of the electron's self-energy in classical electrodynamics by introducing an upper bound of the electric field at the origin.

Show that the equation of motion for the electric field is:

$$\nabla \cdot \left(\frac{\vec{E}}{\sqrt{1 - \frac{E^2}{b^2}}}\right) = 0 \tag{19}$$

In the presence of a point charge

$$\nabla \cdot \left(\frac{\vec{E}}{\sqrt{1 - \frac{E^2}{b^2}}}\right) = Q\delta(r) \tag{20}$$

show that it's solution is

$$\vec{E} = \frac{Q}{\sqrt{r^4 + \frac{Q^2}{b^2}}}\hat{r} \tag{21}$$

The electric field at the origin is not divergent!

5 String ending on a D-brane

Problems 20.6 and 20.7 from Zwiebach (second edition)

6 AdS/CFT: parameters identification

As we discussed in the class IIB string theory in $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ SYM under some parameter identification:

$$4\pi g_s = g_{YM}^2 \qquad \text{and} \qquad \frac{R^4}{\ell_s^4} = \lambda \tag{22}$$

where on the string theory side we have

- the string coupling constant g_s
- the effective string tension $R^2/\alpha' = R^2/\ell_s^2$.

while on the gauge theory side $\lambda = g_{YM}^2 N$

- N is the rank of the gauge color group
- g_{YM} the coupling constant.

Derive (22) by:

1. expanding the DBI action for the D3 brane and comparing it to the YM action and

$$\mathcal{L}_{DBI} = -T_p\sqrt{-\det\left(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right)} \sim T_p\left(2\pi\alpha'\right)^2 F_{\mu\nu}F^{\mu\nu} + \cdots \longrightarrow \mathcal{L}_{YM} = \frac{1}{g_{YM}^2}F_{\mu\nu}F^{\mu\nu} + \cdots \tag{23}$$

2. identifying the tension of the supergravity soliton solution (ADM mass) around N coincident $D3$ Branes (see last lecture and HW 4) with the D-Brane tension $T_p = \frac{1}{(2\pi)^p \ell_{st}^{p+1} g_{st}}$ that we calculated in class using string theory.

In order to expand the DBI action you need the identity

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \cdot \det\left(D - CA^{-1}B\right) = \det D \cdot \det\left(A - BD^{-1}C\right) \tag{24}$$

Prove that this identity is true by first observing that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} \tag{25}$$

7 The effective potential in Schwarzschild background

Consider a real, massless Klein-Gordon scalar field in the background of a Schwarzschild black hole

$$S = \frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi \,. \tag{26}$$

Plug in the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2MG}{r}\right) dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \,. \tag{27}$$

It is very useful to use the tortoise coordinate defined as

$$r^* = r + 2MG \log\left(r - 2MG\right) \,. \tag{28}$$

Define $\psi = r\phi$ and decompose ψ in spherical harmonics $Y_{\ell,m}$. After performing the angular integrals show that the action can be written in the following form

$$S = \sum_{\ell,m} S_{\ell,m} \qquad \text{with} \qquad S_{\ell,m} = \frac{1}{2} \int dt \, dr^* \left[\left(\frac{\partial \psi_{\ell m}}{\partial t} \right)^2 - \left(\frac{\partial \psi_{\ell m}}{\partial r^*} \right)^2 - V_\ell(r^*) \psi_{\ell m}^2 \right] \tag{29}$$

where the effective potential that the scalar field feels is

$$V_\ell(r^*) = \frac{r - 2MG}{r} \left(\frac{\ell\left(\ell+1\right)}{r^2} + \frac{2MG}{r^3} \right) \,. \tag{30}$$

Plot the potential for $\ell = 0, 1, 2$.

Repeat this calculation for a scalar in the background of a Dp-brane.

8 Derive Newton from Einstein

If you have never derived Newton from Einstein, now is the time! All you have to do is to impose that the particles are moving much slower that the speed of light and that the gravitational fields is week and static.

Write down the geodesics equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0 \tag{31}$$

and impose that the particles are moving slowly, i.e.

$$\frac{dx^i}{d\tau} << \frac{dt}{d\tau} \,. \tag{32}$$

The geodesic equations become

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{00} \left(\frac{dt}{d\tau} \right)^2 = 0 \,. \tag{33}$$

For a static field ($\partial_0 g_{\mu\nu} = 0$) show that

$$\Gamma^{\mu}_{00} = -\frac{1}{2} g^{\mu\nu} \partial_\nu g_{00} \,. \tag{34}$$

If the gravitational fields is week you can safely expand around Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{with} \qquad |h_{\mu\nu}| << 1 \,. \tag{35}$$

Using all the above show that

$$\frac{d^2 t}{d\tau^2} = 0 \qquad \Rightarrow \qquad \frac{dt}{d\tau} = \text{const} \tag{36}$$

and then that

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} \partial_i h_{00} \,. \tag{37}$$

This is Newtons equation

$$\vec{a} = -\vec{\nabla} V \tag{38}$$

if you identify $h_{00} = V$ the gravitational potential.