Introduction to the AdS/CFT correspondence
HW set 7

July 6, 2012

1 Reading

1. O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Large N field theories, string theory and gravity, hep-th/9905111 (section 3)
2. E. D'Hoker and D. Z. Freedman, Supersymmetric gauge theories and the AdS/CFT correspondence, hep-th/0201253 (sections 5 and 6 - pay special attention to table 7 page 50)
3. Horatiu Nastase, Introduction to AdS-CFT arXiv:0712.0689 (sections 9 and 10)
4. "Spherical harmonics for the compactification of IIB supergravity on S^5" Peter van Nieuwenhuizen arXiv:1206.2667
5. "The Mass Spectrum of Chiral N=2 D=10 Supergravity on S^5." Kim, Romans & van Nieuwenhuizen Phys.Rev. D32 (1985) 389
6. "The Operator Product Expansion of N=4 SYM and the 4-point Functions of Supergravity", D'Hoker, Mathur, Matusis, Rastelli hep-th/9911222
7. "On short and semi-short representations for four-dimensional superconformal symmetry" Dolan and Osborn hep-th/0209056

2 AdS/CFT dictionary: mass-conformal dimention correspondence

Begin by writing the action of a scalar field of mass m in AdS_{d+1}

$$\frac{1}{2}\int d^{d+1}x\sqrt{g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right] \ . \tag{1}$$

Derive the equation of motion for a scalar field in AdS_{d+1} and deduce it's boundary behavior. It is a good idea to use Poincaré coordinates. Show that the near boundary behavior ($z \to 0$) of the classical solution is

$$\phi(z,\vec{x}) \to z^{d-\Delta}[\alpha(\vec{x}) + \mathcal{O}(z^2)] + z^{\Delta}[\beta(\vec{x}) + \mathcal{O}(z^2)] \ , \tag{2}$$

where Δ is one of the roots of

$$\Delta(\Delta - d) = m^2 L^2 \ . \tag{3}$$

The easiest way is as usual by Fourier decomposition

$$\phi(z, \vec{x}) = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}\, \tilde{\phi}(z, \vec{k})\,. \tag{4}$$

Show that in the momentum space the equation of motion is

$$z^5 \partial_z \left(\frac{1}{z^3} \partial_z \tilde{\phi} \right) - z^2 k^2 \tilde{\phi} - m^2 R^2 \tilde{\phi} = 0 \tag{5}$$

the two independent solutions of which are

$$\tilde{\phi}(z, \vec{k}) = \begin{cases} k^2 z^2 I_{\Delta-2}(kz) \quad \longrightarrow \quad k^{\Delta} z^{\Delta} \quad \text{as} \quad z \to 0 & \text{normalizable} \\ \\ k^2 z^2 K_{\Delta-2}(kz) \quad \longrightarrow \quad k^{4-\Delta} z^{4-\Delta} \quad \text{as} \quad z \to 0 & \text{non-normalizable} \end{cases} \tag{6}$$

where $I_{\Delta-2}$ and $K_{\Delta-2}$ are the modified Bessel functions (see wikipedia).

As we discussed in class $\alpha(\vec{x})$ is regarded as a “source” function while $\beta(\vec{x})$ describes a physical fluctuation.

3 AdS/CFT dictionary: field - operator correspondence

Using the basic AdS/CFT dictionary guess which are the supergravity fields that are dual to the energy momentum tensor, the supersymmetry currents, the $SU(4)$ R-symmetry current as well as the operators $\mathrm{Tr}\,(F_{\mu\nu}F_{\rho\sigma})$ and $\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}_{\rho\sigma}\right)$.

Pick one out of the tree complex scalars (say Z) of $\mathcal{N}=4$ SYM and consider the operators made out of Z^ℓ times the operatos above (all inside a single trace). For example $\mathrm{Tr}\left(Z^\ell T_{\mu\nu}\right)$. Repeat the exercise that you did above to recognize the KK tower of the fields you found before.

4 Conformal invariance constrains 2 and 3 point functions

Show that the 2-point function is

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle = \frac{\delta_{\Delta_1, \Delta_2}}{|x_1 - x_2|^{2\Delta_1}}\,. \tag{7}$$

Using Poincaré symmetry show that it can only depends on $(x_1 - x_2)^2$. Then using inversion symmetry show that it must vanish unless $\Delta_1 = \Delta_2$. Finally, scaling symmetry fixes the exponent. Similarly, check that the 3-point function is given by

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta_3}(x_3) \rangle = \frac{c_{123}(\lambda, N)}{|x_1 - x_2|^{\Delta - 2\Delta_3} |x_2 - x_3|^{\Delta - 2\Delta_1} |x_3 - x_1|^{\Delta - 2\Delta_2}} \tag{8}$$

where $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ and the c_{123} is the OPE coefficient

$$\mathcal{O}_i(x) \mathcal{O}_j(0) = \sum_k c_{ijk} \frac{\mathcal{O}_k(0)}{x^{\Delta_i + \Delta_j - \Delta_k}} \tag{9}$$

that can only depend on λ and N.

4.1 Tree level calculation of 2- and 3- point functions

In this exercise I want you to calculate 2 and 3 point functions (7) and (8) respectively using usual filed theory technics. This calculation will be useful for getting factors of N right from the field theory side.

Compute the correlation functions of the composite operators

$$\mathcal{O}(x) \sim \mathrm{Tr}\left(X^{i_1} \cdots X^{i_\ell}\right)(x) \tag{10}$$

at tree level by performing the Wick contractions. The propagator of a scalar field

$$\langle X^{ma}(x_1) X^{na'}(x_2)\rangle = \frac{\delta^{mn}\delta^{aa'}}{4\pi^2 (x_1 - x_2)^2} \tag{11}$$

a is an index in the adjoint representation of the color $SU(N)$ while $m, n = 1, \cdots, 6$ are the fundamental $SO(6)$ indices. Normalize in such a way that the 2-point function

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{(x_1 - x_2)^{2\Delta_1}} \tag{12}$$

will not come with any factor. What will be the overall normalization of the 3-point function?

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta_3}(x_3)\rangle \sim \frac{1}{(x_1 - x_2)^{\Delta_{12}} (x_2 - x_3)^{\Delta_{23}} (x_3 - x_1)^{\Delta_{31}}} \tag{13}$$

5 2 and 3 point functions from the bulk

Using the bulk-to-boundary propagator $K_\Delta(z, \vec{x})$ for a scalar field with conformal dimension Δ derive the 2 and 3 point functions (7) and (8) respectively from the AdS side.

1. The two point function is extremely simple: just send $z \to 0$ and extract a factor z^Δ.

2. For the 3-point function you have to do an integral over the intermediate interaction point $(z, \vec{x})$ in the bulk

$$\int_{AdS} \frac{dz\, d^4x}{z^5} \prod_{i=1}^{3} C_{\Delta_i} \left(\frac{z}{z^2 + (\vec{x} - \vec{x}_i)^2}\right)^{\Delta_i} \tag{14}$$

   To succeed in doing the integral without even trying (hep-th/9905049) you should take the following steps. First, use a translation to set $\vec{x}_3 = 0$. Then, use an inversion around 0 ($x^\mu \to x^\mu/|x|^2$) to set $\vec{x}_3' = \infty$. The integral simplifies

$$\sim (x_{13}')^{2\Delta_1} (x_{23}')^{2\Delta_2} \int_H \frac{dz\, d^4x}{z_0^5} \frac{z_0^{\Delta_1+\Delta_2+\Delta_3}}{z^{2\Delta_1}[z_0^2 + (\vec{x} - \vec{x}_{13}' - \vec{x}_{23}')^2]^{\Delta_2}}\,. \tag{15}$$

   Finally, using translation invariance and Feynman parametrization for the $\vec{x}$ part of the integral and then carrying out the z integral you can finish the job.