THE ASYMPTOTIC STABILITY OF A NOISY NONLINEAR OSCILLATOR

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The purpose of this work is to obtain an approximation for the top Lyapunov exponent, the exponential growth rate, of the response of a single-well Kramers oscillator driven by either a multiplicative or an additive white noise process. To this end, we consider the equations of motion as dissipative and noisy perturbations of a two-dimensional Hamiltonian system. A perturbation approach is used to obtain explicit expressions for the exponent in the presence of small intensity noise and small dissipation. We show analytically that the top Lyapunov exponent is positive, and for small values of noise intensity $\sqrt{\epsilon}$ and dissipation ϵ the exponent grows in proportion with $\epsilon^{\frac{1}{3}}$.

1 INTRODUCTION

No theorem has had so direct and powerful an influence upon the study of stochastic stability of noisy dynamical systems as the multiplicative ergodic theorem (MET) of Oseledec [1], which established the existence of (typically) finitely many deterministic exponential growth rates called *Lyapunov exponents*. The stability of *linear* stochastic systems based on MET has been well established [2, 3] and the top Lyapunov exponent can be evaluated explicitly with relative ease when the noisy perturbations and dissipation are weak [4, 5].

The primary concern in the analysis of *nonlinear* dynamical systems, is the determination and prediction of steady-states or stationary motions (e.g. invariant measures of the local random dynamical systems), and their corresponding stability. The challenge is to explicitly evaluate the top Lyapunov exponents of these stationary measures asymptotically when the noise is weak, which is the problem we shall address in this paper.

For example, many engineering systems under additive white noise excitations can be expressed as

$$\ddot{x}_t^i + \beta_i \dot{x}_t^i + \frac{\partial U}{\partial x^i}(x_t) = \xi_i(t), \quad i = 1, 2, ..., n, \tag{1}$$

where $\xi_i(t)$'s are stationary stochastic processes, β_i's represent the damping in each mode, and U is the potential. Under the assumptions that $\xi_i(t)$'s are uncorrelated Gaussian processes, and the ratio of the spectral density of each excitation, $\xi_i(t)$, to the corresponding damping, β_i, is the same, i.e.,

$$\gamma = \frac{\beta_i}{\kappa_{ii}} \quad \text{for all } \ i, \quad \text{where} \quad \mathbb{E}[\xi_i(t+\tau)\xi_i(t)] = 2\kappa_{ii}\delta(\tau),$$

the stationary probability density of (1) can be easily written as

$$p(x, \dot{x}) = C \exp\left\{-\gamma\left[\frac{1}{2}\sum_{i=1}^{n}(\dot{x}^i)^2 + U(x)\right]\right\}.$$

Such stationary probability densities exist for an even larger class of multi-dimensional nonlinear systems and there is a vast engineering literature that deals with the determination of such stationary densities (see for example, Lin and Cai [6]). However, there are no concrete results on the sign of the top Lyapunov exponents corresponding to these stationary measures. Hence, their stability is not known. The study of asymptotic stability of nonlinear systems with noise, which we shall address in this paper, opens the door to a host of physically interesting problems in random vibrations, from simple oscillators to noisy autoparametric systems.

Schimansky-Geier and Herzel [7] were the first to consider numerically the Lyapunov exponents of a two dimensional nonlinear system under additive noise. Their work was devoted to the effect of noise on the Kramers oscillator

$$\ddot{x}_t + \epsilon\, \dot{x}_t + U'(x_t)) = \sqrt{2\epsilon}\, \xi(t), \tag{2}$$

where $U(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4,\ a, b > 0$, with double-well potential, which was studied by Kramers in his celebrated work [8]. It was shown [7] that the top Lyapunov exponent is positive, i.e.,

$$\lambda(\epsilon) > 0 \quad \text{for } \epsilon \text{ not too large.}$$

The top Lyapunov exponent is determined by the simultaneous behavior of two neighboring orbits, or the two-point motion of the Kramers oscillator. A positive Lyapunov exponent implies that, while for each single initial condition the corresponding solution trajectory builds-up a nontrivial stationary measure, the distance between any two initial conditions will grow at an exponential rate. Hence, an additive noise in (2) induces an unstable stationary measure. Our task in this paper is to show analytically this remarkable observation for (2) as well for similar systems with multiplicative noise. A brief summary of these results was published in Arnold *et al.* [9]

The Kramers oscillator with double-well potential, considered by Schimansky-Geier and Herzel [7], has multiple fixed points, one of which is connected to itself by a homoclinic orbit. The procedure presented here relies upon an implicit assumption that the instantaneous frequency of the unperturbed motion ($\epsilon = 0$) must be non-zero or the periods of oscillations or rotations are finite. Hence, a subtle treatment is necessary in a neighborhood of the homoclinic orbit where the unperturbed orbits have arbitrarily long periods. In order to remedy this problem, two different models, one which is valid away from the homoclinic orbit, the other valid in a boundary layer about the homoclinic orbit should be introduced and it is beyond the scope of this paper. Thus, we do not consider it fruitful to attempt to make a general theory for all types of two dimensional nonlinear Hamiltonians. Rather, we restrict our development to the case for *Hamiltonians with isolated single elliptic fixed point*, i.e., a weakly perturbed oscillator with a single-well potential

$$U(x) = \frac{a}{2}x^2 + \frac{b}{4}x^4, \quad a, b > 0, \tag{3}$$

excited by a *white noise* process, $\xi(t)$. Here we present a general, effective, systematic approach to determine the asymptotic sample stability of weakly perturbed (dissipatively and stochastically) two dimensional nonlinear Hamiltonian systems. Random perturbations of Hamiltonian systems are of great interest, particularly, in the study of noisy nonlinear mechanical systems. Randomly-perturbed Hamiltonian system on $\mathbb{R}^2$ with multiple fixed points are considered by Freidlin and Wentzell [10] in the context of stochastic averaging and by Freidlin and Wentzell [11] in the context of large deviations techniques. The analysis developed in this paper could be extended with some effort to provide analogous theorems pertaining to Hamiltonians with multiple fixed points. The versatility of the method presented here, will make this method to be adopted to such situations.

In Section 2, we shall place the random vibration problem (1) within the general framework of *random dynamical systems.* In Section 3, we state the mathematical structure of the problem and briefly recall some results obtained by Arnold and Imkeller [12] which are relevant to this paper. In section 4, we introduce the concept of action-angle variables [13], apply the classical results of symplectic transformation and derive the evolution of the action-angle variables. In section 5, due to the nilpotent structure of the linear variational equations, Pinsky and Wihstutz [14] re-scaling is used in the linear variational equations to derive the Furstenberg-Khasminskii formula. In sections 6 and 7 we appeal to the results of Sri Namachchivaya and Van Roessel [5] and Imkeller and Lederer [15] to evaluate the first term in the asymptotic expansion of the top Lyapunov exponent.

2 RANDOM DYNAMICAL SYSTEMS

Here we restrict ourselves to the smooth (i.e. C^∞) case, two-sided continuous time $\mathbf{T} = \mathbb{R}$, and state space $\mathbb{R}^d$. A *smooth random dynamical system* consists of the following two "ingredients"(see Arnold [3]):

1. Model of the noise: A *metric dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ (for short: θ), i.e. a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a measurable flow of measure preserving transformations $\theta_t : \Omega \to \Omega$, i.e. $\theta_0 = id$, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$, $\theta_t\mathbb{P} = \mathbb{P}$, and $(t, \omega) \mapsto \theta_t\omega$ measurable.

2. Model of the system perturbed by noise: A *cocycle* φ over θ of smooth mappings of $\mathbb{R}^d$, i.e. a measurable mapping

$$\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d,\ (t, \omega, x) \mapsto \varphi(t, \omega)x,$$

   for which $(t, x) \mapsto \varphi(t, \omega)x$ is continuous in (t, x) and smooth in x, and φ satisfies the *cocycle property*

$$\varphi(0, \omega) = \mathrm{id}_{\mathbb{R}^d}, \qquad \varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega) \quad \forall s, t \in \mathbb{R} \quad \text{and} \quad \omega \in \Omega.$$

   The cocycle property implies that $\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t\omega)$, i.e. the mapping $\varphi(t, \omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a (smooth) diffeomorphism.

The flow Θ_t on $\Omega \times \mathbb{R}^d$ given by $\Theta_t(\omega, x) := (\theta_t\omega, \varphi(t, \omega)x)$ is called the *skew product flow* corresponding to φ.

Dynamical systems driven by white noise are rigorously dealt with in stochastic analysis and are solutions of (Stratonovich) *stochastic differential equations*

$$dx = f(x)\, dt + g(x) \circ dW_t, \tag{4}$$

where f, g are smooth vector fields in $\mathbb{R}^d$, which is short for

$$\varphi(t,\cdot)x = x + \int_0^t f(\varphi(s,\cdot)x)\,ds + \int_0^t g(\varphi(s,\cdot)x) \circ dW_s.$$

Let us now consider (4), or equivalently (1), in the context of *random dynamical systems. White noise* can be canonically modeled as a metric dynamical system as follows: Let $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}^d) : \omega(0) = 0\}$, $\mathcal{F}$ the Borel sigma-algebra of Ω, and $\mathbb{P}$ the Wiener measure, i.e. the measure generated by the Wiener process (Brownian motion) $(W_t)_{t\in\mathbb{R}}$ in $\mathbb{R}^m$. This process has stationary independent increments with $W_{t+h} - W_t \sim \mathcal{N}(0, |h|I)$, continuous trajectories, and satisfies $W_0 = 0$. The shift $\theta_t\omega(\cdot) := \omega(t+\cdot) - \omega(t)$ leaves $\mathbb{P}$ invariant since the increments are stationary. Then θ is an ergodic metric dynamical system on $(\Omega, \mathcal{F}, \mathbb{P})$ "driving" the stochastic differential equation (4) and $W_t = \omega(t)$.

Theorem 1 (Arnold and Scheutzow [16]) *Let $f, g \in C_b^\infty$. Then the stochastic differential equation (4) has a unique solution $x \mapsto \varphi(t,\omega)x$ which is a smooth random dynamical system. The Jacobian $D\varphi(t,\omega,x)$ is a matrix cocycle over Θ and uniquely solves the* variational equation

$$dv = Df(\varphi(t,\cdot)x)\, v\, dt + Dg(\varphi(t,\cdot)x)\, v \circ dW_t. \tag{5}$$

2.1 INVARIANT MEASURES

For all further steps we need the notion of an invariant measure for a random dynamical system. Let φ be a random dynamical system. A random probability measure $\omega \mapsto \mu_\omega$ on $(\mathbb{R}^d, \mathbb{B}^d)$, where $\mathbb{B}^d$ denotes the space of Borel sets in $\mathbb{R}^d$, is called *invariant* under φ, if

$$\varphi(t,\omega)\mu_\omega = \mu_{\theta_t\omega} \quad \mathbb{P} - a.s. \text{ for all } t \in \mathbb{R}.$$

For random dynamical systems whose one-point motions $\mathbb{R}^+ \ni t \mapsto \varphi(t,\omega)x$ are Markov processes with transition probability $P(t,x,B) = \mathbb{P}\{\omega : \varphi(t,\omega)x \in B\}$ and generator G

(for solutions of stochastic differential equation (4)), a measure ρ on $\mathbb{R}^d$ is called *stationary* if it satisfies for all $t \in \mathbb{R}^+$

$$\rho(\cdot) = \int_{\mathbb{R}^d} P(t, x, \cdot)\, \rho(dx),$$

equivalently, if it solves the *Fokker-Planck equation*

$$G^* \rho = 0, \quad \text{with} \quad G = f + \frac{1}{2} g^2 \tag{6}$$

Here we have written G in the Hörmander form. There is a one-to-one correspondence between stationary ρ's and those invariant μ_ω's for φ which are measurable w.r.t. the past $\mathcal{F}_{-\infty}^0$ of the noise, via the "pullback"

$$\rho \mapsto \mu_\omega = \lim_{t \to \infty} \varphi(-t, \omega)^{-1} \rho, \qquad \mu_\omega \mapsto \mathbb{E}\mu_\cdot = \rho, \tag{7}$$

(see Arnold [3], Sect. 1.7). The procedure of passing from a deterministic stationary measure ρ to a random invariant measure μ_ω described by (7) is called *disintegration* of ρ. There are, however, in general more invariant measures μ_ω than those obtained from stationary measures.

2.2 LYAPUNOV EXPONENTS

The fundamental theorem of Oseledec [1] provides us with the stochastic analogues of a deterministic eigenvalue and eigenspace of a matrix. Let φ be a smooth random dynamical system, and let μ be an ergodic invariant measure. It is clear from Theorem 1 that $D\varphi$ is a linear cocycle over Θ and uniquely solves the linear variational equation (5). The exponential growth rate

$$\lambda(\omega, x, v) := \lim_{t \to \infty} \frac{1}{t} \log \|D\varphi(t, \omega, x)v\|,$$

describes the *Lyapunov exponent* of the solution $v_t(x, v)$, for the the initial condition v ($v \neq 0$) in (5). According to MET [1], λ takes on one of r fixed or non-random values $\lambda_1 < \cdots < \lambda_r$. Which λ_i is realized depends on the initial condition v. The multiplicities of the Lyapunov exponents sum to the dimension of the system, d. The maximum of these, λ_1, determines the almost-sure stability of the random dynamical system $\varphi(t,\omega)$ generated by (4) under the stationary measure ρ [2, 3].

Rewriting the variational equation (5) in polar coordinates

$$s = \frac{v}{\|v\|} \in \mathbb{S}^{d-1}, \qquad r = \|v\| \in (0,\infty)$$

yields

$$dr_t = q_0(x_t, s_t) r_t dt + q_1(x_t, s_t) r_t \circ dW_t, \qquad ds_t = h_0(x_t, s_t)dt + h_1(x_t, s_t) \circ dW_t, \qquad (8)$$

where

$$\begin{aligned} h_0(x,s) &\stackrel{\text{def}}{=} Df(x)s - q_0(x,s)s, \quad q_0(x,s) \stackrel{\text{def}}{=} \langle Df(x)s, s\rangle \\ h_1(x,s) &\stackrel{\text{def}}{=} Dg(x)s - q_1(x,s)s, \quad q_1(x,s) \stackrel{\text{def}}{=} \langle Dg(x)s, s\rangle \end{aligned}$$

and $\langle x, y\rangle$ is the standard scalar product in $\mathbb{R}^d$. In (8), the equation for s_t is decoupled from the one for r_t, so that the pair (x_t, s_t) forms a Markov process with state space $\mathbb{R}^d \times \mathbb{S}^{d-1}$, whose generator for the *additive noise* case simplifies to $L \stackrel{\text{def}}{=} G + h_0(x,s)\frac{\partial}{\partial s}$. Integrating the equation for the radial process r_t in (8) and using the classical ergodic theorem yields the *Furstenberg-Khasminskii formula* [[3], Chap. 6] for the top Lyapunov exponent

$$\lambda = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} Q(x,s)\ \nu(dx, ds) \qquad (9)$$

where Q is some explicitly known function, which for the *additive noise* case simplifies to $Q(x,s) = q_0(x,s)$ and ν is the (to be determined) joint stationary measure for the Markov process (x_t, s_t) on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ with marginal ρ on $\mathbb{R}^d$. The sign of λ in (9) is of particular interest as it determines the stability of the variational equation (5) and in turn the stability of the original nonlinear random dynamical system generated by (4). Formula (9) forms the basis of all asymptotic studies of Lyapunov exponents and particularly the presentation given in this paper.

2.3 SCALAR NOISY NONLINEAR SYSTEMS

Before we proceed further, we should mention in this context some well-known results pertaining to one-dimensional nonlinear stochastic systems. It has been shown that the two point motion of a one dimensional nonlinear stochastic system has a unique property. More precisely if a noisy one dimensional equation,

$$\dot{x}_t = f(x_t) + g(x_t)\xi(t), \qquad x_0 = x \in \mathbb{R}, \tag{10}$$

has a stationary invariant measure with density

$$p(x) = \frac{N}{g(x)} \exp\left\{ \int^x \frac{2f(\eta)}{g^2(\eta)} d\eta \right\}, \tag{11}$$

provided $p(x)$ is normalizable, then as in Arnold [3], the Lyapunov exponent is

$$\lambda = -2 \int_0^\infty \left[\frac{f(x)}{g(x)} \right]^2 p(x)\, dx. \tag{12}$$

The Lyapunov exponent is always negative provided $f(x) \neq 0$. Similar results are also presented by Leng et al. [17].

The challenge has been to extend the existing techniques in order to explicitly evaluate the top Lyapunov exponent of higher ($d \geq 2$) dimensional nonlinear systems with noise, and in particular additive white noise. It is this need and challenge that we shall address in this paper.

3 STATEMENT OF THE PROBLEM

We consider an idealized particle moving in a symmetric single well potential described by a function U defined on $\mathbb{R}$. The Hamiltonian of the system will be given by

$$H(x,y) = U(x) + \frac{y^2}{2}, \quad x, y \in \mathbb{R}.$$

and it is assumed that the Hamiltonian has *an isolated elliptic fixed point.* The purpose of this paper is to examine the asymptotic sample stability of this nonlinear system under random and dissipative perturbations. We restrict to this class of potentials from the beginning to make the calculations of the top Lyapunov exponent become less cumbersome. The particular set of global variables discussed in the subsequent sections of this paper will shed light on this restriction. Formally, we assume

$$U \geq 0, \quad U(0) = 0, \quad U(x) = U(-x), x \in \mathbb{R},$$

and $x \mapsto U(x)$ strictly increasing on $\mathbb{R}_+$. The motion of the corresponding Hamiltonian system is periodic returning to the same point $x, y \in \mathbb{R}$ in the phase space after a period $T(x, y)$. For each $x, y \in \mathbb{R}$, define the return time

$$T(x, y) \stackrel{\text{def}}{=} \inf \{t > 0 : \xi_t(z) = z\} \tag{13}$$

where $\xi_t(z)$ is the Hamiltonian flow for all $(x, y) = z \in \mathbb{R}^2$. It is clear that T depends solely on $H(x, y)$ and that it is nonnegative on $\mathbb{R}^2 \setminus \{0\}$. Thus we start out with a Hamiltonian energy function with a very simple structure.

Assumption 3.1 (Hamiltonian)*: We assume that $H : \mathbb{R}^2 \to \mathbb{R}$ is $\mathbf{C}^\infty$ and nonnegative. We assume also that $H(x, y) = 0$ if and only if $x = 0, y = 0$. Secondly, $H(x, y) = H(-x, -y)$ for all $x, y \in \mathbb{R}$. Thirdly, we assume that*

$$A \stackrel{\text{def}}{=} D^2 H(0)$$

is positive-definite. Finally, we assume that for each $h > 0$, the set $H^{-1}(h)$ is connected and of finite 1-dimensional Hausdorff measure.

Finally, we assume that the particle is *weakly damped* and *weakly perturbed* by a white noise process. The primary concern is the determination of the stability of the stationary invariant measures, which are the stochastic analogue of steady state solutions in non-linear deterministic systems. The perturbations are scaled by appropriate powers of ϵ, ($\epsilon << 1$), in order to obtain the effect of the damping and the noise at the same order. To this end, random perturbation of a two-dimensional Hamiltonian system, with an isolated elliptic fixed point, is precisely given by

$$
\begin{aligned}
dx_t &= y_t \, dt, \\
dy_t &= (-\epsilon \, y_t - U'(x_t)) \, dt + \sqrt{2\epsilon} \, \sigma(x_t, y_t) \circ dW_t.
\end{aligned}
\tag{14}
$$

Here $\sigma : \mathbb{R}^2 \to \mathbb{R}$ is supposed to be a smooth function of sublinear growth. Equation (14) represents the random vibration of single degree of freedom mechanical systems under either parametric or additive white noise excitations. Hence, the typical examples that we consider are given by the *additive noise* case, i.e., $\sigma(x, y) = \sigma = const$, which has been studied extensively in the literature (see for example, Bolotin [18]), or by the *multiplicative noise* coupled to the displacement, i.e. $\sigma(x, y) = x$, or the velocity, i.e. $\sigma(x, y) = y$. Our aim is to obtain an asymptotic expansion of the top Lyapunov exponent of the random dynamical system described in (14) by making use of the prescribed scaling.

Now, we shall place our noisy Hamiltonian system (14) within the general framework of *random dynamical systems* presented in the previous Section 2 and briefly recall some results obtained by Arnold and Imkeller [12] which are relevant to this paper. First, a straightforward application of Arnold and Scheutzow's [16] results on generation of random dynamical systems for continuous time yields the following result.

Theorem 2 *The stochastic differential equation* (14) *uniquely generates a smooth random dynamical system* φ *in* $\mathbb{R}^2$.

Although the random dynamical system φ depends on ε, the invariant measure is independent of ε, that is,

Theorem 3 *The stochastic differential equation* (14) *has a unique stationary measure given by*

$$
p(x, y) = C \exp\left\{ -\frac{1}{2} y^2 - U(x) \right\} \tag{15}
$$

where the properties of the potential function U *satisfy* (3). *Furthermore, the disintegration of* p *is the unique Markov measure.*

We now turn to the asymptotic expansion of the top Lyapunov exponent of the random dynamical system φ corresponding to the stationary measure with density p given by (15).

Theorem 4 (Arnold and Imkeller [12])) *Let φ be the random dynamical system given in Theorem 2 for the Kramers oscillator and $\rho(dx, dy) = p(x, y)dxdy$ be the unique stationary measure* (15). *Parametrizing the unit cicle $\mathbb{S}$ by $s = (\cos\alpha, \sin\alpha)$ and identifying points $\alpha = 0$ and $\alpha = 2\pi$ of the interval $[0, 2\pi]$, we have the angle process of the variational equation*

$$ds_t = h_\epsilon(x_t, y_t, \alpha_t)dt = \left(-\frac{3}{2}x_t^2(1 + \cos 2\alpha_t) + \cos 2\alpha_t - \frac{\epsilon}{2}\sin 2\alpha_t\right) dt,$$

and the Markov process (x_t, y_t, α_t) on $\mathbb{R}^2 \times [0, 2\pi]$ with the generator

$$L_\epsilon = G_\epsilon + h_\epsilon(x, y, \alpha)\frac{\partial}{\partial\alpha} = y\frac{\partial}{\partial x} - (\epsilon y + U'(x))\frac{\partial}{\partial y} + h_\epsilon(x, y, \alpha)\frac{\partial}{\partial\alpha},$$

which has exactly one stationary measure with marginal density p on $\mathbb{R}^2$. This measure has support $\mathbb{R}^2 \times [0, 2\pi]$ and C^∞ density $q_\epsilon(x, y, \alpha)$. The Furstenberg-Khasminskii formula for the top Lyapunov exponent is

$$\lambda = \int_{\mathbb{R}^2}\int_0^{2\pi} \frac{1}{2}\left((2 - 3x^2)\sin 2\alpha + \epsilon(\cos 2\alpha - 1)\right) q_\epsilon(x, y, \alpha)\, d\alpha\, dx\, dy. \qquad (16)$$

It turns out that the cartesian coordinates are not appropriate for the small noise asymptotic expansion of the top Lyapunov exponent (16). In the absence of dissipation and random perturbations ($\epsilon = 0$), system (14) is integrable (Hamiltonian). Unperturbed Hamiltonian dynamics provides amazingly successful descriptions of the nonlinear dynamics and its mathematical theory [13] has evolved alongside the physical understanding, to a point of high sophistication. Hence, we will not use the formula (16) directly but rather change first to action-angle coordinates. The underpinning of the method presented here is a separation of scales. The slowly varying coordinate is the value of the Hamiltonian and the quickly varying coordinate is the position (or angle) in the appropriate level set of the Hamiltonian.

4 ACTION-ANGLE FORMULATION

The random motions consist of fast rotations along the unperturbed trajectories of the deterministic system and slow motion across these trajectories. The nature of our system thus suggests a set of coordinates which splits the two components of motion: *action-angle coordinates.* They are commonly used in the classical perturbation theory of mechanical systems (see Arnold [13]). The *action* part is defined by the area enclosed by the level curves of H. Hence, it captures the slow component of the motion. Whereas the *angle* part describes uniform motion along the level curves, and is therefore related with the fast component.

To this end, we need to transform $H(x,y)$ by means of a canonical transformation into new variables (I,ϕ action-angle) such that the new Hamiltonian is a constant, $h(I)$ and the angle coordinate ϕ increases by 2π after each complete period $T(x,y)=T(I)$ of the motion. To introduce these variables, following Arnold [13], we work with the generating function $S(I,x)$, determined by the requirements

$$y=\frac{\partial S}{\partial x}(I,x), \quad \phi=\frac{\partial S}{\partial I}(I,x), \quad H(x,\frac{\partial S}{\partial x}(I,x))=h(I), \tag{17}$$

$I=I(h)$ is a function of the possible values h of H. The Hamilton-Jacobi equation in (17) is solved for the generating function $S(I,x)$ by letting

$$S(I,x)=\int_{-x_o(I)}^{x} y(I,\xi)\,d\xi, \quad -x_0(I)\leq x\leq x_0(I),$$

where

$$y(I,x)=\sqrt{2(h(I)-U(x))}.$$

It is immediately obvious that $S(I,x_0(I))=\pi I$. Hence, following Arnold [13], we introduce the transformation

$$\phi \;=\; \frac{\partial}{\partial I}\int_{-x_0(I)}^{x}\sqrt{2(h(I)-U(\xi))}\,d\xi = \omega(I)\int_{-x_0(I)}^{x}\frac{1}{\sqrt{2(h(I)-U(\xi))}}\,d\xi, \tag{18}$$

and

$$y(\phi,I)=\pm\sqrt{2(h(I)-U(x(\phi,I)))}. \tag{19}$$

The main point behind the method that is developed here is to use the geometric structure of the unperturbed integrable Hamiltonian problem in order to develop an appropriate set of "coordinates" for studying the perturbed problem. Now that we have developed such symplectic coordinates, let us use (18) and (19) to give some information on the Jacobian of the transformation $(x, y) \mapsto (\phi, I)$ which is essential in deriving the perturbed equations in the new variables (ϕ, I).

Lemma 4.1 *For $I > 0$ we have*

$$\frac{\partial x}{\partial \phi} = \frac{y}{\omega}, \quad \frac{\partial y}{\partial \phi} = -\frac{U'(x)}{\omega}. \tag{20}$$

Moreover,

$$\omega = U'(x)\,\frac{\partial x}{\partial I} + y\,\frac{\partial y}{\partial I}. \tag{21}$$

In particular,

$$\frac{\partial x}{\partial \phi}\frac{\partial y}{\partial I} - \frac{\partial x}{\partial I}\frac{\partial y}{\partial \phi} = 1,$$

i.e. the transformation belongs to a symplectic form.

Proof:

Straight forward. □

Lemma 4.2 *For $I > 0, \phi \in]0, \pi[$ define*

$$\beta(\phi, I) = \int_{\frac{\pi}{2}}^{\phi} [\frac{1}{y^2(\xi, I)} - \frac{\omega'(I)}{\omega^2(I)}]\, d\xi,$$

for $\phi \in [0, \frac{\pi}{2}[$

$$\alpha_0(\phi, I) = \int_0^{\phi} [\frac{U''(x(\xi, I))}{U'(x(\xi, I))^2} - \frac{\omega'(I)}{\omega^2(I)}]\, d\xi,$$

and for $\phi \in]\frac{\pi}{2}, \pi]$

$$\alpha_\pi(\phi, I) = \int_\pi^{\phi} [\frac{U''(x(\xi, I))}{U'(x(\xi, I))^2} - \frac{\omega'(I)}{\omega^2(I)}]\, d\xi.$$

Then we may write for $I > 0$ and $\phi \in]-\pi, \pi[$

$$\frac{\partial(x,y)}{\partial(\phi,I)} = \begin{bmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial I} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial I} \end{bmatrix} = \begin{bmatrix} \frac{y}{\omega} & y\,\beta \\ -\frac{U'(x)}{\omega} & \frac{\omega}{y} - U'(x)\,\beta \end{bmatrix}, \tag{22}$$

for $\phi \in [0, \frac{\pi}{2}[$

$$\frac{\partial(x,y)}{\partial(\phi,I)} = \begin{bmatrix} \frac{y}{\omega} & \frac{\omega}{U'(x)} + y\,\alpha_0 \\ -\frac{U'(x)}{\omega} & -U'(x)\,\alpha_0 \end{bmatrix}, \tag{23}$$

and for $\phi \in]\frac{\pi}{2}, \pi]$

$$\frac{\partial(x,y)}{\partial(\phi,I)} = \begin{bmatrix} \frac{y}{\omega} & \frac{\omega}{U'(x)} + y\,\alpha_\pi \\ -\frac{U'(x)}{\omega} & -U'(x)\,\alpha_\pi \end{bmatrix}. \tag{24}$$

Moreover, for $I > 0, \phi \in [-\pi, \pi]$ we have

$$\begin{aligned} \frac{\partial x}{\partial \phi}(-\phi, I) = -\frac{\partial x}{\partial \phi}(\phi, I) \frac{\partial y}{\partial \phi}(-\phi, I) = \frac{\partial y}{\partial \phi}(\phi, I), \\ \frac{\partial x}{\partial I}(-\phi, I) = \frac{\partial x}{\partial I}(\phi, I) \frac{\partial y}{\partial I}(-\phi, I) = -\frac{\partial y}{\partial I}(\phi, I). \end{aligned} \tag{25}$$

Proof:

Let us first treat the case $-x_0(I) \le x < 0, y \ge 0$ which corresponds to $I > 0, \phi \in [0, \frac{\pi}{2}[$. Integrating (18) by parts and then differentiating with respect to I, we obtain

$$-\phi\frac{\omega'}{\omega^2} = \frac{1}{y}[-\frac{\omega}{U'(x)} + \frac{\partial x}{\partial I}] - \omega \int_{-x_0}^{x} \frac{U''(\xi)}{U'(\xi)^2} \frac{1}{\sqrt{2(h(I) - U(\xi))}} d\xi.$$

Solving this equation for $\frac{\partial x}{\partial I}$ and noting that by Lemma 4.1 we have $d\xi = \frac{\partial \xi}{\partial \phi}\, d\phi = \frac{y}{\omega}\, d\phi$ yields the requested formula for $\frac{\partial x}{\partial I}$.

In case $-x_0(I) < x < x_0(I), y \ge 0$ corresponding to $I > 0, \phi \in]0, \pi[$ symmetry allows us to write the alternative of (18)

$$\phi = \frac{\pi}{2} + \omega \int_0^{x(\phi,I)} \frac{1}{\sqrt{2(h(I) - U(\xi))}}\, d\xi.$$

Now differentiate with respect to I to get

$$(\frac{\pi}{2} - \phi)\frac{\omega'}{\omega^2} = \frac{1}{y}\frac{\partial x}{\partial I} - \omega \int_0^{x} \frac{1}{\sqrt{2(h(I) - U(\xi))}^3} d\xi.$$

This equation is again solved for $\frac{\partial x}{\partial I}$, and the integration in x is replaced by an integration in ϕ. This gives (22).

The case $0 < x < x_0(I), y \geq 0$ is treated as the first case. Finally, (25) is obvious from the definitions. □

The symplectic property of our coordinate change immediately allows us to give formulae for the inverse of the Jacobian. This is an additional advantage of using canonical transformation.

Lemma 4.3 *We have for $I > 0$ and $\phi \in]-\pi, \pi[$*

$$\frac{\partial(\phi, I)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \\ \frac{\partial I}{\partial x} & \frac{\partial I}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\omega}{y} - U'(x)\,\beta & -y\,\beta \\ \frac{U'(x)}{\omega} & \frac{y}{\omega} \end{bmatrix}, \tag{26}$$

for $\phi \in [0, \frac{\pi}{2}[$

$$\frac{\partial(\phi, I)}{\partial(x,y)} = \begin{bmatrix} -U'(x)\,\alpha_0 & -\frac{\omega}{U'(x)} - y\,\alpha_0 \\ \frac{U'(x)}{\omega} & \frac{y}{\omega} \end{bmatrix}, \tag{27}$$

and for $\phi \in]\frac{\pi}{2}, \pi]$

$$\frac{\partial(\phi, I)}{\partial(x,y)} = \begin{bmatrix} -U'(x)\,\alpha_\pi & -\frac{\omega}{U'(x)} - y\,\alpha_\pi \\ \frac{U'(x)}{\omega} & \frac{y}{\omega} \end{bmatrix}, \tag{28}$$

Moreover, for $(x,y) \neq (0,0)$ we have

$$\begin{aligned} \frac{\partial \phi}{\partial x}(x,-y) &= -\frac{\partial \phi}{\partial x}(x,y), \quad \frac{\partial I}{\partial y}(x,-y) = \frac{\partial I}{\partial x}(x,y), \\ \frac{\partial \phi}{\partial y}(x,-y) &= \frac{\partial \phi}{\partial y}(x,y), \quad \frac{\partial I}{\partial y}(x,-y) = -\frac{\partial I}{\partial y}(x,y). \end{aligned} \tag{29}$$

Proof:

This follows directly from Lemma 4.2 and the fact that the Jacobian has determinant 1 due to the symplectic character of the transformation. □

We are now in a position to describe our basic equations (14) in action-angle variables. Differentiating the action-angle variables and making use of Lemma 4.2 and Lemma 4.3 yields,

$$
\begin{aligned}
dI_t &= \frac{U'(x_t)}{\omega(I_t)}\, y_t\, dt + \frac{y_t}{\omega(I_t)}[-\epsilon\, y_t - U'(x_t)]\, dt + \sqrt{2\epsilon}\,\frac{y_t}{\omega(I_t)}\,\sigma(x_t, y_t) \circ dW_t \\
&\stackrel{\text{def}}{=} \epsilon\, f_I(\phi_t, I_t)\, dt + \sqrt{2\epsilon}\, g_I(\phi_t, I_t) \circ dW_t, \qquad (30)\\
\phi_t &= \frac{\partial y_t}{\partial I}\, y_t\, dt - \frac{\partial x_t}{\partial I}[-\epsilon\, y_t - U'(x_t)]\, dt - \sqrt{2\epsilon}\,\frac{\partial x_t}{\partial I}\,\sigma(x_t, y_t) \circ dW_t \\
&\stackrel{\text{def}}{=} \omega(I_t)\, dt + \epsilon\, f_\phi(\phi_t, I_t)\, dt + \sqrt{2\epsilon}\, g_\phi(\phi_t, I_t) \circ dW_t, \qquad (31)
\end{aligned}
$$

where the vector fields appearing in (30) and (31) are renamed as

$$
\begin{aligned}
f_I(\phi, I) &= -\frac{y^2(\phi, I)}{\omega(I)} \quad , \quad f_\phi(\phi, I) = y\,\frac{\partial x}{\partial I}(\phi, I), \\
g_I(\phi, I) &= \frac{(y\,\sigma(x, y))\,(\phi, I)}{\omega(I)} \quad , \quad g_\phi(\phi, I) = -(\frac{\partial x}{\partial I}\,\sigma(x, y))(\phi, I),
\end{aligned}
$$

$I \geq 0, \phi \in [-\pi, \pi]$, to simplify notation in the decomposition of the infinitesimal generator in the following section. For the linearization of our system we need the Jacobian of the vector fields. For convenience we change the order of ϕ and I and the Jacobian is given by

$$
A^f = \begin{bmatrix} A^f_{11} & A^f_{12} \\ A^f_{21} & A^f_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_I}{\partial I} & \frac{\partial f_I}{\partial \phi} \\ \frac{\partial f_\phi}{\partial I} & \frac{\partial f_\phi}{\partial \phi} \end{bmatrix}, \quad A^g = \begin{bmatrix} A^g_{11} & A^g_{12} \\ A^g_{21} & A^g_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial g_I}{\partial I} & \frac{\partial g_I}{\partial \phi} \\ \frac{\partial g_\phi}{\partial I} & \frac{\partial g_\phi}{\partial \phi} \end{bmatrix}. \qquad (32)
$$

Calculations using the preceding Lemmas yield the formulae for each element of the above matrices A^f and A^g. Furstenberg-Khasminskii formula for the top Lyapunov exponent is derived in the next section.

5 SCALING AND PROJECTION

Following the notation of the preceding section we shall now consider the stochastic system in action-angle variables given by in (30) and (31). Our aim is to obtain an asymptotic expansion of the top Lyapunov exponent of the random dynamical system described by (30) and (31). For this purpose we have to study its linearization. Let us denote the linearized variables by (X, Y) and keeping track of the notation introduced in the preceding section, we have

$$\begin{bmatrix} dX_t \\ dY_t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \omega'(I_t) & 0 \end{bmatrix} \begin{bmatrix} X_t \\ Y_t \end{bmatrix} dt + \epsilon A^f(\phi_t, I_t) \begin{bmatrix} X_t \\ Y_t \end{bmatrix} dt + \sqrt{2\epsilon}\, A^g(\phi_t, I_t) \begin{bmatrix} X_t \\ Y_t \end{bmatrix} \circ dW_t. \quad (33)$$

Because of the special structure of the zeroth order terms in equations (30) and (31), the linear variational equations (33) naturally exhibit a nilpotent structure. In order to obtain a formula for the top Lyapunov exponent, one needs the Hörmander's condition for hypoellipticity of the associated generator. Hörmander's condition can be replaced by a sufficient condition in a coordinate-free form, particularly when the deterministic matrix is nilpotent.

Let A represent the matrix corresponding to the stochastic terms. Then for a $d \times d$ nilpotent matrix N and its one-dimensional kernel $v \in \ker N$, the sufficient condition for hypoellipticity is given by

$$\operatorname{rank}\left\{ \left(ad_N^j A \right) v : j = 1, 2, ..., n-1 \right\} = d \quad (34)$$

where we define, $ad_N^0 A = A$, $ad_N^1 A = [N, A] = AN - NA$, $ad_N^j A = [N, ad_N^{j-1}]$ for $j = 1, 2, 3, ...$ It can be easily shown (Pinsky and Wihstutz [14]) that the matrix element $A_{1d} \neq 0$ implies (34). This in turn implies that for our situation the stationary density exists and is smooth if $A_{12}^g \neq 0$. This nilpotent form and the term (A_{12}^g) are responsible for the main results on the asymptotic expansion of its top Lyapunov exponent to be developed in this and the following sections.

5.1 PINSKY-WIHSTUTZ SCALING

In the previous section, it is shown that a smooth stationary density exists, however in order to calculate this we need to make use of the small parameter ϵ that naturally exists in our problem. In terms of polar coordinates $\theta = \tan^{-1}(Y/X)$ and $\rho = \log\sqrt{X^2+Y^2}$, the angular component $\theta_t, t \in \mathbb{R}$, of the process (33) reads

$$d\theta_t = \left\{ \hat{h}_0^\theta(\phi_t, I_t, \theta_t) + \epsilon\, \hat{h}_2^\theta(\phi_t, I_t, \theta_t) \right\} dt + \sqrt{\epsilon}\, \hat{h}_1^\theta(\phi_t, I_t, \theta_t) \circ dW_t, \tag{35}$$

with the generator

$$L^\epsilon = L_0 + \epsilon L_1, \quad L_0 = h_0^\theta(\phi, I, \theta)\frac{\partial}{\partial\theta}, \quad L_1 = h_2^\theta(\phi, I, \theta)\frac{\partial}{\partial\theta} + \frac{1}{2}\left(h_1^\theta(\phi, I, \theta)\frac{\partial}{\partial\phi}\right)^2 \tag{36}$$

where for $I \geq 0$, $\phi \in [-\pi, \pi]$, $\theta \in [0, \pi]$ we have

$$\begin{aligned}
h_0^\theta(\phi, I, \theta) &= \omega'(I)\, cos^2\theta \\
h_2^\theta(\phi, I, \theta) &= A_{21}^f(\phi, I)\, \cos^2\theta + (A_{22}^g - A_{11}^g)(\phi, I)\, \cos\theta\sin\theta - A_{12}^f(\phi, I)\, \sin^2\theta \\
h_1^\theta(\phi, I, \theta) &= A_{21}^g(\phi, I)\, \cos^2\theta + (A_{22}^g - A_{11}^g)(\phi, I)\, \cos\theta\sin\theta - A_{12}^g(\phi, I)\, \sin^2\theta
\end{aligned}$$

In the perturbative form of problem (35), the generator L_0 associated with the nilpotent part vanishes for some $\theta_c = \pm\pi/2$, which is of the order 2 and the measure μ_0 of $L_0^*\mu_0 = 0$ will not have a smooth density, but rather Darac measures at $\pi/2$, i.e., $\mu_0 = \delta_{\pm\pi/2}$. Therefore, we must smooth the measure by applying a suitable scaling as pointed out by Pinsky and Wihstutz [14]. Let us elaborate on this point.

Since the invariant measure of the angular part of the linearization (nilpotent) trivializes in one direction, we appeal to the results of Pinsky and Wihstutz [14]. Pinsky -Wihstutz scaling stretches the coordinates such a way that the leading order diffusion part balances the leading order dift term. This allows us to replace the generalized measure $\mu_0 = 0$ by a smooth measure. Accordingly, the variables (X, Y) are rescaled with a certain fractional power of ϵ, i.e.,

$$X = \epsilon^{\frac{1}{3}} U, \quad Y = V,$$

in order to see the correct asymptotics. In the rescaled variables we obtain the equation

$$\begin{aligned}
\begin{bmatrix} dU_t \\ dV_t \end{bmatrix} &= \begin{bmatrix} \epsilon A_{11}^f(\phi_t, I_t) & \epsilon^{\frac{2}{3}} A_{12}^f(\phi_t, I_t) \\ \epsilon^{\frac{1}{3}}\omega' + \epsilon^{\frac{4}{3}} A_{21}^f(\phi_t, I_t) & \epsilon A_{22}^f(\phi_t, I_t) \end{bmatrix} \begin{bmatrix} U_t \\ V_t \end{bmatrix} dt \\
&+ \sqrt{2} \begin{bmatrix} \epsilon^{\frac{1}{2}} A_{11}^g(\phi_t, I_t) & \epsilon^{\frac{1}{6}} A_{12}^g(\phi_t, I_t) \\ \epsilon^{\frac{5}{6}}\omega'(I_t) + \epsilon^{\frac{1}{2}} A_{21}^g(\phi_t, I_t) & \epsilon A_{22}^g(\phi_t, I_t) \end{bmatrix} \begin{bmatrix} U_t \\ V_t \end{bmatrix} \circ dW_t.
\end{aligned}$$

5.2 FURSTENBERG-KHASMINSKII FORMULA

We next apply the Khasminskii transformation so that the above linear equation is decomposed into radial and angular part. This provides the most convenient setting for the description of the top Lyapunov exponent by means of the so-called Furstenberg-Khasminskii formula. Write

$$U = r\cos\theta, \quad V = r\sin\theta.$$

Then the angular component described by the process $\theta_t, t \in \mathbb{R}$, satisfies the stochastic differential equation

$$d\theta_t = h_0^\theta(\phi_t, I_t, \theta_t)\, dt + h_1^\theta(\phi_t, I_t, \theta_t) \circ dW_t, \tag{37}$$

where for $I \geq 0, \phi \in [-\pi, \pi], \theta \in [0, \pi]$ we have

$$\begin{aligned} h_0^\theta(\phi, I, \theta) &= \epsilon^{\frac{1}{3}}\, \omega'(I)\, cos^2\theta - \epsilon^{\frac{2}{3}}\, A_{12}^f(\phi, I)\, \sin^2\theta \\ &+ \epsilon\, (A_{22}^f - A_{11}^f)(\phi, I)\, \sin\theta\cos\theta + \epsilon^{\frac{4}{3}}\, A_{21}^f(\phi, I)\, \cos^2\theta, \\ h_1^\theta(\phi, I, \theta) &= \sqrt{2}\, [-\epsilon^{\frac{1}{6}}\, A_{12}^g(\phi, I)\, \sin^2\theta \\ &+ \epsilon^{\frac{1}{2}}\, (A_{22}^g - A_{11}^g)(\phi, I)\, \sin\theta\cos\theta + \epsilon^{\frac{5}{6}}\, A_{21}^g(\phi, I)\, \cos^2\theta]. \end{aligned}$$

For the rest of this section we shall be concerned with a calculation of the scaled decomposition of the infinitesimal generator of our 3-dimensional system given by (30), (31) and (35) as well as the functional of the radial part appearing in the representation of Lyapunov exponents in formulae of the Furstenberg-Khasminskii type.

Appropriately adding the drift and the diffusion parts (see Appendix for details), finally yields the infinitesimal generator L^ϵ of our system (30), (31) and (35) as

$$L^\epsilon = L_0 + \epsilon^{\frac{1}{3}} L_1 + \epsilon^{\frac{2}{3}} L_2 + \epsilon L_3 + \epsilon^{\frac{4}{3}} L_4 + \epsilon^{\frac{5}{3}} L_5, \tag{38}$$

where

$$
\begin{aligned}
L_0 &= \omega \frac{\partial}{\partial \phi}, \\
L_1 &= -\omega' u^2 \frac{\partial}{\partial u} + (A^g_{12})^2 \frac{\partial^2}{\partial u^2}, \\
L_2 &= [(A^f_{12} + g_I A^g_{121} + g_\phi A^g_{122})] \frac{\partial}{\partial u} - [2A^g_{12}(A^g_{22} - A^g_{11})]\, u^{\frac{1}{2}} \frac{\partial}{\partial u}(u^{\frac{1}{2}} \frac{\partial}{\partial u}) \\
&+ [2 g_I A^g_{12}] \frac{\partial^2}{\partial I \partial u} + [2 g_\phi A^g_{12}] \frac{\partial^2}{\partial \phi \partial u}, \\
L_3 &= [f_I + g_I\, A^g_{11} + g_\phi\, A^g_{12}] \frac{\partial}{\partial I} + [f_\phi + g_I A^g_{21} + g_\phi A^g_{22}] \frac{\partial}{\partial \phi} \\
&+ g_I^2 \frac{\partial^2}{\partial I^2} + g_\phi^2 \frac{\partial^2}{\partial \phi^2} + 2 g_I g_\phi \frac{\partial^2}{\partial I \partial \phi} \\
&- [A^f_{22} - A^f_{11} + g_I(A^g_{221} - A^g_{111}) + g_\phi(A^g_{222} - A^g_{112})]\, u \frac{\partial}{\partial u} \\
&\qquad + [(A^g_{22} - A^g_{11})^2 - 2A^g_{12}A^g_{21}]\ u \frac{\partial}{\partial u}(u \frac{\partial}{\partial u}) \\
&- [2 g_I (A^g_{22} - A^g_{11})]\, u \frac{\partial^2}{\partial I \partial u} - [2 g_\phi (A^g_{22} - A^g_{11})]\, u \frac{\partial^2}{\partial \phi \partial u}, \\
L_4 &= -[A^f_{21} + g_I A^g_{211} + g_\phi A^g_{212})]\, u^2 \frac{\partial}{\partial u} \\
&\qquad + [2A^g_{21}(A^g_{22} - A^g_{11})]\ u^{\frac{3}{2}} \frac{\partial}{\partial u}(u^{\frac{3}{2}} \frac{\partial}{\partial u}) \\
&- [2 g_I A^g_{21}] u^2 \frac{\partial^2}{\partial I \partial u} - [2 g_\phi A^g_{21}] u^2 \frac{\partial^2}{\partial \phi \partial u}, \\
L_5 &= (A^g_{21})^2\ u^2 \frac{\partial}{\partial u}(u^2 \frac{\partial}{\partial u})
\end{aligned}
\tag{39}
$$

Here and in the sequel we prefer to work with the stereographic projection variable

$$
u = \cot \theta, \quad \theta \in [0, \pi],
$$

for simplicity of presentation.

To represent Lyapunov exponents, we shall make use of a formula of Furstenberg - Khasminskii. In this formula, the following functional of the radial part of the linearization has to be integrated with the invariant measure of our system. Due to the regularity properties of our vector fields, we know that there exists an invariant density p_ϵ. In this case, the formula of Furstenberg-Khasminskii states that the leading Lyapunov exponent

λ_ϵ given by (9) of our system satisfies

$$\lambda_\epsilon = \int_{[-\pi,\pi]\times\mathbb{R}_+\times\mathbb{R}} Q^\epsilon(\phi, I, u)\ p_\epsilon(\phi, I, u)\ d\phi\, dI\, du.$$

As for the infinitesimal generator, our asymptotic analysis requires that we decompose Q^ϵ into fractional powers of $\epsilon^{\frac{1}{3}}$. Similar calculations (see Appendix for details) as for the generator yield

$$Q^\epsilon = \epsilon^{\frac{1}{3}} Q_1 + \epsilon^{\frac{2}{3}} Q_2 + \epsilon Q_3 + \epsilon^{\frac{4}{3}} Q_4 + \epsilon^{\frac{5}{3}} Q_5, \tag{40}$$

where

$$\begin{aligned}
Q_1(.,u) &= \omega' \frac{u}{1+u^2} - (A^g_{12})^2\, \frac{u^2-1}{(1+u^2)^2}, \\
Q_2(.,u) &= [A^f_{12} + g_I A^g_{121} + g_\phi A^g_{122}] \frac{u}{1+u^2} + A^g_{12}(A^g_{22} - A^g_{11})\, \frac{u(u^2-3)}{(1+u^2)^2}, \\
Q_3(.,u) &= [A^f_{11} + g_I A^g_{111} + g_\phi A^g_{112}]\, \frac{u^2}{1+u^2} + [A^f_{22} + g_I A^g_{221} + g_\phi A^g_{222}]\, \frac{1}{1+u^2} \\
&\qquad + (A^g_{22} - A^g_{11})^2\, \frac{2u^2}{(1+u^2)^2} + A^g_{12} A^g_{21} \frac{(u^2-1)^2}{(1+u^2)^2}, \\
Q_4(.,u) &= [A^f_{21} + g_I A^g_{211} + g_\phi A^g_{212}]\, \frac{u}{1+u^2} + A^g_{21}(A^g_{22} - A^g_{11})\, \frac{u(3u^2-1)}{(1+u^2)^2} \\
Q_5(.,u) &= (A^g_{21})^2\, \frac{u^2(u^2-1)}{(1+u^2)^2}.
\end{aligned} \tag{41}$$

6 ASYMPTOTIC EXPANSION

We construct a formal expansion of the invariant measure, i.e.,

$$p_\varepsilon = p_0 + \epsilon^{\frac{1}{3}} p_1 + \epsilon^{\frac{2}{3}} p_2 + \cdots + \varepsilon^{\frac{N}{3}} p_N + \cdots$$

Substituting this expansion and the expansion for L^ε into the Fokker-Planck equation yields the following sequence of Poisson equations to be solved for p_0, p_1, p_2, ... :

$$\begin{aligned}
L_0^* p_0 &= 0 \\
L_0^* p_1 &= -L_1^* p_0 \\
L_0^* p_2 &= -L_1^* p_1 - L_2^* p_0 \\
L_0^* p_3 &= -L_1^* p_2 - L_2^* p_1 - L_3^* p_0 \\
&\vdots
\end{aligned}$$

This yields the following expression for the maximal Lyapunov exponent:

$$\lambda^\varepsilon = \epsilon^{\frac{1}{3}} \langle Q_1, p_0 \rangle + \epsilon^{\frac{2}{3}} \left[\langle Q_2, p_0 \rangle + \langle Q_1, p_1 \rangle \right] + \cdots$$

As in [5], a proof that this expansion is, in fact, asymptotic begins with the construction of the *adjoint problem*

$$L^\epsilon f_\epsilon = Q_\epsilon - \Lambda_\epsilon \tag{42}$$

with Q^ϵ, L^ϵ as defined above and

$$\begin{aligned}
f^\epsilon &= f_0 + \epsilon^{\frac{1}{3}} f_1 + \epsilon^{\frac{2}{3}} f_2 + \epsilon f_3 + \cdots + \varepsilon^{\frac{N}{3}} f_N, \\
\Lambda^\epsilon &= \Lambda_0 + \epsilon^{\frac{1}{3}} \Lambda_1 + \epsilon^{\frac{2}{3}} \Lambda_2 + \epsilon \Lambda_3 + \cdots + \varepsilon^{\frac{N}{3}} \Lambda_N.
\end{aligned}$$

Contrary to the usual form, we allow $\Lambda^\epsilon, \Lambda_i, i \geq 0$ to be functions of I alone. By using our formulae for L_i and Q_i and identifying terms in the corresponding expansion following from (42) then produces a set of Poisson-Type equations. Hence, Λ_i's are chosen so that the sequence of equations

$$\begin{aligned}
L_0 f_0 &= -\Lambda_0 \\
L_0 f_1 &= Q_1 - \Lambda_1 - L_1 f_0 \\
L_0 f_2 &= Q_2 - \Lambda_2 - L_1 f_1 - L_2 f_0 \\
&\vdots \\
L_0 f_N &= -\Lambda_N - \sum_{i=1}^{i=5} L_i f_{N-i}
\end{aligned} \tag{43}$$

are solvable. Next we define the truncated density $\tilde{p}^{\varepsilon} = p_0 + \epsilon^{\frac{1}{3}} p_1 + \epsilon^{\frac{2}{3}} p_2 + \cdots + \varepsilon^{\frac{N}{3}} p_N$ and assume $\nu(I)$ as $I-$marginal of both p_ε and $\tilde{p}^{\varepsilon}$. Then, the error $\langle Q_\varepsilon, p_\varepsilon\rangle - \langle Q_\varepsilon, \tilde{p}^{\varepsilon}\rangle$ introduced by truncating λ^{ε} at an arbitrary order $N \geq 0$ can be evaluated as in [5]. Suppose that the functions $p_0, p_1, \cdots, p_N$ and $f_0, f_1, \cdots, f_N$ are constructed such that all inner products in the expressions for the error are well defined and bounded, then it it can be shown as in [5] that the expansion for a fixed $N \geq 0$ is a valid asymptotic expansion. In the subsequent section, we compute the leading term

$$\lambda_1 = \langle Q_1, p_0\rangle \tag{44}$$

along with the estimate of the remainder term in the asymptotic expansion of the top Lyapunov exponent.

7 CALCULATION OF THE FIRST TERM λ_1

In this section we shall compute the leading terms in the asymptotic expansion of the top Lyapunov exponent of our system, based on its representation in the Furstenberg-Khasminskii formula. The invariant density of our three dimensional system is the unique lift of the density $\nu(I)$ of the $I-$motion. The density $\nu(I)$ is given as the solution of the adjoint equation

$$-\frac{d}{dI}[(\overline{f_I} + \overline{g_I A_{11}^g} + \overline{g_\phi A_{12}^g})\nu] + \frac{d^2}{dI^2}[\overline{g_I^2}\,\nu] = 0, \tag{45}$$

where for convenience the average of functions k over $\phi \in [-\pi, \pi]$ is denoted by $\overline{k}$. We can easily calculate $\nu(I)$ for the three cases we are mostly interested in, i.e., $\sigma = const$ or $\sigma(x, y) = x$ or y, for $x, y \in \mathbb{R}$.

Lemma 7.1 *Let $c \geq 0$ be given such that*

$$c_1 \;=\; \int_0^\infty \exp\left(-\int_c^I \omega(J)\left[\frac{\overline{y^2}(J)}{\overline{y^2\sigma^2}(J)} + \frac{\overline{y\sigma\sigma_y}(J)}{\overline{y^2\sigma^2}(J)}\right] dJ\right) dI \;< \infty.$$

Then

$$\nu(I) \;=\; c_1 \exp\left(-\int_c^I \omega(J)\left[\frac{\overline{y^2}(J)}{\overline{y^2\sigma^2}(J)} + \frac{\overline{y\sigma\sigma_y}(J)}{\overline{y^2\sigma^2}(J)}\right] dJ\right), \tag{46}$$

gives the marginal density in I of p_ϵ. In particular, if $\sigma =$ constant (i.e., additive noise), we have

$$\nu(I) = c_1 \exp\left\{-\frac{h(I)}{\sigma^2}\right\}, \quad I \geq 0, \tag{47}$$

Proof:

Equation (45) may be written equivalently, with some constant $c \in \mathbb{R}$

$$(\overline{f_I} - \overline{g_I A^g_{11}} + \overline{g_\phi A^g_{12}})\ \nu + c = \overline{g_I^2}\, \frac{d}{dI}\nu.$$

Now an easy calculation gives the formulae

$$\begin{aligned} A^g_{11} &= \frac{\sigma + y\,\sigma_y}{\omega}\,\frac{\partial y}{\partial I} + \frac{y\,\sigma_x}{\omega}\,\frac{\partial x}{\partial I} - y\,\sigma\frac{\omega'}{\omega^2}, \\ A^g_{12} &= \frac{y^2\,\sigma_x - (\sigma + y\,\sigma_y)\,U'(x)}{\omega^2}. \end{aligned}$$

Use these and periodicity to derive

$$\overline{g_\phi A^g_{12}} = -\overline{g_I A^g_{22}} = \overline{g_I A^g_{11}} - \overline{g_I \sigma_y}.$$

Hence the homogeneous part of the equation determining ν is given by

$$\nu_0(I) = c_1\ \exp(-\int_c^I \omega(J)\,[\frac{\overline{y^2}(J)}{\overline{y^2\sigma^2}(J)} + \frac{\overline{y\,\sigma\,\sigma_y}(J)}{\overline{y^2\sigma^2}(J)}]\ dJ),$$

with an arbitrary constant c_1. We may assume that $c_1^{-1} = \int_0^\infty \nu_0(I)\,dI < \infty$ and choose $c = 0$ and get the desired formula. $\Box$

Since for the convergence of our algorithm the following condition,

$$(\mathbf{F}) \qquad \omega'(I) > 0 \quad \text{for a.e.} \quad I \geq 0,$$

is important, we shall make this general assumption throughout out this paper.

For reasons which will become clear, in the computation of the leading term in the asymptotic expansion of the top Lyapunov exponent, we shall solve for a density $p_0(I,\theta)$ which satisfies both $L_0^* p_0 = 0$, the zeroth order term in the expansion of the Fokker-Planck equation, and $\overline{L_1}^* p_0 = 0$, the solvability of the first order term in the expansion, i.e.,

$$L_0^* p_1 = -L_1^* p_0.$$

Proposition 7.1 *For $I, s \geq 0$ let $a(I) = \overline{(A_{12}^g)^2}(I)$, $\mu(I,s) = (\frac{3a(I)\,s}{2\,\omega'(I)})^{\frac{1}{3}}$ and let $n(\mu,\sigma^2)(u)$ denote the Gaussian density with mean μ, variance σ^2, evaluated at $u \in \mathbb{R}$, and*

$$q_\gamma(s) = \Gamma(\gamma)^{-1} s^{\gamma-1} \exp(-s), \quad s \geq 0,$$

the density of a Gamma law with parameter $\gamma > 0$. Let for $I \geq 0, u \in \mathbb{R}$

$$p_0(I,u) = \int_0^\infty n(\mu(I,s), \frac{1}{4\frac{\omega'(I)}{a(I)}\mu(I,s)})\,(u)\, q_{\frac{1}{6}}(s)\, ds. \tag{48}$$

Then we have

$$L_0^* p_0 = 0, \quad \overline{L_1}^* p_0 = 0.$$

Proof:

Since p_0 is just a function of I, u, the first equation is obvious. To derive the second, note first that

$$\overline{L_1} = -\omega'\, u^2 \,\frac{\partial}{\partial u} + a\,\frac{\partial^2}{\partial u^2}.$$

We fix $I \geq 0$ and have to solve for $p_0(I,.)$ satisfying

$$\omega'(I)\, u^2\; p_0(I,.) + a(I)\,\frac{d}{du} p_0(I,.) = c(I)$$

for a constant $c(I)$ which is determined by the normalization condition for $p_0(I,.)$. Denote $\alpha = \frac{\omega'(I)}{3a(I)}$. The obvious solution of the above differential equation given by

$$p_0(I,u) = c(I)\, \exp(-\alpha u^3) \int_{-\infty}^u \exp(\alpha\, v^3) dv$$

has to be described in an alternative form. For this purpose we may write, setting $v = u - h$ with $h \geq 0$

$$-\alpha(u^3 - v^3) = -3\alpha h[u - \frac{h}{2}]^2 - \frac{1}{4}\alpha h^3.$$

Using this in the integral representation, and changing variables once again, setting $s = \frac{1}{4}\alpha h^3$, we obtain the alternative expression

$$p_0(I,u) = c(I)\,\frac{1}{3}(\frac{4}{\alpha})^{\frac{1}{3}} \int_0^\infty \exp(-3\alpha\,(\frac{4s}{\alpha})^{\frac{1}{3}}[u - \frac{1}{2}(\frac{4s}{\alpha})^{\frac{1}{3}}]^2)\exp(-s)\,s^{-\frac{2}{3}}\,ds.$$

Now observe that the renormalization of the quadratic exp $-$factor in terms of a Gaussian density will produce an $s^{-\frac{1}{6}}$, so that the Gaussian densities with mean $\frac{1}{2}(\frac{4s}{\alpha})^{\frac{1}{3}} = \mu(I,s)$ and variance $\frac{1}{6\alpha(\frac{4s}{\alpha})^{\frac{1}{3}}} = \frac{1}{4\frac{\omega'}{a}\mu(I,s)}$ have to be averaged by a probability measure with density $C \cdot s^{-\frac{5}{6}}\exp(-s)$, i.e. the density of a $\Gamma(\frac{1}{6})$ law. This completes the proof. $\square$

We now start our asymptotic analysis with the Ansatz of *adjoint expansion* (42). In order to obtain the first term in the asymptotic expansion of the top Lyapunov exponent, the first three of (43) have to be analyzed carefully in the sequel. They are given by

$$L_0 f_0 \;=\; -\Lambda_0, \tag{49}$$

$$L_0 f_1 + L_1 f_0 \;=\; Q_1 - \Lambda_1, \tag{50}$$

$$L_0 f_2 + L_1 f_1 + L_2 f_0 \;=\; Q_2 - \Lambda_2. \tag{51}$$

We first obtain

$$\int \Lambda_0 p_0 d(\phi,I,u) \;=\; -\int L_0 f_0\, p_0 d(\phi,I,u) = \int f_0 L_0^* p_0\, d(\phi,I,u) = 0, \tag{52}$$

since $L_0^* p_0 = 0$. This expresses the fact that the zeroth term λ_0 in the development of λ^ϵ vanishes. Moreover, we have

$$L_0 f_0 = \omega\,\frac{\partial}{\partial\phi} f_0 = -\Lambda_0,$$

hence for $I \geq 0, \phi \in [-\pi,\pi], u \in \mathbb{R}$

$$f_0(\phi,I,u) = -\Lambda_0(I)\,\phi + g(I,u).$$

But by periodicity in ϕ, this in turn implies that

$$\Lambda_0 = 0. \tag{53}$$

Hence f_0 is just a function of I and u. Let us next use this knowledge to analyze (50). Since $L_0^* p_0 = 0 = \overline{L_1}^* p_0$ we get

$$\begin{aligned}
0 &= -\int f_1 \, L_0^* p_0 \, d(\phi, I, u) = \int L_0 f_1 \, p_0 \, d(\phi, I, u) \qquad (54)\\
&= \int [Q_1 - \Lambda_1 - L_1 f_0] \, p_0 \, d(\phi, I, u)\\
&= \int [Q_1 - \Lambda_1] \, p_0 \, d(\phi, I, u) - \int f_0 \, \overline{L_1}^* p_0 \, d(I, u)\\
&= \int [Q_1 - \Lambda_1] \, p_0 \, d(\phi, I, u).
\end{aligned}$$

Equation (54) gives us the leading term in the development of the top Lyapunov exponent of our system. It can also be interpreted as the solvability condition for (50).

Theorem 7.1 *We have*

$$\lambda_1 = (\frac{3}{2})^{\frac{1}{3}} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})} \int_0^\infty \omega'(I)^{\frac{2}{3}} \; a(I)^{\frac{1}{3}} \; \nu(I) \; dI. \tag{55}$$

In particular, $\lambda_1 > 0$.

Proof:

We solve the averaged form of equation (54) for Λ_1 and use the equality $\overline{L_1}^* p_0 = 0$ to write for every $I \geq 0$

$$\begin{aligned}
\Lambda_1(I) &= \int_{\mathbb{R}} (\overline{Q_1} - \overline{L_1} f_0)(I, u) \, p_0(I, u) du \qquad (56)\\
&= \int_{\mathbb{R}} \overline{Q_1}(I, u) \, p_0(I, u) du.
\end{aligned}$$

We next use split off $\overline{Q_1}$ one part which lies in the range of $\overline{L_1}$. This is done in the following way. Setting

$$f(u) = -\frac{1}{2} \ln(1 + u^2),$$

we calculate

$$f'(u) = -\frac{u}{1+u^2}, \quad f''(u) = \frac{u^2-1}{(1+u^2)^2}$$

and therefore

$$L_1 f(.,u) = \omega' \, u \frac{u^2}{1+u^2} + (A_{12}^g)^2 \frac{u^2-1}{(1+u^2)^2}$$

which by the formula given for Q_1 leads to

$$(L_1 f + Q_1)(.,u) = (\overline{L_1} f + \overline{Q_1})(.,u) = \omega' \, u.$$

Hence we obtain with another appeal to the equation $\overline{L_1}^* p_0 = 0$ and Proposition 7.1

$$\begin{aligned} \Lambda_1(I) &= \omega'(I) \int_{\mathbb{R}} u \, p_0(I,u) \, du \\ &= \omega'(I) \int_0^\infty \mu(s,I) \, q_{\frac{1}{6}}(s) \, ds \\ &= (\frac{3}{2})^{\frac{1}{3}} \, \omega'(I)^{\frac{2}{3}} \, a(I)^{\frac{1}{3}} \int_0^\infty s^{\frac{1}{3}} \, q_{\frac{1}{6}}(s) \, ds \\ &= (\frac{3}{2})^{\frac{1}{3}} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})} \, \omega'(I)^{\frac{2}{3}} \, a(I)^{\frac{1}{3}}. \end{aligned} \tag{57}$$

It remains to integrate Λ_1 with the density ν to obtain the formula claimed. □

The difficult part of these calculations is to show that the expansion is, in fact, asymptotic, so that the computational algorithm that is developed here is indeed convergent. For this, we need the estimation of the remainder terms in our asymptotic expansion, i.e., we need some more information on f's. The proof that such an *algorithm of computation is convergent* will be presented in Arnold *et al.* [19].

8 CONCLUSIONS

In this paper we extend the work by Arnold and Imkeller [12] on the Kramers oscillator. To this end, we made use of the classical results on action-angle variables [13], and more recent results on Lyapunov exponents by Arnold, Papanicolaou and Wihstutz [4], Pinsky and Wihstutz [14], Sri Namachchivaya and Van Roessel [5] and Imkeller and Lederer [15].

An asymptotic expansion for the maximal Lyapunov exponent, the exponential growth rate, of the response of single-well Kramers oscillator driven by either an additive or multiplicative white noise process was constructed. However, only the first term of the asymptotic expansion was analytically evaluated. Based on this, it was shown that the top Lyapunov exponent is positive, and for small values of noise intensity $\sqrt{\epsilon}$ and dissipation ϵ the exponent grows proportionally to $\epsilon^{\frac{1}{3}}$. A similar result is proved by Baxendale and Goukasian [20] for the multiplicative case, where calculations for the linearized process are done with respect to a moving frame. The idea behind such a moving frame is to use instead of the coordinates which remains parallel to (x, y), a new coordinate system (u, v) with one axis u moving so as to remain tangent to the unperturbed trajectory, while the other axis v remain perpendicular to the unperturbed trajectory, which in the dynamical systems literature is known as Diliberto [21] transformations. We only presented the main results and the proofs of the main theorem. The fact that such an *algorithm of computation is convergent* is presented in Arnold *et al.* [19].

In closing, it seems appropriate to make the following remarks regarding the implications of the positive top Lyapunov exponent of the stationary measure for the Kramers Oscillator. Since the corresponding Markov process $(x_t, \dot{x}_t)$ generated by (2) (so-called one-point motion of the Kramers Oscillator) is positive recurrent, the stationary measure can be viewed as the occupation measure, i.e., the proportion of time spent by a typical solution of (2) in the volume element $dx\, dy$. The top Lyapunov exponents which deal with stability on the other hand, are determined by the behavior of two neighboring orbits or the two point motion of the Kramers Oscillator. In this context, the positivity of the top Lyapunov exponent has remarkable implications. While for each initial condition the solution trajectory asymptotically approaches the volume element in the state space giving rise to a nontrivial stationary measure, the distance between any two initial conditions will grow at an exponentially fast rate. Furthermore, the growth of two dimensional volume under the solution flow is determined by the sum of the two Lyapunov exponents which is $-\epsilon$ and thus negative. Hence, as t goes to ∞ the original two dimensional volume under the solution flow will shrink, but will be continuously stretched in one direction (and folded in a complicated manner).

In addition, a positive Lapunov exponent is also an indication of the fact that via Pesin's entropy formula, the system under the stationary measure has positive entropy.

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APPENDIX: CALCULATION OF L^ϵ and Q^ϵ

The infinitesimal generator L^ϵ of our system (30), (31) and (35) is given by

$$\begin{aligned}
L^\epsilon &= h_0^I \frac{\partial}{\partial I} + h_0^\phi \frac{\partial}{\partial \phi} + h_0^\theta \frac{\partial}{\partial \theta} + \frac{1}{2} (h_1^I \frac{\partial}{\partial I} + h_1^\phi \frac{\partial}{\partial \phi} + h_1^\theta \frac{\partial}{\partial \theta})^2 \\
&= [h_0^I + \frac{1}{2}(h_1^I \frac{\partial h_1^I}{\partial I} + h_1^\phi \frac{\partial h_1^I}{\partial \phi} + h_1^\theta \frac{\partial h_1^I}{\partial \theta})] \frac{\partial}{\partial I} \\
&+ [h_0^\phi + \frac{1}{2}(h_1^I \frac{\partial h_1^\phi}{\partial I} + h_1^\phi \frac{\partial h_1^\phi}{\partial \phi} + h_1^\theta \frac{\partial h_1^\phi}{\partial \theta})] \frac{\partial}{\partial \phi} \\
&+ [h_0^\theta + \frac{1}{2}(h_1^I \frac{\partial h_1^\theta}{\partial I} + h_1^\phi \frac{\partial h_1^\theta}{\partial \phi} + h_1^\theta \frac{\partial h_1^\theta}{\partial \theta})] \frac{\partial}{\partial \theta} \\
&+ \frac{1}{2}(h_1^I)^2 \frac{\partial^2}{\partial I^2} + \frac{1}{2}(h_1^\phi)^2 \frac{\partial^2}{\partial \phi^2} + \frac{1}{2}(h_1^\theta)^2 \frac{\partial^2}{\partial \theta^2} \\
&+ h_1^I h_1^\phi \frac{\partial^2}{\partial I \partial \phi} + h_1^I h_1^\theta \frac{\partial^2}{\partial I \partial \theta} + h_1^\phi h_1^\theta \frac{\partial^2}{\partial \phi \partial \theta}.
\end{aligned} \tag{58}$$

We shall now collect the main steps in the evaluation of L^ϵ, starting with the drift part. Ordering according to powers of $\epsilon^{\frac{1}{3}}$, the contributions to the drift part of (58) yield the formula

$$\begin{aligned}
&[h_0^I + \frac{1}{2}(h_1^I \frac{\partial h_1^I}{\partial I} + h_1^\phi \frac{\partial h_1^I}{\partial \phi} + h_1^\theta \frac{\partial h_1^I}{\partial \theta})](.,\theta) \frac{\partial}{\partial I} \\
&+ [h_0^\phi + \frac{1}{2}(h_1^I \frac{\partial h_1^\phi}{\partial I} + h_1^\phi \frac{\partial h_1^\phi}{\partial \phi} + h_1^\theta \frac{\partial h_1^\phi}{\partial \theta})](.,\theta) \frac{\partial}{\partial \phi} \\
&+ [h_0^\theta + \frac{1}{2}(h_1^I \frac{\partial h_1^\theta}{\partial I} + h_1^\phi \frac{\partial h_1^\theta}{\partial \phi} + h_1^\theta \frac{\partial h_1^\theta}{\partial \theta})](.,\theta) \frac{\partial}{\partial \theta} \\
&= \omega \frac{\partial}{\partial \phi} + \epsilon^{\frac{1}{3}} \Big[[\omega' \cos^2\theta + 2\,(A_{12}^g)^2 \sin^3\theta \cos\theta]\frac{\partial}{\partial \theta}\Big] \\
&- \epsilon^{\frac{2}{3}} \Big[[(A_{12}^f + g_I A_{121}^g + g_\phi A_{122}^g)\sin^2\theta + A_{12}^g(A_{22}^g - A_{11}^g)(3\sin^2\theta\cos^2\theta - \sin^4\theta)]\frac{\partial}{\partial \theta}\Big] \\
&+ \epsilon\Big[[f_I + g_I\, A_{11}^g + g_\phi\, A_{12}^g]\frac{\partial}{\partial I} + [f_\phi + g_I A_{21}^g + g_\phi A_{22}^g]\frac{\partial}{\partial \phi} \\
&\quad + [(A_{22}^f - A_{11}^f + g_I(A_{221}^g - A_{111}^g) + g_\phi(A_{222}^g - A_{112}^g))\sin\theta\cos\theta \\
&\qquad + ((A_{22}^g - A_{11}^g)^2 - 2A_{12}^g A_{21}^g)(\sin\theta\cos^3\theta - \sin^3\theta\cos\theta)]\frac{\partial}{\partial \theta}\Big] \\
&+ \epsilon^{\frac{4}{3}}\Big[[A_{21}^f + g_I A_{211}^g + g_\phi A_{212}^g)\cos^2\theta + A_{21}^g(A_{22}^g - A_{11}^g)(\cos^4\theta - 3\sin^2\theta\cos^2\theta]\frac{\partial}{\partial \theta}\Big] \\
&- \epsilon^{\frac{5}{3}} \Big[[2\,(A_{21}^g)^2 \sin\theta\cos^3\theta]\frac{\partial}{\partial \theta}\Big].
\end{aligned} \tag{59}$$

After some tedious calculations, we obtain the following formula for the diffusion part of the infinitesimal generator (58).

$$
\begin{aligned}
&\frac{1}{2}(h_1^I)^2\,\frac{\partial^2}{\partial I^2}+\frac{1}{2}(h_1^\phi)^2\,\frac{\partial^2}{\partial \phi^2}+\frac{1}{2}(h_1^\theta)^2\,\frac{\partial^2}{\partial \theta^2}\\
&+h_1^I\,h_1^\phi\,\frac{\partial^2}{\partial I\partial\phi}+h_1^I\,h_1^\theta\,\frac{\partial^2}{\partial I\partial\theta}+h_1^\phi\,h_1^\theta\,\frac{\partial^2}{\partial\phi\partial\theta}\\
&=\epsilon^{\frac{1}{3}}\Big[[(A_{12}^g)^2\sin^4\theta]\frac{\partial^2}{\partial\theta^2}\Big]-\epsilon^{\frac{2}{3}}\Big[2A_{12}^g(A_{22}^g-A_{11}^g)\sin^3\theta\cos\theta]\frac{\partial^2}{\partial\theta^2}\\
&\qquad+[2g_IA_{12}^g\sin^2\theta]\frac{\partial^2}{\partial I\partial\theta}+[2g_\phi A_{12}^g\sin^2\theta]\frac{\partial^2}{\partial\phi\partial\theta}\Big]\\
&+\epsilon\Big[g_I^2\frac{\partial^2}{\partial I^2}+g_\phi^2\frac{\partial^2}{\partial\phi^2}+[((A_{22}^g-A_{11}^g)^2-2A_{12}^gA_{21}^g)\sin^2\theta\cos^2\theta]\frac{\partial^2}{\partial\theta^2}\\
&\qquad+2g_Ig_\phi\frac{\partial^2}{\partial I\partial\phi}+[2g_I(A_{22}^g-A_{11}^g)\sin\theta\cos\theta]\frac{\partial^2}{\partial I\partial\theta}\\
&\qquad+[2g_\phi(A_{22}^g-A_{11}^g)\sin\theta\cos\theta]\frac{\partial^2}{\partial\phi\partial\theta}\Big]\\
&+\epsilon^{\frac{4}{3}}\Big[2A_{21}^g(A_{22}^g-A_{11}^g)\sin\theta\cos^3\theta]\frac{\partial^2}{\partial\theta^2}\\
&\qquad+[2g_IA_{21}^g\cos^2\theta]\frac{\partial^2}{\partial I\partial\theta}+[2g_\phi A_{21}^g\cos^2\theta]\frac{\partial^2}{\partial\phi\partial\theta}\Big]\\
&+\epsilon^{\frac{5}{3}}\Big[[(A_{21}^g)^2\cos^4\theta]\frac{\partial^2}{\partial\theta^2}\Big].
\end{aligned}
\tag{60}
$$

Adding the drift and the diffusion parts finally yields the following decomposition of the infinitesimal generator

$$
L^\epsilon=L_0+\epsilon^{\frac{1}{3}}L_1+\epsilon^{\frac{2}{3}}L_2+\epsilon L_3+\epsilon^{\frac{4}{3}}L_4+\epsilon^{\frac{5}{3}}L_5,
$$

where

$$
\begin{aligned}
L_0 &= \omega \frac{\partial}{\partial \phi}, \\
L_1 &= [\omega' \cos^2\theta + 2\,(A^g_{12})^2 \sin^3\theta\cos\theta] \frac{\partial}{\partial\theta} + [(A^g_{12})^2 \sin^4\theta] \frac{\partial^2}{\partial\theta^2}, \\
L_2 &= -[(A^f_{12} + g_I A^g_{121} + g_\phi A^g_{122}) \sin^2\theta \\
&\qquad + A^g_{12}(A^g_{22} - A^g_{11})(3\sin^2\theta\cos^2\theta - \sin^4\theta)] \frac{\partial}{\partial\theta} \\
&\quad - [2A^g_{12}(A^g_{22} - A^g_{11}) \sin^3\theta\cos\theta] \frac{\partial^2}{\partial\theta^2} \\
&\quad - [2g_I A^g_{12} \sin^2\theta] \frac{\partial^2}{\partial I \partial\theta} - [2g_\phi A^g_{12} \sin^2\theta] \frac{\partial^2}{\partial\phi\partial\theta}, \\
L_3 &= [f_I + g_I\, A^g_{11} + g_\phi\, A^g_{12}] \frac{\partial}{\partial I} \\
&\quad + [f_\phi + g_I A^g_{21} + g_\phi A^g_{22}] \frac{\partial}{\partial\phi} \\
&\quad + [(A^f_{22} - A^f_{11} + g_I(A^g_{221} - A^g_{111}) + g_\phi(A^g_{222} - A^g_{112})) \sin\theta\cos\theta \\
&\qquad + ((A^g_{22} - A^g_{11})^2 - 2A^g_{12}A^g_{21})(\sin\theta\cos^3\theta - \sin^3\theta\cos\theta)] \frac{\partial}{\partial\theta} \\
&\quad + g_I^2 \frac{\partial^2}{\partial I^2} + g_\phi^2 \frac{\partial^2}{\partial\phi^2} + [((A^g_{22} - A^g_{11})^2 - 2A^g_{12}A^g_{21}) \sin^2\theta\cos^2\theta] \frac{\partial^2}{\partial\theta^2} \\
&\quad + 2g_I g_\phi \frac{\partial^2}{\partial I \partial\phi} + [2g_I(A^g_{22} - A^g_{11}) \sin\theta\cos\theta] \frac{\partial^2}{\partial I\partial\theta} \\
&\quad + [2g_\phi(A^g_{22} - A^g_{11}) \sin\theta\cos\theta] \frac{\partial^2}{\partial\phi\partial\theta}, \\
L_4 &= [A^f_{21} + g_I A^g_{211} + g_\phi A^g_{212}) \cos^2\theta \\
&\qquad + A^g_{21}(A^g_{22} - A^g_{11})(\cos^4\theta - 3\sin^2\theta\cos^2\theta)] \frac{\partial}{\partial\theta} \\
&\quad + \Big[2A^g_{21}(A^g_{22} - A^g_{11}) \sin\theta\cos^3\theta] \frac{\partial^2}{\partial\theta^2} \\
&\quad + [2g_I A^g_{21} \cos^2\theta] \frac{\partial^2}{\partial I\partial\theta} + [2g_\phi A^g_{21} \cos^2\theta] \frac{\partial^2}{\partial\phi\partial\theta}, \\
L_5 &= [2\,(A^g_{21})^2 \sin\theta\cos^3\theta] \frac{\partial}{\partial\theta} + [(A^g_{21})^2 \cos^4\theta] \frac{\partial^2}{\partial\theta^2}.
\end{aligned}
$$

As explained in Section 2, to evaluate the top Lyapunov exponent, we shall make use of formula (9) of Furstenberg-Khasminskii which requires the calculation of

$$Q^\epsilon = q_0 + \frac{1}{2}[h_1^I \frac{\partial q_1}{\partial I} + h_1^\phi \frac{\partial q_1}{\partial \phi} + h_1^\theta \frac{\partial q_1}{\partial \theta}].$$

which decomposes into fractional powers of $\epsilon^{\frac{1}{3}}$ as

$$Q^\epsilon = \epsilon^{\frac{1}{3}} Q_1 + \epsilon^{\frac{2}{3}} Q_2 + \epsilon Q_3 + \epsilon^{\frac{4}{3}} Q_4 + \epsilon^{\frac{5}{3}} Q_5,$$

where

$$\begin{aligned}
Q_1(.,\theta) &= \omega' \sin\theta\cos\theta - (A_{12}^g)^2 \sin^2\theta(\cos^2\theta - \sin^2\theta), \\
Q_2(.,\theta) &= A_{12}^f \sin\theta\cos\theta + A_{12}^g(A_{22}^g - A_{11}^g) \sin\theta\cos\theta(\cos^2\theta - 3\sin^2\theta) \\
&\quad + (g_I A_{121}^g + g_\phi A_{122}^g)\sin\theta\cos\theta, \\
Q_3(.,\theta) &= A_{11}^f \cos^2\theta + A_{22}^f \sin^2\theta + (A_{22}^g - A_{11}^g)^2\, 2\sin^2\theta\cos^2\theta \\
&\quad + A_{12}^g A_{21}^g(\cos^2\theta - \sin^2\theta)^2 + (g_I A_{111}^g + g_\phi A_{112}^g)\cos^2\theta \\
&\quad + (g_I A_{221}^g + g_\phi A_{222}^g)\sin^2\theta,
\end{aligned}$$

$$\begin{aligned}
Q_4(.,\theta) &= A_{21}^f \sin\theta\cos\theta + A_{21}^g(A_{22}^g - A_{11}^g) \sin\theta\cos\theta(3\cos^2\theta - \sin^2\theta) \\
&\quad + (g_I A_{211}^g + g_\phi A_{212}^g)\sin\theta\cos\theta, \\
Q_5(.,\theta) &= (A_{21}^g)^2 \cos^2\theta(\cos^2\theta - \sin^2\theta).
\end{aligned}$$

Making use of the stereographic projection

$$u = \cot\theta, \quad \theta \in [0,\pi]$$

for simplicity, the infinitesimal generator L^ϵ and Q^ϵ are expressed in (38) and (40).

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