The Kramers Oscillator Revisited

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Dedicated to our highly esteemed colleague and friend

Lutz Schimansky-Geier

on the occasion of his 50th birthday

Abstract

In their 1993 paper [16], Schimansky-Geier and Herzel discovered numerically that the Kramers oscillator (which is identical with the Duffing oscillator forced by additive white noise) has a positive top Lyapunov exponent in the low damping regime.

In this paper, we study the Kramers oscillator from the point of view of random dynamical systems, to which we give a brief introduction. In particular, we confirm the findings in the paper [16] about the Lyapunov exponent by performing more precise simulations, revealing that the Lyapunov exponent is positive up to a critical value of the damping, from which on it remains negative.

We then show that the Kramers oscillator has a global random attractor which in the stable regime (large damping) is just a random point and in the unstable regime (small damping) has very complicated geometrical structure. In the latter case the invariant measure supported by the attractor is a Sinai-Ruelle-Bowen measure with positive entropy. The Kramers oscillator hence undergoes a stochastic bifurcation at the critical value of the damping parameter.

Key words and phrases: Kramers oscillator, noisy Duffing oscillator, random dynamical system, Lyapunov exponent, invariant measure, random attractor,

stochastic bifurcation.

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1 Introduction

Since its introduction by Duffing in 1918, the (forced) nonlinear oscillator with a cubic stiffness term given by

$$\ddot{x}_t + \gamma \dot{x}_t + U'(x_t) = f(t), \quad U(x) = -a\frac{x^2}{2} + b\frac{x^4}{4}, \quad a, b > 0, \ \gamma \geq 0, \tag{1.1}$$

has been one of the paradigms of nonlinear dynamics (see e.g. Guckenheimer and Holmes [6]).

If the forcing is chosen to be

$$f(t) = \sqrt{2\varepsilon\gamma}\, \xi(t), \quad \xi(t) \text{ white noise,}$$

(1.1) describes stochastic motion of a particle in a bistable potential, where $\varepsilon > 0$ stands for the temperature of a heat bath. This model was proposed and studied by Kramers in his celebrated paper [11], hence the name "Kramers oscillator".

We put $\varepsilon = a = b = 1$ and write the Langevin equation (1.1) correctly as a stochastic differential equation (SDE) (see [1]) for x_t, $y_t = \dot{x}_t$ as

$$\begin{aligned} dx_t &= y_t\, dt, \\ dy_t &= (-\gamma y_t - U'(x_t))\, dt + \sqrt{2\gamma}\, dW_t, \end{aligned} \tag{1.2}$$

where W_t is a standard Wiener process. For every initial value (x_0, y_0) the solution of (1.2) exists and forms a Markov process possessing a unique stationary measure with density

$$p(x, y) = N \exp\left(-U(x) - \frac{y^2}{2}\right), \tag{1.3}$$

where N is a norming constant, i.e. p is the unique probability density solving the Fokker-Planck equation $G^* p = 0$, where

$$G^* = -y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}((\gamma y + U'(x))(\cdot)) + \gamma\frac{\partial^2}{\partial y^2} \tag{1.4}$$

is the formal adjoint of the generator

$$G = y\frac{\partial}{\partial x} - (\gamma y + U'(x))\frac{\partial}{\partial y} + \gamma\frac{\partial^2}{\partial y^2} \tag{1.5}$$

of the Markov process. Note that p is independent of the friction (or damping) coefficient γ which will serve as a bifurcation parameter. This means that the family of SDE (1.2) does *not* undergo a P-bifurcation (see Arnold [2, Chap. 9]), i.e. there is no qualitative change of the stationary density as γ varies.

The Lyapunov exponents $\lambda_1(\gamma) \geq \lambda_2(\gamma)$ corresponding to the measure p are defined to be the two possible exponential growth rates of the solutions of the variational equation (linearization) of (1.2) given by

$$dv_t = \begin{pmatrix} 0 & 1 \\ -U''(x_t) & -\gamma \end{pmatrix} v_t dt = \begin{pmatrix} 0 & 1 \\ 1 - 3x_t^2 & -\gamma \end{pmatrix} v_t dt, \qquad (1.6)$$

where x_t is the x component of the solution of (1.2) starting with random initial values (x_0, y_0) having density p.

By the trace formula (see (2.8)), one can immediately read off from (1.6) that

$$\lambda_1(\gamma) + \lambda_2(\gamma) = -\gamma < 0 \quad \text{for all} \quad \gamma > 0\,,$$

meaning that the solution of the variational equation contracts volume at an exponential rate $-\gamma$.

Schimansky-Geier and Herzel [16] now observed the remarkable fact that

$$\lambda_1(\gamma) > 0 \quad \text{for } \gamma \text{ not too large.} \qquad (1.7)$$

This is, to our knowledge, the first example given of a dynamical system for which *additive* noise produces an *unstable* stationary measure.

Our aim in this paper is to have a new look at the Kramers oscillator from the point of view of random dynamical systems. It will turn out that the Kramers oscillator has a global random attractor for all values of γ, but it undergoes a stochastic bifurcation at that value of the friction coefficient at which $\lambda_1(\gamma)$ changes sign, with dramatic changes of the topological structure of the attractor and the invariant measure supported by the attractor.

We stress that nothing of this is visible on the level of the Fokker-Planck equation, i.e. if we stay in the Markovian context, but only emerges from looking at the Kramers oscillator as a (random) dynamical system.

2 Random Dynamical Systems

We will now very briefly explain those concepts of the theory of random dynamical systems which are relevant for our case. We refer the reader to Arnold [2] for a more detailed and systematic treatment.

The definition of a random dynamical system is tailor-made to cover the most important families of dynamical systems under the influence of randomness which are currently of interest, in particular random and stochastic ordinary and partial differential equations and random difference equations. Randomness could describe environmental or parametric perturbations, internal fluctuations (as in the case of the Kramers oscillator), measurement errors, or just lack of knowledge. One of the advantages of our approach is the fact that we study randomness in the framework of classical dynamical systems theory, enabling us to utilize all its powerful machinery.

2.1. Definition (Random Dynamical System (RDS)). A (smooth) random dynamical system (RDS) φ on the state space $\mathbb{R}^d$ with two-sided continuous time $\mathbb{R}$ is a pair of the following objects:

(i) Model of the noise: A metric dynamical system $\theta=(\theta_t)_{t\in\mathbb{R}}$ in the sense of ergodic theory on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. $(t,\omega) \mapsto \theta_t\omega$ is measurable and the family of selfmappings $\theta_t : \Omega \to \Omega$ of Ω forms a flow (i.e. $\theta_0 = \mathrm{id}$, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$) which leaves the measure $\mathbb{P}$ invariant (i.e. $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$, where $(\theta_t\mathbb{P})(B) := \mathbb{P}(\theta_t^{-1}(B))$ is the image of the measure $\mathbb{P}$ under the mapping θ_t). We also assume for simplicity that θ is ergodic.

(ii) Model of the system perturbed by noise: A *cocycle* φ over θ, i.e. a measurable mapping $\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, $(t, \omega, x) \mapsto \varphi(t, \omega, x)$, for which $(t, x) \mapsto \varphi(t, \omega, x)$ is continuous and the family $\varphi(t, \omega) : \mathbb{R}^d \to \mathbb{R}^d$, $x \mapsto \varphi(t, \omega, x)$, of random selfmappings of $\mathbb{R}^d$ are $\mathcal{C}^1$ (continuously differentiable) and form a cocycle over θ, i.e. satisfy the *cocycle property*

$$\varphi(0, \omega) = \mathrm{id}\,, \ \varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega) \text{ for all } t, s \in \mathbb{R} \text{ and } \omega \in \Omega\,. \quad (2.1)$$

Here $\circ$ means composition of mappings. ■

It follows from the cocycle property that all mappings $\varphi(t, \omega)$ are diffeomorphisms, with inverse

$$\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t\omega).$$

Note also that $\Theta_t(\omega, x) := (\theta_t\omega, \varphi(t, \omega, x))$ is a dynamical system (flow) on $\Omega \times \mathbb{R}^d$, called the *skew product flow* corresponding to φ.

We now explain how the solution of an SDE fits into this framework.

2.2. Example (SDE as an RDS). Let

$$dx_t = f(x_t)dt + g(x_t) \circ dW_t \quad (2.2)$$

be a (Stratonovich) SDE in $\mathbb{R}^d$ with m-dimensional Wiener process W. We can assume without loss of generality that time is two-sided, i.e. we solve (2.2) from $t = 0$ forwards as well as backwards in time.

We claim that, modulo conditions [2, Sect. 2.3], the random mappings which assign to each initial value $x \in \mathbb{R}^d$ the solution $\varphi(t, \omega, x)$ of (2.2) at time t form an RDS.

To this end, we model white noise resp. W as a metric dynamical system as follows: Let Ω be the space of continuous functions $\omega : \mathbb{R} \to \mathbb{R}^m$ which satisfy $\omega(0) = 0$, let $\mathcal{F}$ be the Borel σ-algebra in Ω corresponding to the topology of uniform convergence on compacts, and let $\mathbb{P}$ be the Wiener measure (distribution of W) on $\mathcal{F}$. Define the shift on Ω by $\theta_t\omega(s) := \omega(t+s) - \omega(t)$, reflecting the fact that the Wiener process has stationary increments rather than being stationary. Then θ is an ergodic metric dynamical system on $(\Omega, \mathcal{F}, \mathbb{P})$ "driving" the SDE (2.2), and $W_t(\omega) = \omega(t)$. ∎

Invariant measures are of fundamental importance for an RDS as they encapsulate its long-term and ergodic behavior. Hence to find and describe them is one of the primary tasks.

2.3. Definition (Invariant and Stationary Measure). Let φ be an RDS.

(i) A random probability measure $\omega \mapsto \mu_\omega$ on $(\mathbb{R}^d, \mathcal{B}^d)$, $\mathcal{B}^d$ the Borel sets in $\mathbb{R}^d$, is said to be *invariant* under φ if for all $t \in \mathbb{R}$

$$\varphi(t, \omega)\mu_\omega = \mu_{\theta_t\omega} \quad \mathbb{P}\text{-a.s.}$$

(ii) A probability measure ρ on $(\mathbb{R}^d, \mathcal{B}^d)$ is called *stationary* for an SDE (2.2) if it is invariant under the Markov semigroup $P(t, x, B) = \mathbb{P}\{\omega : \varphi(t, \omega, x) \in B\}$ generated by the solution of the SDE for time $\mathbb{R}^+$, i.e. if

$$\rho(\cdot) = \int_{\mathbb{R}^d} P(t, x, \cdot)\rho(dx) \quad \text{for all } t > 0.$$

This is equivalent to the infinitesimal condition $G^*\rho = 0$ (Fokker-Planck equation), where G^* is the formal adjoint of the generator G of $P(t, x, B)$ given by

$$G = \sum_{i=1}^{d} f_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{k,l=1}^{d} (g(x)g(x)^*)_{kl}\frac{\partial^2}{\partial x_k \partial x_l}\,.$$

∎

The concept of a stationary measure of the Markov semigroup corresponding to (2.2) is older and more restrictive than the concept of an invariant measure for the RDS generated by (2.2). There is, however, the following one-to-one correspondence.

2.4. Theorem (Characterization of Markov Measures). *Let φ be the RDS generated by the SDE (2.2). Then there is a one-to-one correspondence between the stationary measures ρ and those invariant measures μ_ω which are measurable with respect to the past $\mathcal{F}_{-\infty}^0 := \sigma(W_t, t \le 0)$ of the Wiener process (so-called Markov measures), the correspondence being given by*

$$\rho \mapsto \mu_\omega := \lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)\rho, \quad \mu_\omega \mapsto \rho := \mathbb{E}\mu_\cdot\,. \tag{2.3}$$

For a proof see [2, Sect. 1.7].

The procedure of passing from a deterministic stationary measure ρ to a random invariant measure μ_ω described by (2.3) is called *disintegration* of ρ.

The so-called *first method* proposed by Lyapunov in his 1892 thesis to study the long-term behavior of nonlinear systems was by means of the exponential growth rates (today called Lyapunov exponents) of the solutions of the variational equation (linearization). This method was filled with new life in 1968 when Oseledets proved his celebrated Multiplicative Ergodic Theorem. We present this theorem just for the particular case of an RDS which is generated by an SDE with additive noise, and for a stationary measure [2, Theorem 4.2.13].

2.5. Theorem (Multplicative Ergodic Theorem for SDE). *Let $f \in \mathcal{C}^2$ be a vector field in $\mathbb{R}^d$, let σ be a fixed $d \times m$ matrix and let W be an m-dimensional Wiener process. Assume that the SDE*

$$dx_t = f(x_t)dt + \sigma dW_t \tag{2.4}$$

generates a unique RDS φ. Let $p(x) > 0$ be the density of a stationary measure $\rho(dx) = p(x)dx$, i.e. p satisfies

$$G^* p = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}(f_i(x)p(x)) + \frac{1}{2}\sum_{k,l=1}^{d} (\sigma\sigma^*)_{k,l} \frac{\partial^2 p(x)}{\partial x_k \partial x_l}\,.$$

Let $Df(x) := \left(\frac{\partial f_i(x)}{\partial x_j}\right)$ be the Jacobian of f,

$$dv_t = Df(x_t)v_t\, dt \tag{2.5}$$

the variational equation of (2.4), and assume that the following integrability condition is satisfied:

$$\int_{\mathbb{R}^d} \|Df(x)\|\, p(x)dx < \infty. \tag{2.6}$$

Then there exist d real numbers, called the Lyapunov exponents of φ under ρ,

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d \tag{2.7}$$

which are the possible exponential growth rates of the solutions $v_t(\omega, x_0, v_0) = D\varphi(t, \omega, x_0)v_0$ of the variational equation (2.5), i.e. for $\mathbb{P}$-almost all ω, ρ-almost all $x_0 \in \mathbb{R}^d$ and all $v_0(\omega, x_0) \in \mathbb{R}^d$

$$\lim_{t\to\infty} \frac{1}{t} \log \|v_t(\omega, x_0, v_0(\omega, x_0))\| = \lambda(\omega, x_0, v_0(\omega, x_0))$$

exists and takes on values from the finite list (2.7). Furthermore, we have the trace formula

$$\sum_{i=1}^{d} \lambda_i = \int_{\mathbb{R}^d} (\text{trace}\, Df(x))p(x)\, dx\,. \tag{2.8}$$

The sign of the top Lyapunov exponent λ_1 hence determines the stability of the variational equation, and in turn (by appealing to the stable manifold theorem for RDS, see [2, Chap. 7]) the stability of the original nonlinear RDS φ.

It is thus important to obtain quantitative information about λ_1 which often can be drawn from a formula obtained by rewriting the variational equation (2.7) in polar coordinates $r = \|v\| \in (0,\infty)$, $s = \frac{v}{\|v\|} \in S^{d-1}$ (the unit sphere in $\mathbb{R}^d$) as a system

$$ds_t = h(x_t, s_t)dt, \quad dr_t = Q(x_t, s_t)r_t dt, \tag{2.9}$$

where $h(x,s) := Df(x)s - Q(x,s)s$, $Q(x,s) := \langle Df(x)s, s\rangle$, and $\langle x, y\rangle := \sum_{i=1}^d x_i y_i$ is the standard scalar product in $\mathbb{R}^d$.

Note that the equation for s_t is decoupled from the one for r_t, so that the pair (x_t, s_t) forms a Markov process with state space $\mathbb{R}^d \times S^{d-1}$ and generator $L = G + h\frac{\partial}{\partial s}$.

Integrating the equation for r_t in (2.9) by separating variables and using the ergodic theorem yields the following result.

2.6. Theorem (Furstenberg-Khasminskii formula). *Assume the situation of Theorem 2.5. Then there exists a stationary measure ν for the Markov process (x_t, s_t) on $\mathbb{R}^d \times S^{d-1}$ whose marginal on $\mathbb{R}^d$ is ρ such that*

$$\lambda_1 = \int_{\mathbb{R}^d \times S^{d-1}} \langle Df(x)s, s\rangle\, \nu(dx, ds). \tag{2.10}$$

Another basic object of a deterministic dynamical system is an attractor, i.e. a compact invariant set which attracts all orbits in the course of time, on which thus the “essential” dynamics of the system takes place. We generalize this concept to the random case. It is quite natural that an attractor of an RDS will be a random rather than a deterministic set.

2.7. Definition (Attractor of RDS). Let φ be an RDS and let $A(\omega)$ be a random compact set, i.e. $\omega \mapsto A(\omega)$ is a measurable mapping into the space of non-void compact subsets of $\mathbb{R}^d$. Such a set is called *invariant* if $\varphi(t,\omega)A(\omega) = A(\theta_t\omega)$ for all t and ω.

An invariant random compact set A is called a (global random) attractor of φ if for any random variable X

$$\lim_{t\to\infty} \mathbb{P}\{\omega : d(\varphi(t,\omega,X), A(\theta_t\omega)) > \varepsilon\} = 0 \quad \text{for all} \quad \varepsilon > 0, \tag{2.11}$$

where $d(x, A) := \inf_{y\in A} \|x - y\|$ is the distance of x and A. ∎

Note that in the definition of an attractor the orbit $\varphi(t,\omega,X)$ is compared with $A(\theta_t\omega)$, but both objects are continuously moving with time. It turns out that asking for $\mathbb{P}$-almost sure convergence of the distance would be too restrictive – hence the convergence in probability in (2.11).

The existence of an attractor $A(\omega)$ helps locating invariant measures, since by a result of Ochs [13, Theorem 2]

$$\mu_\omega(A(\omega)) = 1 \quad \mathbb{P}\text{-a.s.} \tag{2.12}$$

for any invariant measure μ_ω.

The next section of the paper will consist of applying the above concepts to the Kramers oscillator.

3 Analysis of the Kramers Oscillator

Consider the SDE (1.2) describing the Kramers oscillator. Recall that without loss of generality we make the choice $\varepsilon = a = b = 1$. We choose γ as the parameter which we would like to vary.

As the coefficients of (1.2) do not satisfy global Lipschitz conditions, there is the possibility of explosion of solutions in finite time. We first have to make sure that the SDE indeed generates a global RDS φ.

3.1. Theorem (Existence of RDS). *The SDE (1.2) for the Kramers oscillator generates a $\mathcal{C}^\infty$ RDS φ in $\mathbb{R}^2$ (this RDS will henceforth also be called the Kramers oscillator).*

This is a particular case of a result of Schenk-Hoppé [14, Theorem 5.8] for the Duffing oscillator with additive as well as multiplicative noise (see also [2, Sect. 9.4]). The idea of the proof is to convert (1.2) into a random differential equation (perturbed by real noise) by means of a random coordinate transformation.

To assure the reader that the statement of Theorem 3.1 is not so obvious let us mention that it does in general *not* suffice for the existence of an RDS that none of the solutions $\varphi(t, \omega, x_0)$, $x_0 \in \mathbb{R}^d$ any initial value, explodes in finite time.

The next question is about invariant and stationary measures. As pointed out in the Introduction, it is a striking feature of the Kramers oscillator (U could be replaced by a more general potential) that the Fokker-Planck equation $G^*p = 0$, G^* given by (1.4), can be explicitly solved to yield the density (1.3), and this density turns out to be independent of the friction parameter γ.

We claim that $\rho(dx, dy) = p(x, y)dxdy$ is the unique stationary measure. This would be immediate if G^* were elliptic (i.e. if the diffusion matrix $\sigma\sigma^*$ were positive-definite). For the Kramers oscillator, however, the diffusion matrix

$$\sigma\sigma^* = \begin{pmatrix} 0 & 0 \\ 0 & 2\gamma \end{pmatrix}$$

has rank 1, so we are in the non-elliptic case.

To prove uniqueness of the stationary measure in the non-elliptic case we check whether the Lie algebra generated by the two vector fields $f(x, y) =$

$(y, -\gamma y - U'(x))$ and $g(x,y) = (0, \sqrt{2\gamma})$ in (1.2) has full dimension 2. If yes, this implies that G and G^* are hypoelliptic, hence all solutions of $G^*q = 0$ are C^∞. However, by a result of Kliemann [10], a second measure with a smooth density cannot coexist with a measure whose density is positive everywhere.

The Lie bracket $[f,g]$ can be easily calculated to be

$$[f,g](x,y) := \begin{pmatrix} f_1 \frac{\partial g_1}{\partial x} + f_2 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial f_1}{\partial x} - g_2 \frac{\partial f_1}{\partial y} \\ f_1 \frac{\partial g_2}{\partial x} + f_2 \frac{\partial g_2}{\partial y} - g_1 \frac{\partial f_2}{\partial x} - g_2 \frac{\partial f_2}{\partial y} \end{pmatrix} = -\sqrt{2\gamma} \begin{pmatrix} 1 \\ -\gamma \end{pmatrix}. \tag{3.1}$$

At each $(x,y) \in \mathbb{R}^2$ the vectors $g(x,y)$ and $[f,g](x,y)$ are clearly linearly independent proving that the corresponding Lie algebra is full.

We summarize our findings in the following theorem.

3.2. Theorem (Existence and Uniqueness of Stationary Measure). *The Kramers oscillator has the unique stationary measure*

$$\rho(dx,dy) = p(x,y)dxdy, \quad p(x,y) = N \exp\left(-U(x) - \frac{y^2}{2}\right), \tag{3.2}$$

where N is a norming constant.

Although ρ is independent of γ (i.e. the long-term behavior of one trajectory $\varphi(t,\omega,x)$ does not "feel" γ), the RDS φ (i.e. the simultaneous motion of two and more points) does depend on γ, and so does the disintegration

$$\mu_\omega(dx,dy) = \lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)\rho(dx,dy)$$

(see (2.12)). We will see that for small damping, φ has many more invariant measures, due to the fact that φ has a highly nontrivial attractor. However, μ_ω is the unique Markov measure.

We next clarify the existence of an attractor.

3.3. Theorem (Existence and Uniqueness of Attractor). *The Kramers oscillator has a unique global random attractor $A(\omega)$ for any value of $\gamma > 0$.*

This was proved by Imkeller and Schmalfuss [8, Sect. 3.4], using the only presently available technique for the white noise case which we briefly describe (following Ochs [12]):

(i) Transform the SDE (1.2) into a random differential equation (containing real noise only) by using a linear random coordinate transformation keeping x_t and replacing y_t by $z_t = y_t - \sqrt{2\gamma}\, u(\theta_t\omega) + \frac{\gamma}{2} x_t$, where

$$u_t(\omega) = u(\theta_t\omega) := e^{-(\gamma/2)t} \int_{-\infty}^{t} e^{(\gamma/2)s} dW_s(\omega)$$

is the stationary Ornstein-Uhlenbeck process, i.e. the unique stationary solution of $du = -\frac{\gamma}{2}udt + dW$. The transformed equations are

$$\dot{x}_t = z_t - \frac{\gamma}{2}x_t + \sqrt{2\gamma}u_t, \quad \dot{z}_t = -\frac{\gamma}{2}z_t - U'(x_t) + \frac{\gamma^2}{4}x_t, \tag{3.3}$$

where the white noise term has dropped out.

(ii) Use the "Lyapunov function" $V(x,z) := U(x) + \frac{z^2}{2}$ and prove that $\dot{V}_t \leq \alpha(\theta_t\omega)V_t + \beta(\theta_t\omega)$ with $\int \alpha \, d\mathbb{P} < 0$.

(iii) The affine equation $\dot{v}_t = \alpha(\theta_t\omega)v_t + \beta(\theta_t\omega)$ has a unique stationary solution, hence by (ii) and the comparison principle the equation (3.3) has an absorbing set. This implies the existence of a unique attractor for (3.3).

(iv) The transformation in (i) is "tempered", hence can be inverted to yield an attractor for the original SDE.

We will numerically determine the attractor after our analysis of the Lyapunov exponents.

3.4. Theorem (Lyapunov Exponents). *Let φ be the Kramers oscillator and $\rho(dx,dy) = p(x,y)dxdy$ be the unique stationary measure.*

(i) The integrability condition (2.6) for the variational equation (1.6) is satisfied, so that the Multiplicative Ergodic Theorem (2.5) for φ and ρ applies. In particular,

$$\lambda_1(\gamma) + \lambda_2(\gamma) = -\gamma < 0,$$

thus

$$\lambda(\gamma) := \lambda_1(\gamma) \geq -\frac{\gamma}{2} \geq \lambda_2(\gamma).$$

(ii) Stationary measure on $\mathbb{R}^2 \times S^1$: Parametrizing the unit circle S^1 by $s = (\cos\alpha, \sin\alpha)$ and identifying points $\alpha = 0$ and $\alpha = 2\pi$ of the interval $[0, 2\pi]$, we have for the angle of the variational equation

$$d\alpha_t = h(x_t, \alpha_t)dt = (-\frac{3}{2}x_t^2 \cos 2\alpha_t - \frac{\gamma}{2}\sin 2\alpha_t)dt. \tag{3.4}$$

The Markov process (x_t, y_t, α_t) on $\mathbb{R}^2 \times [0, 2\pi]$ having generator

$$L = G + h(x,\alpha)\frac{\partial}{\partial\alpha},$$

where G is the generator of (x_t, y_t) given by (1.5), has exactly two stationary measures with marginal density p on $\mathbb{R}^2$. These measures have support $\mathbb{R}^2 \times [-\frac{\pi}{4}, 0]$ and C^∞ density $q(x,y,\alpha)$, resp. support $\mathbb{R}^2 \times [\frac{3\pi}{4}, \pi]$ and density $q(x,y,\alpha-\pi)$.

(iii) Furstenberg-Khasminskii formula: We have

$$\lambda(\gamma) = \int_{\mathbb{R}^2}\int_{-\frac{\pi}{4}}^{0} \left((2-3x^2)\sin 2\alpha + \gamma(\cos 2\alpha - 1)\right) q(x,y,\alpha)dxdyd\alpha. \tag{3.5}$$

Proof. (i) The norm of the Jacobian

$$Df(x) = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & -\gamma \end{pmatrix}$$

satisfies $||Df(x)|| \le C_1 + C_2 x^2$ which is clearly integrable with respect to ρ as the density decays exponentially fast.

(ii) Note first that all functions involved are π-periodic so that we can restrict ourselves to the α interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, identifying again the endpoints.

We first verify that the Lie algebra generated by the vector fields $X = (y, -\gamma y - U'(x), h(x,\alpha))$ and $Y = (0, \sqrt{2\gamma}, 0)$ on $\mathbb{R}^2 \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ has full rank 3. This can be done with the recipe (3.1) by calculating $[X,Y]$, $[X,[X,Y]]$, $[X,[X,[X,Y]]]$ and $[[X,Y],[X,[X,Y]]]$ which we leave as an exercise.

We conclude that L^*, though not elliptic, is hypoelliptic, hence all solutions of $L^*\nu = 0$ are C^∞. To prove uniqueness of the solution we utilize a result of Kliemann [10] saying that the possible supports of stationary measures are the so-called invariant control sets C of the deterministic control system

$$\dot{\xi}_t = X(\xi_t) + u(t)Y(\xi_t),$$

where $u(t)$ is a piecewise constant control function with values in $\mathbb{R}$, and on such a C the stationary measure is unique. By directly inspecting the function $h(x,\alpha)$ we find that $C = \mathbb{R}^2 \times [-\frac{\pi}{4}, 0]$ is the unique invariant control set.

(iii) This is just formula (2.10) applied to our case and taking into account that due to the π-periodicity the integral over $[0, 2\pi]$ is twice the integral over $[-\frac{\pi}{2}, \frac{\pi}{2}]$. □

Formula (3.5) can be used as the starting point for the asymptotic analysis of $\lambda(\gamma)$ for small as well as large γ. For example, applying the method of adjoint asymptotic expansion (see e.g. Arnold, Doyle and Sri Namachchivaya [3]) to formula (3.5) we dare to make the following conjecture. For details see Arnold and Imkeller [4].

3.5. Conjecture (Asymptotic Expansion of Top Lyapunov Exponent). *Assume the situation of Theorem 3.4. Then*

$$\lambda(\gamma) = \Lambda_1 \gamma^{1/3} + o(\gamma^{1/3}) \quad \textit{for} \quad \gamma \to 0, \tag{3.6}$$

where $\Lambda_1 > 0$ *and* $\lim_{\gamma\to 0} o(\gamma^{1/3})/\gamma^{1/3} = 0$.

In particular, there exists some $\gamma_0 > 0$ *such that* $\lambda(\gamma) > 0$ *for all* $\gamma \in (0, \gamma_0)$.

Figures 1 and 2 depict the top Lyapunov exponent $\lambda(\gamma)$ as a function of γ. Figure 1 shows the range $0 \le \gamma \le 0.7$ and Figure 2 the range $0 \le \gamma \le 100$. The data support that $\lambda(\gamma)$ has a steep growth with vertical slope from $\lambda(0) = 0$ to a positive maximum at $\gamma_1 \approx 0.9$. Then $\lambda(\gamma)$ decays and crosses 0 at $\gamma_2 \approx 0.53$, reaches its minimum at $\gamma_3 \approx 4$ and then seems to increase and converge to 0 for $\gamma \to \infty$.

For the calculation we used the definition of $\lambda(\gamma)$ as the exponential growth rate of a typical solution of (1.6). The SDE (1.2) was numerically integrated by an Euler scheme with 10^6 steps of size 0.0005. We calculated 300 values of $\lambda(\gamma)$ for Figure 1 and 600 values for Figure 2.

Figure 1: The top Lyapunov exponent $\lambda(\gamma)$ of the Kramers oscillator for $0 \leq \gamma \leq 0.7$

Figure 2: The top Lyapunov exponent $\lambda(\gamma)$ of the Kramers oscillator for $0 \leq \gamma \leq 100$

We now turn to the numerical computation of the attractor. For this we use a subdivision algorithm developed by Dellnitz and Hohmann [5] for deterministic dynamical systems and adapted to the random case by Keller and Ochs [9, 12].

As the top Lyapunov exponent $\lambda(\gamma)$ changes sign at $\gamma_2 \approx 0.53$, we expect a stochastic bifurcation (qualitative change of φ, see [2, Chap. 9]) on the level of attractors and invariant measures if γ moves across γ_2.

Figure 3: The random attractor $A(\omega)$ of the Kramers oscillator in the unstable regime ($\gamma = 0.25$)

This can indeed be observed numerically:

(i) Stable regime: For all parameter values $\gamma > \gamma_2$ the global random attractor consists of just one random point, $A(\omega) = \{a(\omega)\}$. It follows by (2.12) that the unique invariant measure of φ is the random Dirac measure $\mu_\omega(dx, dy) = \delta_{a(\omega)}(dx, dy)$ which is also the disintegration of the unique stationary measure ρ in the sense of Theorem 2.4.

(ii) Unstable regime: For all parameter values $\gamma \in (0, \gamma_2)$, the random attractor is a "chaotic" object with Cantor-set-type transversal intersections, see Figure 3 for a particular realization of $A(\omega)$. Figure 3 was produced by the subdivision algorithm starting with a box of size 7×7 and stopping after 15 subdivisions in each direction. The fact that $A(\omega)$ is complicated is quite feasable because in this case $\lambda_1(\gamma) > 0 > \lambda_2(\gamma)$, thus a typical point of the attractor has a one-dimensional unstable manifold which necessarily has to belong to the attractor (see Schenk-Hoppé [15, Theorem 7.3]).

We refer the reader to Ochs [12] and the forthcoming paper by Arnold and Imkeller [4] for more details.

We also have obtained numerical information on the disintegration μ_ω of ρ supported by $A(\omega)$ in the unstable case. It turns out that μ_ω is so extremely complex that it cannot be reproduced without using colors. We claim that in this case μ_ω is a Sinai-Ruelle-Bowen measure, and φ has positive fibre entropy $h_\mu(\varphi)$ under μ_ω. The fibre entropy only measures the information production by the cocycle φ and suppresses the one by θ, see Gundlach [7]. More precisely,

$$h_\mu(\varphi) = \lambda(\gamma) > 0\,,$$

where we have used Pesin's entropy formula (which has never been rigorously proved for non-compact state spaces).

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