The smoothness of laws of random flags and Oseledets spaces of linear stochastic differential equations

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Abstract

The Oseledets spaces of a random dynamical system generated by a linear stochastic differential equation are obtained as intersections of the corresponding nested invariant spaces of a forward and a backward flag, described as the stationary states of flows on corresponding flag manifolds. We study smoothness of their laws and conditional laws by applying Malliavin's calculus. If the Lie algebras induced by the actions of the matrices generating the system on the manifolds span the tangent spaces at any point, laws and conditional laws are seen to be C^∞-smooth. As an application we find that the semimartingale property is well preserved if the Wiener filtration is enlarged by the information present in the flag or Oseledets spaces.

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Introduction

The treatment of many problems related to asymptotic properties of the random dynamical systems induced by sde with the tools of stochastic analysis faces one essential difficulty: the random invariant measures involved are not adapted with respect to the history of the driving Brownian motion. One way to deal with it was initiated in [20], and employs essentially the well known tool of "enlargement of filtrations". Instead of the natural filtration of the Wiener process one works with the family of its σ-algebras enlarged by the information present in the Oseledets spaces which carry the invariant measures. According to a general result of Jacod [22] the semimartingale property is preserved if the enlarging random elements possess conditional laws absolutely continuous with respect to some common reference measure. In our case this calls for an investigation of the smoothness of the conditional laws of the Oseledets spaces - or the elements of two flags of random invariant spaces creating them - with respect to Riemannian volume on the Grassmannian manifolds in which they take their values.

We thus enter a domain which is clearly interesting in its own right. The laws of the random subspaces of the forward flag for example are just the eventually unique invariant measures of certain backward Kolmogorov equations induced by our system on appropriate Grassmann manifolds. Their smoothness has been investigated in purely analytical terms using Hörmander's approach. For the vast literature and, in particular, the control theoretic aspect see for example Sussmann [40], Arnold, Kliemann, Oeljeklaus [4].

Our interest in conditional laws, however, calls for an approach of smoothness properties by the methods of Malliavin's calculus. As a by-product, we reprove Hörmander's theorem for the laws with stochastic methods. We start by identifying the Grassmannian $G_k(d)$ of k-dimensional linear subspaces of $\mathbf{R}^d$ with the orthogonal projectors on them, thus describing the random linear spaces in the flags as random orthogonal projectors. Then the action of the linear flow $(\Phi_t)_{t\in\mathbf{R}}$ induced by the underlying sde on $p \in G_k(d)$ creates flows $P_t, t \in \mathbf{R}$, on $G_k(d)$ which possess the random projectors Q in the flags as their stationary states. So properties of the law of $P_t, t \in \mathbf{R}$, will reveal similar properties of the law of Q. The general link between hypoellipticity and smoothness of laws via Malliavin's calculus has been investigated in a number of papers. See for example Malliavin [27], Stroock [38], [39], Kusuoka and Stroock [24], Bismut and Michel [10], Bell [8], Taniguchi [42], Coquio [12], and the book of Nualart [30] for more references. Our conditions read: if the actions of the matrices generating the linear flow on the Grassmannians create Lie algebras which span the whole tangent space in any point, the flows $(P_t)_{t\in\mathbf{R}}$ and thus Q have C^∞ conditional laws. The richness condition for the Lie algebras is satisfied for instance if the Lie group generated by the matrices is the general or special linear group.

The material is organized as follows. In section 1 we discuss a decomposition of the action of the linear flow on the Grassmannians, and prove that the projectors in the flags are invariant states of corresponding flows. In section 2 we give the problem of smoothness of the laws of the projectors in the forward flag a purely stochastic treatment. Its essential tool is the proof that the inverse of a Malliavin bilinear form is sufficiently well integrable. This study is basic to section 3, in which we extend the analysis to treat smoothness of conditional laws. Finally, in section 4 the previous results on absolute continuity of laws of projectors on spaces in the forward and backward flags are combined to yield in a purely analytical reasoning the absolute continuity of the laws of the Oseledets spaces.

Smoothness properties of the laws of invariant subspaces are interesting in their own right (see Ledrappier, Young [25], p. 532). Our treatment may be refined to discuss for example positivity, support and decay of the densities. The enlargement of filtrations and smoothness results could find applications in the study of various asymptotic properties of random dynamical systems: for example in the investigation of a concept of random rotation numbers (see Arnold and San Martin [6], Ruffino [34]).

Notations and preliminaries

Our basic probability space is the m-dimensional canonical Wiener space $(\Omega, \mathbf{F}, \mathbf{P})$, enlarged such as to carry an m-dimensional "Brownian motion" indexed by $\mathbf{R}$. More precisely, $\Omega = C(\mathbf{R}, \mathbf{R}^m)$ is the set of continuous functions on $\mathbf{R}$ with values in $\mathbf{R}^m$, $\mathbf{F}$ the σ-algebra of Borel sets with respect to uniform convergence on compacts of $\mathbf{R}$, $\mathbf{P}$ the probability measure on $\mathbf{F}$ for which the *canonical Wiener process* $W_t = (W_t^1, ..., W_t^m), t \in \mathbf{R}$, satisfies that both $(W_t)_{t\geq 0}$ and $(W_{-t})_{t\geq 0}$ are usual m-dimensional Brownian motions. The natural filtration $\{\mathbf{F}_s^t = \sigma(W_u - W_v : s \leq u, v \leq t) : \mathbf{R} \ni s \leq t \in \mathbf{R}\}$ of W is assumed to be completed by the $\mathbf{P}$-completion of $\mathbf{F}$. For $t \in \mathbf{R}$, let $\theta_t : \Omega \to \Omega, \omega \mapsto \omega(t+\cdot) - \omega(t)$, the "shift" on Ω by t. It is well known that θ_t preserves Wiener measure $\mathbf{P}$ for any $t \in \mathbf{R}$ and is even ergodic. Hence $(\Omega, \mathbf{F}, \mathbf{P}, (\theta_t)_{t\in\mathbf{R}})$ is an ergodic metric dynamical system (see Arnold [1]). The shift is filtered in the sense that

$$\theta_u^{-1}\, \mathbf{F}_s^t = \mathbf{F}_{s+u}^{t+u}, \quad s, t, u \in \mathbf{R}.$$

As usual, we use a "$\circ$" to denote Stratonovich integrals with respect to Wiener process. Since our parameter space is $\mathbf{R}$ and not just $\mathbf{R}_+$ let us briefly recall the conventions for the definition of the Stratonovich integral. The forward Stratonovich integral of a continuous $(\mathbf{F}_0^t)_{t\geq 0}$-semimartingale $(f_t)_{t\geq 0}$ w. r. t. a one dimensional Wiener process $(W_t)_{t\geq 0}$ is defined as usual and denoted by

$$\int_0^t f_s \circ dW_s, \quad t \geq 0,$$

the backward Stratonovich integral of a continuous backward $(\mathbf{F}_t^0)_{t\leq 0}$-semimartingale $(g_t)_{t\leq 0}$ w. r. t. a backward one dimensional Wiener process $(W_t)_{t\leq 0}$ by

$$\int_t^0 g_s \circ dW_s, \quad t \leq 0.$$

For a process $(f_t)_{t\in\mathbf{R}}$ such that $(f_t)_{t\geq 0}$ is a continuous forward $(\mathbf{F}_0^t)_{t\geq 0}$-semimartingale and $(f_t)_{t\leq 0}$ a continuous backward $(\mathbf{F}_t^0)_{t\leq 0}$-semimartingale we introduce the convention

$$\int_0^t f_s \circ dW_s = \begin{cases} \int_0^t f_s \circ dW_s, & t \geq 0, \\ -\int_t^0 f_s \circ dW_s, & t \leq 0. \end{cases}$$

We briefly recall the basic concepts of Malliavin's calculus we need. See Nualart [30] for a more detailed treatment. For $T \geq 0, 1 \leq j \leq m$, we shall denote by D^j the derivative operator which, for a smooth random variable of the form

$$F = f(W_{t_1}, ..., W_{t_n}), \quad f \in C_b^\infty((\mathbf{R}^m)^n), \quad t_1, ..., t_n \in \mathbf{R}_+,$$

takes the form

$$D_s^j F = \sum_{i=1}^n \frac{\partial}{\partial x_{i,j}} f(W_{t_1}, ..., W_{t_n}) 1_{[0,t_i]}(s).$$

For each $p \geq 1, \mathbf{D}_{p,1}([0,T])$ will denote the Banach space of random variables on the Wiener space, defined as the closure of the set of smooth random variables with respect to the norm

$$||F||_{p,1} = ||F||_p + \sum_{1\leq j\leq m} E([\int_0^T |D_s^j F|^2 \, ds]^{\frac{p}{2}})^{\frac{1}{p}},$$

to which D^j extends in a natural way.

For $k \in \mathbf{N}$, we define the H-derivative of order k by k-fold iteration of the above derivative on the space of smooth random variables. For such a random variable F, the expression $D_{s_1...s_k}^{j_1...j_k} F$ for $1 \leq j_i \leq m, 1 \leq i \leq k, s_1, ..., s_k \in \mathbf{R}_+$ is self explanatory. For $T \geq 0, p \geq 1, k \in \mathbf{N}$, we denote by $\mathbf{D}_{p,k}([0,T])$ the Banach space given by the completion of the set of smooth random variables with respect to the norm

$$\| F \|_{p,k} = \| F \|_p + \sum_{1\leq j\leq k} \sum_{1\leq l_i\leq m, 1\leq i\leq j} [E([\int_0^T (D_{s_1...s_j}^{l_1...l_j} F)^2 \, ds_1...ds_j]^{\frac{p}{2}})]^{\frac{1}{p}}.$$

Finally, let $\mathbf{D}_\infty$ be the intersection of the preceding Sobolev spaces over all $p \geq 1, k \in \mathbf{N}, T \geq 0$.

We shall deal with matrices and operators in $\mathbf{R}^d$, equipped with standard basis and standard scalar product, mainly. By I we denote the d-dimensional unit matrix (identity operator). The transpose of a matrix A is written A^*, its trace $\mathrm{tr} A$, its rank $\mathrm{rk}\, A$..

We shall be concerned with the *Oseledets spaces* of a linear system of stochastic differential equations, objects described in the multiplicative ergodic theorem of Oseledets. We shall briefly recall its main assertions in the framework of linear sde. Let $A_0, A_1, ..., A_m$ be $d \times d$ matrices, and consider the linear flow $(\Phi_t)_{t\in\mathbf{R}}$ on $Gl(d; \mathbf{R})$ generated by

$$\begin{aligned} d\Phi_t &= A_0 \, \Phi_t \, dt + \sum_{i=1}^m A_i \, \Phi_t \circ dW_t^i, \\ \Phi_0 &= I. \end{aligned} \tag{1}$$

Then the *Ruelle matrix* $\Phi^\pm$ of the system given by $\Phi^\pm = \lim_{t\to\pm\infty} [\Phi_t^* \, \Phi_t]^{\frac{1}{2|t|}}$ has a spectral decomposition

$$\Phi^\pm = \sum_{i=1}^r e^{\pm\lambda_i} S_i^\pm$$

with some r, $1 \leq r \leq d$, random orthogonal projectors $S_i^\pm$ of deterministic rank d_i such that $d_1 + \cdots + d_r = d$. The *Lyapunov exponents* λ_i describe the asymptotic exponential growth rate of the trajectories of the solutions of the linear equation: the orthogonal projectors $Q_i = S_i^+ + \cdots + S_r^+$ and $P_i = S_1^- + \cdots + S_i^-$ determine those subspaces of $\mathbf{R}^d$ on which trajectories have exponential growth rates $\leq \lambda_i$ as $t \to \infty$ resp. $\geq \lambda_i$ as $t \to -\infty$. More formally,

$$x \in Q_i(\mathbf{R}^d) \quad \text{iff} \quad \lim_{t\to\infty} \frac{1}{t} \ln ||\Phi_t \, x|| \leq \lambda_i,$$

and

$$x \in P_i(\mathbf{R}^d) \quad \text{iff} \quad \lim_{t \to -\infty} \frac{1}{t} \ln ||\Phi_t\, x|| \geq \lambda_i.$$

The intersections $E_i = Q_i(\mathbf{R}^d) \cap P_i(\mathbf{R}^d)$ of these two families, the *Oseledets spaces* correspondingly characterize those subspaces on which trajectories have exponential growth rate λ_i both at ∞ and at $-\infty$, thus generalizing the concept of eigenspaces of linear algebra. We equally associate orthogonal projectors with them which will be denoted R_i and have (P-a.s.) rank d_i, $1 \leq i \leq r$. The *forward flag* of orthogonal projectors associated with our system is the family $(Q_1, \cdots, Q_r)$, the *backward flag* the family $(P_1, \cdots, P_r)$.

So the random elements the laws of which we study can be considered as taking their values in some Grassmannian manifold of linear subspaces of $\mathbf{R}^d$, if we identify the latter with the space of orthogonal projectors in $\mathbf{R}^d$ of a certain fixed rank. Thus for dimension k, $1 \leq k \leq d$, we let

$$G_k(d) = \{p : p \in \mathbf{R}^{d \times d}, \mathrm{rk}\, p = k, p^2 = p,\ p^* = p\}.$$

This simple identification provides an isometric embedding of the Grassmannian manifold, usually described as

$$O(d)/O(k) \times O(d-k)$$

of k-dimensional subspaces of $\mathbf{R}^d$ into the space $\mathbf{R}^{d \times d}$ of $d \times d$ matrices. Fixing a canonical basis $(e_1, \cdots, e_d)$ of $\mathbf{R}^d$, and denoting by p_k the orthogonal projector identifying e_i for $i \leq k$ and annihilating e_i for $i > k$, we can describe $G_k(d)$ as

$$G_k(d) = \{o\, p_k\, o^* : o \in O(d)\}.$$

Recalling that for $o \in O(d)$ the tangent space $T_o O(d)$ consists of the linear operators $o\, A$ where A is skew symmetric, we may use this description to identify $T_p G_k(d)$ as

$$T_p G_k(d) = \{(I-p)\, A\, p + p\, A^*\, (I-p) :\ A \in \mathbf{R}^{d \times d}\},$$

for $p \in G_k(d)$. Indeed, if $p = o\, p_k\, o^*$, a tangent vector at p comes from a tangent vector at o in $O(d)$, i.e. is of the form

$$o\, A p_k\, o^* + o\, p_k A^*\, o^* = B\, p + p\, B^* = (I-p)\, B\, p + p\, B^*\, (I-p)$$

with $B = o\, A\, o^*$ skew symmetric. Moreover, if $A \in \mathbf{R}^{d \times d}$, we can choose a skew symmetric B with $h_A(p) = h_B(p)$, if $h_C(p) = (I-p)\, C\, p + p\, C\, (I-p)$ for $C \in \mathbf{R}^{d \times d}$. For example, $B = (I-p)\, A\, p - p\, A^*\, (I-p)$ is appropriate.

Hence the compact manifolds M we shall be working with are embedded in Euclidean spaces and inherit their Riemannian metrics from the usual Euclidean scalar product which shall be denoted by $\langle ., . \rangle$ irrespective of the dimension of the underlying space. The (co-) tangent bundle of M will be denoted by $TM (T^*M)$. Lie brackets are written $[., .]$.

1 Flows of diffeomorphisms on Grassmannians, their invariant states and smoothness of laws

For $p \in G_k(d)$, our linear cocycle $(\Phi_t)_{t\in\mathbf{R}}$ generates a process $(\Phi_t\, p)_{t\in\mathbf{R}}$ of linear operators of rank k which does not necessarily live on $G_k(d)$. But we can decompose it multiplicatively into a process living on $G_k(d)$ and another one living on $Gl(d;\mathbf{R})$. If we require that the latter has the same action on p, this decomposition is unique. It plays a central role in our investigation of the laws and conditional laws of the Oseledets spaces. The component of the decomposition living on the Grassmannian creates a flow which eventually approaches, as $t \to \infty$, an invariant state that must be given by a projector belonging to the forward flag of our system. This way properties of the law of the invariant projector can be related to properties of the laws of the Grassmann components of $(\Phi_t\, p)_{t\in\mathbf{R}}$. In fact smoothness properties of the law follow from well known Hörmander conditions for related vector fields on the Grassmannian in a purely analytic classical reasoning, as we shall explain.

We start with giving the multiplicative decomposition which rests upon the following very simple purely analytical considerations. Fix $1 \leq k \leq d$, and denote $M = G_k(d)$.

Lemma 1.1 *Let $A \in Gl(d;\mathbf{R}), p \in M$. Then there exists a unique pair (q, B) of linear operators such that $q \in M, B \in Gl(d;\mathbf{R})$ and such that the following equations hold*

$$A\,p = q\,B, \quad (I-q)\,A = B\,(I-p), \tag{2}$$

$$A\,p = B\,p. \tag{3}$$

Proof:

To prove existence, let q be the orthogonal projector onto the range of $A\,p$, and define

$$B = q\,A\,p + (I-q)\,A\,(I-p).$$

By definition, we have $q\,A\,p = A\,p$. This implies

$$(I-p)\,A^{-1}\,q(\mathbf{R}^d) = (I-p)\,A^{-1}\,q\,A\,p(\mathbf{R}^d) = (I-p)\,p(\mathbf{R}^d) = 0.$$

Therefore we obtain the equation $(I-p)\,A^{-1} = (I-p)\,A^{-1}\,(I-q)$. Consequently, setting

$$C = p\,A^{-1}\,q + (I-p)\,A^{-1}\,(I-q),$$

we calculate

$$C\,B = p\,A^{-1}\,q\,A\,p + (I-p)\,A^{-1}\,(I-q)\,A\,(I-p) = p + (I-p) = I.$$

This means $C = B^{-1}$, and $B \in Gl(d;\mathbf{R})$. The definition of B immediately yields

$$q\,B = B\,p = q\,A\,p = A\,p,$$

as well as

$$B\,(I-p) = (I-q)\,A\,(I-p) = (I-q)\,A.$$

Hence the pair (q, B) fulfills (2) and (3).

To prove uniqueness, assume (q', B') is a pair of linear operators such that $q' \in M, B' \in Gl(d; \mathbf{R})$ which fulfills (2) and (3). Then $q'\, B' = q\, B$, hence by nonsingularity of B, B' $q'(\mathbf{R}^d) = q(\mathbf{R}^d)$ whence $q' = q$, both being orthogonal projectors. Now, from (2) and (3)

$$B\, p = A\, p = B'\, p, \quad B\,(I - p) = (I - q)\, A = B'\,(I - p).$$

This obviously implies $B = B'$, and we are done. □

Let us call in the sequel the unique pair (q, B) existing according to lemma 1.1 the *left decomposition* of the pair (A, p), and refer to q as the *angular part*, and B as the *radial part*. This terminology is justified at least in case $k = 1$, i.e. $M = P^{d-1}$. In this case q is indeed the angular component of the one dimensional vector $A\, p$, yet without orientation, while (3) includes the statement $p\, A\, p = p\, B\, p$, i.e. B carries the information about the radial action of A. In the case of projective space, more precisely the case of the sphere, the radial-angular decomposition of stochastic flows has been used for example in Khasminskii [23], Arnold, Oeljeklaus, Pardoux [5], and Arnold [1]. We view our decomposition as a generalization of the radial-angular one to higher dimensions.

To see how flows in $Gl(d; \mathbf{R})$ acting on M are transported via the left decomposition on the generator level, let us next see how vector fields are transformed in the angular and radial component.

Lemma 1.2 *Let $A \in Gl(d; \mathbf{R}), p \in M$, and let $(A_t)_{t \in I}$ be a smooth curve through $A_0 = A$ with tangent vector $\dot{A}$. Then the left decomposition of $((A_t, p))_{t \in I}$ induces smooth curves $(q_t)_{t \in I}$ on M, and $(B_t)_{t \in I}$ on $Gl(d; \mathbf{R})$ through $q = q_0$ resp $B = B_0$. For the tangent vectors $\dot{q}$ and $\dot{B}$ at 0 we have*

$$\dot{q} = (I - q)\, \dot{A} A^{-1}\, q + q\, (\dot{A} A^{-1})^*\, (I - q), \tag{4}$$

$$\dot{B} B^{-1} = (I - q)\, \dot{A} A^{-1}\, q - q\, (\dot{A} A^{-1})^*\, (I - q) + q\, \dot{A} A^{-1}\, q + (I - q)\, \dot{A} A^{-1}\, (I - q). \tag{5}$$

Proof:

Since $Gl(d; \mathbf{R})$ is a Lie group, we may assume $A_0 = B_0 = I, q_0 = p$. To get $\dot{q}$, note first that $q^2 = q, q^* = q$ implies

$$\dot{q} = (I - p)\, \dot{q}\, p + p\, \dot{q}\, (I - p).$$

Now use (2) to see $\dot{q} + p\, \dot{B} = \dot{A}\, p$. This equation immediately implies $(I - p)\, \dot{q}\, p = (I - p)\, \dot{A}\, p$. Hence we obtain (4). For (5), use the equation

$$B = q\, A\, p + (I - q)\, A\, (I - p),$$

as well as (4), to get

$$\begin{aligned} \dot{B} &= \dot{q}\, [p - (I - p)] + p\, \dot{A}\, p + (I - p)\, \dot{A}\, (I - p) \\ &= (I - p)\, \dot{A}\, p - p\, \dot{A}^*\, (I - p) + p\, \dot{A}\, p + (I - p)\, \dot{A}\, (I - p). \end{aligned}$$

This completes the proof. □

Lemma 1.2 indicates that the left decomposition shares another feature of the radial-angular decomposition on the sphere: differential equations determining the angular part and the radial part "decouple", i.e. the equation for the angular part does not contain the radial one. Since we deal with Stratonovich equations, lemma 1.2 implies immediately that given $p \in M$, the left decomposition (P_t, R_t) of (Φ_t, p) is generated by the pair of stochastic differential equations

$$dP_t = h_{A_0}(P_t)\, dt + \sum_{i=1}^{m} h_{A_i}(P_t) \circ dW_t^i, \quad P_0 = p, \tag{6}$$

$$dR_t = g_{A_0}(P_t)\, R_t dt + \sum_{i=1}^{m} g_{A_i}(P_t)\, R_t \circ dW_t^i, \quad R_0 = I, \tag{7}$$

with the vector fields

$$h_A(p) = (I - p)\, A\, p + p\, A^* \,(I - p), \tag{8}$$

$$g_A(p) = (I - p)\, A\, p - p\, A^* \,(I - p) + p\, A\, p + (I - p)\, A\, (I - p), \tag{9}$$

$p \in M$. To see this, one just has to note that (1) can be written

$$d\Phi_t\, \Phi_t^{-1} = A_0\, dt + \sum_{i=1}^{m} A_i \circ dW_t^i.$$

Moreover, the uniqueness of the left decomposition implies immediately that the processes $(P_t)_{t\in\mathbf{R}}$ induce a flow of diffeomorphisms on M and even a cocycle in the sense of Arnold [1], which we shall henceforth denote by $(P(t, .)\cdot)_{t\in\mathbf{R}}$, defined by $P(t, \omega)\, p = P_t(\omega)$, if $(P_t)_{t\in\mathbf{R}}$ is the angular part of $((\Phi_t,\, p) : t \in \mathbf{R})$.

Using the dynamical characterization of the projectors of the forward flag associated with our linear system given by the theorem of Oseledets, one can easily see that these projectors are the stationary states of the flow $(P(t, .))_{t\in\mathbf{R}}$ on the corresponding Grassmannian. So suppose from now on that we fix an orthogonal projector Q in the forward flag, omitting its index i, which takes its values in $M = G_k(d)$.

Theorem 1.1 *For $t \in \mathbf{R}$, let $(\hat{P}_t, \hat{R}_t)$ be the left decomposition of (Φ_t, Q). Then we have $\hat{P}_t = Q \circ \theta_t$.*

Proof:
By Arnold [1], p. 137, for $t \in \mathbf{R}, \omega \in \Omega$ outside a set of measure 0 we have

$$\Phi_t(\omega)\, Q(\omega)\, (\mathbf{R}^d) = Q(\theta_t \omega)\, (\mathbf{R}^d).$$

By definition $\Phi_t\, Q = \hat{P}_t\, \hat{R}_t$, and since $\hat{R}_t$ is nonsingular, $\hat{P}_t(\mathbf{R}^d) = Q \circ \theta_t\, (\mathbf{R}^d)$, so that finally the projector property gives the desired $\hat{P}_t = Q \circ \theta_t$. □

The "integrated version" of theorem 1.1 states essentially that the law of Q is a Markovian invariant measure and can be obtained as the unique solution of a Fokker-Planck equation on M. Thus its smoothness properties follow readily from Hörmander's theorem under well known hypoellipticity conditions for the vector fields associated with the angular component of our linear flow, as described in lemma 1.2 above. To formulate these conditions, let $\mathcal{L}^k$ be the Lie algebra generated by the vector fields $h_{A_0}, \cdots, h_{A_m}$, $\mathcal{I}^k$ the ideal in $\mathcal{L}^k$ generated by the vector fields $h_{A_1}, \cdots, h_{A_m}$, i.e.

$$\begin{aligned} \mathcal{I}^k &= \operatorname{span}\{h_{A_1}, ..., h_{A_m}, [h_{A_{i_p}}, [h_{A_{i_{p-1}}}, [...[h_{A_{i_1}}, h_{A_{i_0}}]...] : \\ & \quad 0 \leq i_j \leq m, 1 \leq j \leq p, p \in \mathbf{N}\}, \end{aligned}$$

The following conditions are then crucial for smoothness of the law

$$\text{(SH)} \qquad \mathcal{L}_p^k = T_p M, \quad p \in M,$$

$$\text{(WH)} \qquad \mathcal{I}_p^k = T_p M, \quad p \in M.$$

We note first that except in trivial cases, (SH) and (WH) are equivalent due to properties of our manifolds.

Theorem 1.2 *Except in case $d = 2$ and $h_{A_1} = \cdots = h_{A_m} = 0$, (SH) and (WH) are equivalent.*

Proof:
In case $d > 2$ our Grassmannians M have compact universal covering spaces $G_k^+(d)$, the oriented Grassmannian of k-dimensional linear spaces in $\mathbf{R}^d$. Hence due to a result of Sussmann, Jurdjevic [41], theorem 4.9 and corollaries 4.6 and 4.7, (SH) and (WH) are equivalent. In case $d = 2$, and if at least one of $h_{A_1}, ..., h_{A_m}$ does not vanish identically, the conditions are equally equivalent. Since we are in the analytic situation then (see Ichihara, Kunita [18], supplements and theorem 2^*, and Arnold, Oeljeklaus, Pardoux [5], p.138), if for example $h_{A_i}(p) \neq 0$ for some $p \in M$, (WH) is satisfied everywhere on M and implies (SH). In case $d = 1$, the two conditions are trivially satisfied. Hence except in case $d = 2$ and $h_{A_1} = ... = h_{A_m} = 0$, (SH) and (WH) are equivalent. □

Remark: Let us briefly consider the case excluded above and show that it is indeed trivial for our purposes. If $d = 2$ and h_{A_i} vanishes identically for $1 \leq i \leq m$, A_0 has a pair of complex conjugate eigenvalues, and thus $h_{A_0} \neq 0$ everywhere. In this case the Lyapunov exponents vanish and there is just one trivial Oseledets space. Hence all our analysis goes trivially through as well.

We may therefore in the sequel suppose throughout that (SH) and (WH) are equivalent, and denote this condition simply by (H).

By analytical arguments, (H) is seen to imply the smoothness of the law of Q. We shall briefly recall this well known fact together with some arguments. Below in section 2 we shall also give a probabilistic proof using Malliavin's calculus. The main ingredients of the probabilistic proof are needed when we tackle the smoothness of the conditional laws in section 3.

Theorem 1.3 *Under (H) the law of Q possesses a C^∞-density with respect to Riemannian volume on M.*

Proof:
According to San Martin [35], theorem 3.1., if the subgroup G of $Gl(d;\mathbf{R})$ generated by $A_0,\cdots,A_m$ acts transitively on M, there is a unique invariant control set on M. But under (H), G acts transitively. Hence the Fokker-Planck equation on M, given by $L_k^*\nu_k = 0$, induced by the infinitesimal generator L_k given by $L_k = h_{A_0} + \frac{1}{2}\sum_{j=1}^m h_{A_j}^2$, has a unique solution, which, still by Hörmander's theorem, is smooth with respect to Riemannian volume. But this solution is our law of Q: Crauel [14], Le Jan [26] imply that the solutions of the Fokker-Planck equation are in one-to-one correspondence with the random invariant measures μ_k induced by the cocycle $(\Phi_t)_{t\in\mathbf{R}}$ on M; yet, according to theorem 1.1, is just $\mu_k = \delta_Q$, the point mass in Q. Hence the law of Q is smooth. □

Remark: In the same way, we could get smoothness results for the whole forward (and backward) flag of orthogonal projectors associated with our linear system. This would just mean to require (H) for all dimensions appearing in the flags. Then, another purely analytical reasoning based upon Federer's co-area formula, which is made precise in section 4, leads to the joint smoothness of the Oseledets spaces. We shall formulate corresponding results in the subsequent sections for the conditional laws which, of course, include the smoothness properties for the laws, and prove them using Malliavin's calculus. One might think that the left decomposition $(Q\circ\theta_t,\hat{R}_t)$ of (Φ_t,Q) and independence properties of the Wiener process provide a direct analytical argument for obtaining the smoothness of conditional laws from the one of the laws. But the radial component $\hat{R}_t$ is only $\mathbf{F}_0^\infty$-measurable, and the reasoning in section 3 will essentially consist in controlling its influence on the Malliavin bilinear form.

2 The H-derivative of the flow of projectors; a stochastic proof of smoothness of laws

Let as in the preceding section k be fixed, $1\le k\le d$, and consider a random orthogonal projector Q in the forward flag taking its values in $M = G_k(d)$. To prove smoothness of the law of Q with the tools of Malliavin's calculus, positivity criteria for the Malliavin bilinear form have to be established. H-derivatives of our random orthogonal projectors were studied in [3]. We start giving a simple description based upon the preceding section. To tackle positivity properties, the Malliavin bilinear form of Q and its eigenvalues are compared to corresponding quantities for the flow of diffeomorphisms related to the angular components of $(\Phi_t,p), p\in M$, as described above. It will turn out that all we have to do is refine the well known arguments needed for the solution of a sde on a compact manifold to obtain positivity *uniformly in* p on coordinate neighborhoods of M. For $p\in M$ denote again the left decomposition of (Φ_t,p) by (P_t,R_t), $t\in\mathbf{R}$.

We begin by calculating the H-derivatives. Recall the definition of the vector fields h_A, g_A in (8), (9).

Proposition 2.1 *For* $t\in\mathbf{R}$, *we have* $\Phi_t, P_t, R_t\in\mathbf{D}_\infty$, *and for* $1\le j\le m$ *and* $0\le r\le t$

$$(D_r^j\,\Phi_t)\,\Phi_t^{-1} \;=\; A_j^{r,t},$$

$$\begin{aligned} D^j_r\, P_t &= h_{A^{r,t}_j}(P_t), \\ (D^j_r\, R_t)\, R^{-1}_t &= g_{A^{r,t}_j}(P_t), \end{aligned}$$

where $A^{r,t} = (\Phi_t\, \Phi^{-1}_r)\, A\, (\Phi_t\, \Phi^{-1}_r)^{-1}$ *for* $A \in \mathbf{R}^{d\times d}$. *Moreover,* $Q \in \mathbf{D}_\infty$, *and for* $1 \le j \le m, r \ge 0$ *and with* $A^r = A^{r,0}, A \in \mathbf{R}^{d\times d}$, *we have*

$$D^j_r\, Q = -h_{A^r_j}(Q).$$

Proof:
$\Phi_t \in \mathbf{D}_\infty$ and the first formula are well known. Recall that the H-derivative can be seen as a Frechet derivative, use the formula for the H-derivative of Φ_t and lemma 1.2 to obtain that $P_t, R_t \in \mathbf{D}_\infty$ and the asserted formulas for their derivatives. Let us argue for the last assertion. Note that Crauel [14] and Le Jan [26] give

$$Q = \lim_{t\to\infty} P_{-t} \circ \theta_t,$$

and that the cocycle property of the linear flow yields

$$\Phi_{-t} \circ \theta_t = \Phi^{-1}_t.$$

Hence $(P_{-t} \circ \theta_t, R_{-t} \circ \theta_t)$ is the left decomposition of (Φ^{-1}_t, p). Moreover, for $1 \le j \le d, 0 \le r \le t$, we have

$$[D^j_r\, \Phi^{-1}_t]\, \Phi_t = -\Phi^{-1}_t\, [D^j_r\, \Phi_t] = -A^r_j,$$

and hence from lemma 1.2 it follows

$$D^j_r\, [P_{-t} \circ \theta_t] = -h_{A^r_j}(P_{-t} \circ \theta_t). \tag{10}$$

(10) implies in particular that the H-derivative of $P_{-t} \circ \theta_t$ is bounded in the $\mathbf{D}_{1,2}$-norm for $t \in \mathbf{R}^+$. Hence the closedness of the H-derivative implies that $Q \in \mathbf{D}_\infty$ and the formula

$$D^j_r\, Q = -h_{A^r_j}(Q).$$

This completes the proof. □

Conditions for the smoothness of the law of Q are formulated in terms of the following bilinear forms on the cotangent bundle of M. For $S, T \in [0, \infty], S \le T, p \in M$ let as before $(P_t)_{t\in\mathbf{R}}$ be the angular component of $(\Phi_t\, p)_{t\in\mathbf{R}}$ and

$$\begin{aligned} \langle\langle DP_t, DP_t\rangle\rangle^T_S \;&: T^*_{P_t}M \times T^*_{P_t}M \;\to \mathbf{R}, \\ (u, v) \;&\mapsto \sum_{j=1}^m \int_S^T \langle D^j_r P_t, u\rangle \langle D^j_r P_t, v\rangle\, dr, \end{aligned}$$

and define $\langle\langle DQ, DQ\rangle\rangle^T_S$ analogously on $T^*_Q M \times T^*_Q M$. We recall that in these formulas $\langle ., .\rangle$ is just the usual scalar product in $\mathbf{R}^{d\times d}$. Another important quantity to be used in the sequel will be the smallest eigenvalue of the positive definite quadratic forms just defined. We denote it by $\Lambda(\langle\langle DP_t, DP_t\rangle\rangle_S^T)$ or $\Lambda(\langle\langle DQ, DQ\rangle\rangle_S^T)$. For example, we have the following relationship

$$\Lambda(\langle\langle DQ, DQ\rangle\rangle_S^T) = \inf_{v\in T_Q^*M, \|v\|=1} \int_S^T \sum_{j=1}^m \langle D_s^j Q, v\rangle^2 \, ds.$$

Our main argument will rest upon a comparison of the laws of $\det\langle\langle DQ, DQ\rangle\rangle_0^t$ and $\det\langle\langle DP_t, DP_t\rangle\rangle_0^t$. For this purpose, let us next consider more carefully the invariance properties of Q. This will provide an easy access to both positivity and integrability properties of its Malliavin bilinear form. We shall write $(P(t,.)\, p)_{t\in\mathbf{R}}$ for the projector p moved by the flow at time t. We obtain the following invariance property.

Proposition 2.2 *Let $t\in\mathbf{R}$, ρ be the law of Q. Then the law of $\det\langle\langle DQ, DQ\rangle\rangle_0^t$ under $\mathbf{P}$ is identical to the law of $\det\langle\langle DP(t,.)\cdot, DP(t,.)\cdot\rangle\rangle_0^t$ under $\mathbf{P}\otimes\rho$. A similar statement holds true for the smallest eigenvalues $\Lambda(\langle\langle DQ, DQ\rangle\rangle_0^t)$ and $\Lambda(\langle\langle DP(t,.)\cdot, DP(t,.)\cdot\rangle\rangle_0^t)$ of the Malliavin bilinear forms.*

Proof:
Fix $t\in\mathbf{R}_+$. First of all, theorem 1.1 yields

$$P(-t,\theta_t\cdot)\, Q\circ\theta_t = Q.$$

We consider the $\mathbf{P}$-law of the family of random variables $(h_{A_j^r}(Q) : 0\le r\le t)$. Since $Q\circ\theta_t$ is $\mathbf{F}_t^\infty$-measurable, $P(-t,\theta_t\cdot)$ $\mathbf{F}_0^t$-measurable as well as $(A_j^r : 0\le r\le t)$, we see that the $\mathbf{P}$-law of $(h_{A_j^r}(Q) : 0\le r\le t)$ equals the $\mathbf{P}\otimes\rho$-law of $(h_{A_j^r}(P(-t,\theta_t\cdot)\cdot\circ\theta_t) : 0\le r\le t)$. By symmetry of the law of the two-sided Wiener process with respect to 0, and shift invariance of $\mathbf{P}$, this again equals the law of $(h_{A_j^{-r}\circ\theta_t}(P(t,\cdot)\cdot) : 0\le r\le t)$. It remains to remark that $\Phi_{-r}\circ\theta_t = \Phi_{t-r}\circ\Phi_t^{-1}$, so that

$$A_j^{-r}\circ\theta_t = \Phi_t\,\Phi_{t-r}^{-1}\, A_j\,\Phi_{t-r}\,\Phi_t^{-1} = A_j^{t-r,t},$$

to see that this is the law of $(h_{A_j^{t-r,t}}(P(t,\cdot)\cdot) : 0\le r\le t)$. This evidently implies the equality of laws claimed. □

The main problem for obtaining smoothness of the density of the law of Q under Hörmander's condition on M, is the proof of the existence of all moments of the inverse of $\det\langle\langle DQ, DQ\rangle\rangle_0^t$ for some $t>0$, or, more strictly, of the inverse of $\Lambda(\langle\langle DQ, DQ\rangle\rangle_0^t)$. As proposition 2.2 shows, this boils down to the proof of the existence of all moments of the inverse of $\det\langle\langle DP(t,.)\cdot, DP(t,.)\cdot\rangle\rangle_0^t$ or of $\Lambda(\langle\langle DP(t,.)\cdot, DP(t,.)\cdot\rangle\rangle_0^t)$ for some $t>0$ with respect to the probability measure $\mathbf{P}\otimes\rho$. In fact, we shall use local coordinates on M to prove such an integrability statement. Fix for the rest of the section a family $(U_i)_{1\le i\le l}$ of coordinate neighborhoods covering M. In these terms, it is enough to prove

$$\sup_{p\in M} E(1_{\{P(t,.)\,p\in U_i\}}\,(\det\langle\langle DP(t,.)\,p, DP(t,.)\,p\rangle\rangle_0^t)^{-q}) < \infty \tag{11}$$

for any $q\ge 1, 1\le i\le l$. To prove (11), choose local coordinates $\phi_i : U_i\to\mathbf{R}^r$ (here actually $r=k(d-k)$). Since M is a submanifold of $\mathbf{R}^{d\times d}$, we may assume that ϕ_i is the restriction of some smooth mapping $\hat{\phi}_i : \mathbf{R}^{d\times d} \to \mathbf{R}^{d\times d}$ which is a diffeomorphism of some open neighborhood of U_i in $\mathbf{R}^{d\times d}$, and has compact support.

Let us now fix $i, 1 \le i \le l$, and omit it as a subscript. We transfer the vector fields h_A on M to $\mathbf{R}^r$ by putting

$$\phi\, h_A(y) = \phi_*\, h_A(\phi^{-1}(y)), \quad y \in \phi(U), A \in \mathbf{R}^{d\times d},$$

and set

$$\begin{aligned}\phi\,\mathcal{L}_y \;&=\; \text{span}\,\{\phi h_{A_1}(y), ..., \phi h_{A_m}(y), [\phi h_{A_{i_p}}, [\phi h_{A_{i_{p-1}}}, [...[\phi h_{A_{i_1}}, \phi h_{A_{i_0}}]...](y) :\\ &\quad\; 0 \le i_j \le m, 1 \le j \le p, p \in \mathbf{N}\}.\end{aligned}$$

Then, since ϕ_* is pointwise a homomorphism with respect to the Lie algebra structure, we obtain

$$\phi\,\mathcal{L}_y = \phi_*\,(\mathcal{L}_{\phi^{-1}(y)}), \quad y \in \phi(U). \tag{12}$$

Assuming that Hörmander's condition holds for M is therefore translated into the condition

$$\phi\,\mathcal{L}_y = \mathbf{R}^r, \quad y \in \phi(U). \tag{13}$$

In these terms we may consider the following stochastic differential equations in Euclidean space $\mathbf{R}^r$

$$\begin{aligned} dX_t \;&=\; \phi h_{A_0}(X_t)\, dt + \sum_{i=1}^m \phi h_{A_i}(X_t) \circ dW_t^i, \qquad (14)\\ X_0 \;&=\; x \in \mathbf{R}^r, \end{aligned}$$

$$\begin{aligned} dZ_t \;&=\; D(\phi h_{A_0})(X_t)\, Z_t\, dt + \sum_{i=1}^m D(\phi h_{A_i})(X_t)\, Z_t \circ dW_t^i, \qquad (15)\\ Z_0 \;&=\; I, \end{aligned}$$

where I is the $r \times r$ unit matrix. By smoothness and boundedness of the vector fields involved, (14) and (15) generate cocycles of diffeomorphisms $(X(t,.))_{t\in\mathbf{R}}$ of $\mathbf{R}^r$ and $(Z(t,.))_{t\in\mathbf{R}}$ of $\mathbf{R}^{r\times r}$ the values of which at $x \in \mathbf{R}^r$ shall be written $X(t,.)\,x$ etc., as usually. To obtain a coordinate version of (11), we shall have to extend the results of Taniguchi [42], pp. 276-283, slightly. To begin with, the Malliavin bilinear form in coordinates $\langle\langle DX(t,.)\,x, DX(t,.)\,x\rangle\rangle_S^T$ for $0 \le S \le T \le \infty$ now becomes a bilinear form on $\mathbf{R}^r \times \mathbf{R}^r$, the object usually called Malliavin's matrix (see for example Nualart [30]). It is well known (see for example Ikeda, Watanabe [19], p. 340), that for $s \ge 0, 1 \le j \le m, x \in \mathbf{R}^r$

$$D_s^j\, X(t,.)\, x = Z(t,.)\, x\;\, Z(s,.)^{-1}\, x\;\, \phi h_{A_j}\,(X(t,.)\, x).$$

But the cocycle property (on the skew product) yields the equations

$$Z(-s, \theta_t\cdot)^{-1}\, X(t,.)\, x = Z(t,.)\, x\;\, Z(s,.)^{-1}\, x,$$

and

$$X(-s,\theta_t\cdot)^{-1}\,X(t,.)\,x = X(t-s,.)\,x,$$

$0\le s\le t, x\in \mathbf{R}^r$. Now abbreviate $B_j=\phi h_{A_j}, 1\le j\le m$. Then by virtue of the above equations, if in accordance with Taniguchi [42] we set

$$a(s,x,.)(v)=\sum_{i=1}^m \langle v, Z(-s,\theta_t\cdot)^{-1}\,x\,B_j(X(-s,\theta_t\cdot)^{-1}\,x)\rangle^2,$$

and

$$\tau(x)=\inf\{s\ge 0: X(-s,\theta_t\cdot)^{-1}\,x\notin\phi(U)\}\wedge t,$$

$0\le s\le t, x\in\mathbf{R}^r, v\in S^{r-1}$, we obtain

$$\begin{aligned}&\langle\langle DX(t,.)\,x, DX(t,.)\,x\rangle\rangle_0^t \\ &\quad=\sum_{j=1}^m\int_0^t \langle Z(-s,\theta_t\cdot)^{-1}\,X(t,.)\,x\;\; B_j\,(X(-s,\theta_t\cdot)^{-1}\,x),.\rangle \\ &\qquad\langle Z(-s,\theta_t\cdot)^{-1}\,X(t,.)\,x\;\; B_j\,(X(-s,\theta_t\cdot)^{-1}\,x),.\rangle\,ds,\end{aligned}\tag{16}$$

considered as a bilinear form on $\mathbf{R}^r\times\mathbf{R}^r$, and consequently

$$\begin{aligned}\det\langle\langle DX(t,.)\,x, DX(t,.)\,x\rangle\rangle_0^t &\ge \Lambda(\langle\langle DX(t,.)\,x, DX(t,.)\,x\rangle\rangle_0^t) \\ &\ge [\inf_{v\in S^{r-1}}\int_0^{\tau(X(t,.)\,x)} a(s,X(t,.)\,x,.)(v)\,ds]^r.\end{aligned}\tag{17}$$

Now the essential step in Taniguchi's [42] arguments generalizes in the following way.

Proposition 2.3 *For any $q\ge 1, t>0$ we have*

$$\sup_{x\in\mathbf{R}^r} E(1_{\{X(t,.)\,x\in\phi(U)\}}\,\Lambda(\langle\langle DX(t,.)\,x, DX(t,.)\,x\rangle\rangle_0^t)^{-q})<\infty.\tag{18}$$

Proof:

If we fix $t>0,\epsilon>0$ and set

$$\sigma(x)=\inf\{s\ge 0:\mid X(s,.)\,x-x\mid>\epsilon\}\wedge t,$$

lemmas 3.3 and 3.4 of Taniguchi [42] yield

$$\begin{aligned}&\sup_{x\in\mathbf{R}^r}\mathbf{P}(\int_0^{\tau(X(t,.)\,x)} a(s,X(t,.)\,x,.)(v)\,ds\le k^{-m_0-6},\; X(t,.)\,x\in\phi(U)) \\ &\quad\le \mathbf{P}(\inf_{x\in\phi(U)}\int_0^{\frac{1}{k}} a(s,x,.)(v)\,ds\le k^{-m_0-6})+\mathbf{P}(\inf_{x\in\phi(U)}\sigma(x)>\frac{1}{k}) \\ &\quad= O(k^{-q})\end{aligned}\tag{19}$$

for $q\ge 1$ as $k\to\infty$, uniformly in $v\in S^{r-1}$. Here $m_0\in\mathbf{N}$ is chosen according to Taniguchi [42], lemma 3.2, such that

$$\mathbf{P}(\int_0^{\frac{1}{k}} a(s,x,.)(v)\,ds\le k^{-m_0})=O(k^{-q})$$

for any $q \geq 1, v \in S^{r-1}, x \in \phi(U)$, as $k \to \infty$. Due to the boundedness of the vector fields involved and the compactness of M, the arguments given in Taniguchi [42], pp. 282, 283, do equally not depend on $x \in \phi(U)$. Therefore they apply to sharpen (19) to

$$\sup_{x \in \mathbf{R}^r} \mathbf{P}(\inf_{v \in S^{r-1}} \int_0^{\tau(X(t,.)\,x)} a(s, X(t,.)\,x, .)(v)\, ds \leq k^{-m_0-7},\, X(t,.)\,x \in \phi(U)) = O(k^{-q}), \tag{20}$$

for all $q \geq 1$, as $k \to \infty$. We emphasize that to be able to apply the reasoning of Taniguchi [42], we essentially had to apply Hörmander's condition in the coordinate form of (13). Now (20) implies that for $q \geq 1$

$$\sup_{x \in \mathbf{R}^r} E([\inf_{v \in S^{r-1}} \int_0^{\tau(X(t,.)\,x)} a(s, X(t,.)\,x, .)(v)\, ds]^{-q}\, 1_{\{X(t,.)\,x \in \phi(U)\}}) < \infty. \tag{21}$$

In view of (21), it just remains to recall (17) to get the desired result. □

With the preceding result we finally obtain integrability of the inverse Malliavin bilinear form.

Proposition 2.4 *Suppose that (H) is satisfied for M. Then for any $p \geq 1, t > 0$ we have*

$$(\Lambda(\langle\langle DQ, DQ \rangle\rangle_0^t))^{-1} \in L^p(\Omega, \mathbf{F}, \mathbf{P}),$$

and in particular

$$(\det \langle\langle DQ, DQ \rangle\rangle_0^t)^{-1} \in L^p(\Omega, \mathbf{F}, \mathbf{P}).$$

Proof:
Returning the results of proposition 2.3 to M (see Taniguchi [42], pp. 277, 278), we arrive at (11). Combine this with proposition 2.2 to obtain the asserted integrability. □

Due to proposition 2.4, we are now in a position to complete the stochastic proof of smoothness of the law of Q. For the sequel, we shall only need the result of proposition 2.4.

Theorem 2.1 *Suppose that Q is a random orthogonal projector on a linear space in the forward flag of the linear cocycle generated by the stochastic differential equation (1) which takes its values in $M = G_k(d)$ for some $1 \leq k \leq d$. Assume that (H) is satisfied for M. Then the law of Q possesses a C^∞-density.*

Proof:
Modulo a standard passage to local coordinates, according to Nualart [30], pp. 83, 91-93, it is enough to show that there exists $t > 0$ such that

$$Q \in \mathbf{D}_{p,k}([0,t]) \quad \text{for} \quad p \geq 1, k \in \mathbf{N}, \tag{22}$$

$$(\det \langle\langle DQ, DQ \rangle\rangle_0^t)^{-1} \in L^p(\Omega, \mathbf{F}, \mathbf{P}) \quad \text{for} \quad p \geq 1. \tag{23}$$

Indeed, in (24) on p. 83 of Nualart [30] one may choose $DF^i\, 1_{[0,t]}$ instead of DF^i, $1 \leq i \leq m$, set

$$\gamma_F^t = (\int_0^t D_s F^i\, D_s F^j\, ds)_{1\leq i,j\leq m},$$

to obtain

$$\partial_i\, \phi(F) = \sum_{j=1}^m \int_0^t D_s\phi(F)\, D_s F^j\, ds\; (\gamma_F^t)^{-1}_{ji},$$

in the terms used there. The duality relation between the H-derivative and Skorokhod's integral has then equally to be applied to the functions $DF^i\, 1_{[0,t]}, 1 \leq i \leq m$, instead, and a similar reasoning for higher derivatives.

Now, (22) follows from proposition 2.1, whereas (23) is a consequence of proposition 2.4. □

We finally wish to obtain smoothness of the law for the entire forward (or backward) flag of orthogonal projectors of our system. For this purpose recall from above that our forward flag $(Q_r, \cdots, Q_1)$ takes its values in a flag manifold $F_{k_r,\cdots,k_1}(d)$ where $k_i = d_i + \cdots + d_r, 1 \leq i \leq r$. We may realize the flag manifold in the same way as the Grassmannians by

$$F_{k_r,\ldots,k_1}(d) = \{(o\, p_{k_r}\, o^*,\; o\, p_{k_{r-1}}\, o^*,\; \ldots, I) :\; o \in \mathbf{O}(d)\}.$$

We recall that p_k denotes the orthogonal projector identifying the first k vectors of a canonical basis of $\mathbf{R}^d$ and annihilating the remaining ones. The Hörmander conditions corresponding to the flag manifold are the following

$$(SHF) \qquad T_{(p_r,\ldots,p_1)}\, F_{k_r,\ldots,k_1}(d) = \mathcal{L}_{p_r}^{k_r} \times \ldots \times \mathcal{L}_{p_1}^{k_1} \quad \text{for any} \quad (p_r, \ldots, p_1) \in F_{k_r,\ldots,k_1}(d),$$

$$(WHF) \qquad T_{(p_r,\ldots,p_1)}\, F_{k_r,\ldots,k_1}(d) = \mathcal{I}_{p_r}^{k_r} \times \ldots \times \mathcal{I}_{p_1}^{k_1} \quad \text{for any} \quad (p_r, \ldots, p_1) \in F_{k_r,\ldots,k_1}(d).$$

By (HF) we denote the condition (WHF) or equivalently (SHF), excluding from the beginning a trivial case in which our reasoning is trivially true (see the remarks in section 1). Here we put indices k on the vector fields h^k in order to indicate in which Grassmannians they operate.

Theorem 2.2 *Suppose that (HF) is satisfied for the flag manifold $F_{k_r,\ldots,k_1}(d)$. Then the law of $(Q_r, \ldots, Q_1)$ possesses a C^∞- density with respect to the Riemannian volume on $F_{k_r,\ldots,k_1}(d)$.*

Proof:

We just have to note that according to proposition 2.4 we have

$$\prod_{i=1}^r (\det\langle\langle DQ, DQ\rangle\rangle_0^t)^{-1} \in L^p(\Omega, \mathbf{F}, \mathbf{P})$$

for any $p \geq 1$. This implies the desired result, given the manifold structure, according to Imkeller [21]. □

3 Smoothness of conditional laws of orthogonal projectors in the forward flag

We shall now extend the treatment the laws of orthogonal projectors were given in the preceding section to conditional laws. We start with recalling the key observation made in section 1. For the following fix again k, $1 \le k \le d$, and let Q be a random orthogonal projector in the forward flag taking its values in $M = G_k(d)$. Then the left decomposition of (Φ_t, Q) was seen in theorem 1.1 to lead to the dual equations

$$Q \circ \theta_t \, \hat{R}_t = \hat{R}_t \, Q = \Phi_t \, Q, \tag{24}$$

$$\hat{R}_t \, (I - Q) = (I - Q) \circ \theta_t \, \hat{R}_t = (I - Q) \circ \theta_t \, \Phi_t. \tag{25}$$

Since for t fixed, $Q \circ \theta_t$ is independent of $\mathbf{F}_0^t$, and thus of $(\Phi_s : s \le t)$, (24) indicates that proving smoothness of the conditional law of Q given $\mathbf{F}_0^t$ essentially boils down to controlling the influence of $\hat{R}_t$, which, unfortunately, is not independent of $\mathbf{F}_0^t$. This will be done by controlling the eigenvalues of $\hat{R}_t \, \hat{R}_t^*$ and using the integrability of their moments. This way the Malliavin bilinear form on $[t, \infty[$ can be controlled by the one on $[0, \infty[$, shifted by θ_t, and thus the criterion for smoothness of the conditional laws deduced from the one for the laws discussed in section 2.

The following lemma will be crucial for estimating the influence of the radial component in the left decomposition.

Lemma 3.1 *For $t \in \mathbf{R}, A \in \mathbf{R}^{d \times d}$, let $v = h_A(Q), v_t = h_{\Phi_t \, A \, \Phi_t^{-1}}(Q \circ \theta_t)$. Moreover, let $\lambda_0(t)$ be the smallest, $\lambda_1(t)$ the biggest eigenvalue of $\hat{R}_t \, \hat{R}_t^*$, and denote $c_t = \frac{\lambda_0(t)}{\lambda_1(t)}$. Then c_t as well as c_t^{-1} possesses moments of all orders and we have*

$$c_t \, \langle v_t, v_t \rangle \le \langle v, v \rangle \le c_t^{-1} \, \langle v_t, v_t \rangle.$$

Proof:

Since the vector fields (9) in the stochastic differential equations describing $\hat{R}_t, \hat{R}_t^{-1}$ are smooth and bounded, both $\hat{R}_t$ and $\hat{R}_t^{-1}$ possess moments of all orders. Consequently, $\lambda_0(t)^{-1}$, the biggest eigenvalue of $(\hat{R}_t^*)^{-1} \, \hat{R}_t^{-1}$, and $\lambda_1(t)$, the biggest of $\hat{R}_t \, \hat{R}_t^*$, equally possess moments of all orders. Hence so do c_t and c_t^{-1}. Now note that lemma 1.1 and theorem 1.1 yield the following equation, where in the last line, for brevity, we omit the subscript t

$$\begin{aligned} \langle v, v \rangle &= 2 \, \mathrm{tr}((I - Q) \, A \, Q \, A^* \, (I - Q)) \qquad (26) \\ &= 2 \, \mathrm{tr}(\hat{R}^{-1} \, (I - Q) \circ \theta \, \Phi A \Phi^{-1} \, Q \circ \theta \, \hat{R} \hat{R}^* \, (\Phi A \Phi^{-1})^* \, (I - Q) \circ \theta \, (\hat{R}^*)^{-1}). \end{aligned}$$

Now use a diagonalization of $\hat{R}\hat{R}^*$, for example, and recall the definition of v_t to see from (26) that

$$\langle v, v \rangle \le \frac{\lambda_1(t)}{\lambda_0(t)} \, \langle v_t, v_t \rangle, \quad \text{and} \quad \langle v, v \rangle \ge \frac{\lambda_0(t)}{\lambda_1(t)} \, \langle v_t, v_t \rangle.$$

This completes the proof. □

By means of lemma 3.1 we can prove that the criterion of Malliavin's calculus for the smoothness of the conditional laws can be traced back to the one for smoothness of the laws.

Lemma 3.2 *For $T > t > 0$, let $\lambda_0(t)$ resp. $\lambda_1(t)$ be the smallest resp. biggest eigenvalues of $\hat{R}_t \hat{R}_t^*$, and $c_t = \frac{\lambda_0(t)}{\lambda_1(t)}$. Then*

$$\begin{aligned}\det\langle\langle DQ, DQ\rangle\rangle_t^T &\geq \Lambda(\langle\langle DQ, DQ\rangle\rangle_t^T) \\ &\geq c_t^4 \, \Lambda(\langle\langle DQ, DQ\rangle\rangle_0^{T-t}) \circ \theta_t.\end{aligned}$$

Proof:
The first inequality follows trivially from the definitions. To estimate $\Lambda(\langle\langle DQ, DQ\rangle\rangle_t^T)$, fix $1 \leq j \leq m, r \geq t, v \in T_Q^* M$ such that $||v|| = 1$. Let $B \in \mathbf{R}^{d\times d}$ such that $v = h_B(Q)$. Then an argument similar to the one given in the preceding proof yields

$$\begin{aligned}&\langle D_r^j Q, v\rangle^2 \qquad (27)\\ &\quad = 4\mathrm{tr}((I-Q)\, A_j^r \, Q \, B^* \, (I-Q))^2 \\ &\quad = 4\mathrm{tr}((\hat{R}^{-1}\,(I-Q)\circ\theta \, A_j^{r-t} \circ \theta \, Q \circ \theta \, \hat{R}\hat{R}^* \, Q \circ \theta \, (\Phi B^* \Phi^{-1})^* \, (I-Q)\circ\theta \, (\hat{R}^*)^{-1}) \\ &\quad \geq c_t^2 \, \langle D_{r-t}^j Q \circ \theta_t, v_t\rangle^2,\end{aligned}$$

where $v_t = h_{\Phi_t B \Phi_t^{-1}}(Q \circ \theta_t)$, and in line 3 the subscript t is omitted. Now (27) and lemma 3.1 yield

$$\Lambda(\langle\langle DQ, DQ\rangle\rangle_t^T) \geq c_t^4 \, \Lambda(\langle\langle DQ, DQ\rangle\rangle_0^{T-t}) \circ \theta_t,$$

which completes the proof. □

This allows us to formulate our main result.

Theorem 3.1 *Suppose that Q is a random orthogonal projector on a linear space in the forward flag of the linear system induced by the stochastic differential equation (1), which takes its values in M. Assume that M satisfies (H). Then for any $t \geq 0$ the conditional law of Q given $\mathbf{F}_0^t$ possesses a density which is C^∞ ($\mathbf{P}$-a.s.).*

Proof:
According to [21] we have to show that there is $T > t$ such that for all $p \geq 1$

$$(\det\langle\langle DQ, DQ\rangle\rangle_t^T)^{-1} \in L^p(\Omega, \mathbf{F}, \mathbf{P}). \qquad (28)$$

By lemma 3.2 and θ_t-invariance of $\mathbf{P}$ this reduces to a corresponding integrability condition for c_t^{-1} and $(\Lambda(\langle\langle DQ, DQ\rangle\rangle_0^{T-t}))^{-1}$. Lemma 3.1 is used to treat c_t^{-1}, while the inverse smallest eigenvalue is treated by proposition 2.4. This completes the proof. □

If we require (HF), we obtain smoothness for the conditional laws of the entire forward flag of orthogonal projectors.

Theorem 3.2 *Suppose that (HF) is satisfied for the flag manifold $F_{k_r,\ldots,k_1}(d)$. Then for any $t \geq 0$ the conditional law of $(Q_r, ..., Q_1)$ possesses ($\mathbf{P}$-a.s.) a C^∞- density with respect to the Riemannian volume on $F_{k_r,\ldots,k_1}(d)$.*

Proof:
Apply (28) simultaneously to all $Q_r, ..., Q_1$. □

As a consequence of theorem 3.2 we now obtain that the semimartingale property is preserved, if the Wiener filtration is enlarged by the entire forward flag. Moreover, a priori estimates for stochastic integrals with integrands only adapted for the larger filtration follow.

Theorem 3.3 *Suppose that (HF) is satisfied for the flag manifold $F_{k_r,\ldots,k_1}(d)$. Let for $t \geq 0$*

$$\mathcal{G}_0^t = \mathbf{F}_0^t \vee \sigma(Q_r, ..., Q_1).$$

Then any $\mathbf{F}_0^t$-semimartingale is a $\mathcal{G}_0^t$-semimartingale. Moreover, for any $T > t \geq 0, 1 < r < q$ there exist constants $c_{r,q,T}$ such that for any $\mathcal{G}_0^t$-adapted $\mathbf{P}$-a.s. locally square integrable process $u = (u^1, ..., u^m)$ we have

$$\| \sup_{0\leq s\leq t} | \sum_{j=1}^{m} \int_0^s u_v^i \, dW_v^i \, |\|_r \leq c_{r,q,T} \ \| [\int_0^t \ \| u_v \|^2 \, dv] \|_q \ .$$

Proof:
Apply theorem 3 of [21], which is possible due to theorem 3.2 and (28). □

Remark: The Euclidean form of the criterion of Malliavin's calculus for the absolute continuity of the conditional laws we have been using above is also known from *conditional Malliavin's calculus*. See Nualart, Zakai [32], or Bouleau, Hirsch [11].

4 The smoothness of the laws of the Oseledets spaces

Recall now that the families of projectors $(Q_r, \cdots, Q_1)$ of the forward and $(P_r, \cdots, P_1)$ of the backward flag of our linear system (1) are independent, and that the family $(R_r, \cdots, R_1)$ of orthogonal projectors on the Oseledets spaces are just given by intersecting corresponding spaces of the forward and backward flags. In this section, we shall show that the smoothness properties of the flags are inherited by the Oseledets spaces. For this purpose a purely analytical argument based upon the co-area formula of Federer [15], theorem 3.2.22., is sufficient.

For the first part we fix i, $1 \leq i \leq r$, and note that according to our conventions, Q_i takes its values in $G_{d_i+\cdots+d_r}(d)$, P_i in $G_{d_1+\cdots+d_i}(d)$, and R_i in $G_{d_i}(d)$. To abbreviate, denote $k = d_i + \cdots + d_r, l = d_1 + \cdots + d_i, c = d_i$. The dimensions of the Grassmannians we work with are denoted by $m = k\,(d-k), n = l\,(d-l), z = c(d-c)$. To use Federer's formula in appropriate terms, we shall employ Hausdorff measure

H^m, H^n, H^z on the respective Grassmannians instead of Riemannian volume. Up to a multiplicative constant, Riemannian volume is identical to Hausdorff measure (see Federer [15], p.171). We shall apply the co-area formula to the mapping which associates to $(q,p) \in G_k(d) \times G_l(d)$ the projector $s \in G_c(d)$ on the intersection of the linear spaces onto which q and p project. Since this intersection may be a linear space of dimension bigger than c on a set of $H^m \otimes H^n$-measure zero, we fix $s_0 \in G_c(d)$ and set

$$f : G_k(d) \times G_l(d) \to G_c(d), \quad (q,p) \mapsto s \quad \text{resp.} \quad s_0,$$

if the intersection of the spaces onto which q and p project is represented by the projector s, s_0 if it is more than c-dimensional. We shall make use of the fact that the Jacobian mapping Jf of f (see Federer [15], p. 258) is invariant under orthogonal transformations of $\mathbf{R}^d$. For $o \in \mathbf{O}(d)$, orthogonal transformations on $G_k(d)$ by o in our setting are given by $T_o^k : G_k(d) \to G_k(d), p \mapsto opo^*, \quad 1 \leq k \leq d$.

Lemma 4.1 *Let $o \in \mathbf{O}(d)$. For $H^m \otimes H^n$-a.e. $(q,p) \in G_k(d) \times G_l(d)$ we have*

$$Jf(T_o^k q, T_o^l p) = Jf(q,p).$$

Proof:

We shall use repeatedly the invariance of Hausdorff measure with respect to orthogonal transformations on Grassmannians which follows immediately from the definition. It relies upon the fact that diameters of sets in a Euclidean space are invariant for orthogonal transformations. Let $0 \leq g$ be a bounded function measurable with respect to the Borel sets of $G_k(d) \times G_l(d)$. Then the co-area formula yields the equation

$$\begin{aligned} &\int_{G_k(d)\times G_l(d)} g(q,p)\, Jf(q,p)\, H^m(dq)\, H^n(dp) \\ &\quad = \alpha \int_{G_c(d)} (\int_{f^{-1}(s)} g(q,p)\, H^{m+n-z}(d(q,p)))\, H^z(ds), \end{aligned} \tag{29}$$

where the constant α comes from the normalization of the Hausdorff measures. It will be present in all subsequent applications of the co-area formula. We write the same formula for the function $g \circ (T_{o^*}^k, T_{o^*}^l)$ and use invariance of H^{m+n-z} as well as H^z for orthogonal transformations on $G_k(d) \times G_l(d)$ resp. $G_c(d)$ to obtain the following chain of equalities

$$\begin{aligned} &\int_{G_k(d)\times G_l(d)} g(T_{o^*}^k q, T_{o^*}^l p)\, Jf(q,p)\, H^m(dq)\, H^n(dp) \\ &\quad = \alpha \int_{G_c(d)} (\int_{f^{-1}(s)} g(T_{o^*}^k q, T_{o^*}^l p)\, H^{m+n-z}(d(q,p)))\, H^z(ds) \\ &\quad = \alpha \int_{G_c(d)} (\int_{f^{-1}(T_o^1 s)} g(q,p)\, H^{m+n-z}(d(q,p)))\, H^z(ds) \\ &\quad = \alpha \int_{G_c(d)} (\int_{f^{-1}(s)} g(q,p)\, H^{m+n-z}(d(q,p)))\, H^z(ds). \end{aligned} \tag{30}$$

We next use the invariance of $H^m \otimes H^n$ for orthogonal transformations on $G_k(d) \times G_l(d)$ to deduce from (29) and (30)

$$\begin{aligned} &\int_{G_k(d)\times G_l(d)} g(q,p)\, Jf(T_o^k q, T_o^l p)\, H^m(dq)\, H^n(dp) \\ &\quad = \int_{G_k(d)\times G_l(d)} g(q,p)\, Jf(q,p)\, H^m(dq)\, H^n(dp). \end{aligned} \tag{31}$$

(31) evidently implies the desired equation. □

We are now in a position to prove that smoothness properties of the laws of our random projectors in the forward and backward flags are inherited by the random projectors on the Oseledets spaces.

Theorem 4.1 *Suppose that the forward and backward Ruelle matrices of the linear system induced by the stochastic differential equation (1) are given by*

$$\Phi^+ = \lim_{t\to\infty} [\Phi_t^* \, \Phi_t]^{\frac{1}{2t}} = \sum_{i=1}^r e^{\lambda_i} \, S_i^+,$$

$$\Phi^- = \lim_{t\to-\infty} [\Phi_t^* \, \Phi_t]^{\frac{1}{2|t|}} = \sum_{i=1}^r e^{-\lambda_i} \, S_i^-,$$

with Lyapunov exponents $\lambda_1 > ... > \lambda_r$, *and orthogonal projectors in the forward resp. backward flag given by*

$$Q_i = \sum_{j\geq i} S_j^+, \quad P_i = \sum_{j\leq i} S_j^-,$$

$0 \leq i \leq r+1$ *(where the empty sum is trivially interpreted),* R_i *the orthogonal projectors on the intersections of the random subspaces onto which* Q_i *and* P_i *project,* $1 \leq i \leq r$. *Suppose further that* $1 \leq i \leq r$ *is given and (H) is satisfied with respect to the Grassmannians* $G_{d_1+\cdots+d_i}(d)$ *as well as* $G_{d_i+\cdots+d_r}(d)$. *Then the law of* R_i *possesses a* C^∞*-density with respect to Riemannian volume on* $G_{d_i}(d)$.

Proof:

Use the same notation as above, and omit the subscript i in Q_i, P_i, R_i for simplicity of notation. Let h_Q resp. h_P be densities of Q resp. P with respect to Riemannian volume on $G_k(d)$ resp. $G_l(d)$. Then Federer's co-area formula again yields the equation

$$\int_{G_k(d)\times G_l(d)} g(f(q,p))\, h_Q(q)\, h_P(p)\, H^m(dq)\, H^n(dp) \tag{32}$$
$$= \alpha \int_{G_c(d)} (\int_{f^{-1}(s)} Jf(q,p)^{-1}\, g(s)\, h_Q(q)\, h_P(p)\, H^{m+n-z}(d(q,p)))\, H^z(ds),$$

this time for nonnegative bounded functions g on $G_c(d)$ which are measurable with respect to the Borel sets of the Grassmannian. Equation (32) evidently qualifies the function

$$h_R(s) = \alpha \int_{f^{-1}(s)} Jf(q,p)^{-1}\, h_Q(q)\, h_P(p)\, H^{m+n-z}(d(q,p)), \tag{33}$$

$s \in G_c(d)$, as a density of R with respect to H^z on $G_c(d)$. Hence all that remains to be done is show that h_R is a C^∞- function on $G_c(d)$. To do this, we make use once again of the invariance of H^{m+n-z} for orthogonal transformations and lemma 4.1 to describe h_R in an alternative way. For this purpose, fix $s_0 \in G_c(d)$ and for $s \in G_c(d)$ let o_s be an orthogonal matrix which satisfies $o_s\, s = s_0$. Then, the invariance properties we referred to give

$$h_R(s) = \alpha \int_{f^{-1}(s_0)} Jf(q,p)^{-1}\, h_Q(T_{o_s}^k q)\, h_P(T_{o_s}^l p)\, H^{m+n-z}(d(q,p)), \tag{34}$$

$s \in G_c(d)$. Now note that $f^{-1}(s_0)$ is compact, and recall that due to our hypotheses h_Q and h_P are C^∞-functions on $G_k(d)$ resp. $G_l(d)$. Hence by a standard argument using dominated convergence the desired C^∞-property of h_R boils down to the smoothness of the mappings

$$s \mapsto T^k_{o_s}\, q, \quad q \in G_k(d), \quad 1 \le k \le r.$$

To get this we just have to make sure that the mapping $s \mapsto o_s$ can be chosen C^∞. But this is obvious, and the proof is complete. □

By using a procedure similar to the preceding proof we can finally extend the result of theorem 4.1 to the whole family of Oseledets spaces. As before, one just has to make sure that Hörmander's condition is satisfied on the flag manifold instead of the Grassmannian.

Theorem 4.2 *Suppose that the forward and backward Ruelle matrices of the linear system induced by the stochastic differential equation (1) are given by*

$$\Phi^+ = \lim_{t\to\infty} [\Phi_t^* \, \Phi_t]^{\frac{1}{2t}} = \sum_{i=1}^r e^{\lambda_i}\, S_i^+,$$

$$\Phi^- = \lim_{t\to-\infty} [\Phi_t^* \, \Phi_t]^{\frac{1}{2|t|}} = \sum_{i=1}^r e^{-\lambda_i}\, S_i^-,$$

with Lyapunov exponents $\lambda_1 > ... > \lambda_r$, and orthogonal projectors in the forward resp. backward flag given by

$$Q_i = \sum_{j\ge i} S_j^+, \quad P_i = \sum_{j \le i} S_j^-,$$

$0 \le i \le r+1$ *(where the empty sum is trivially interpreted), R_i the orthogonal projectors on the intersections of the random subspaces onto which Q_i and P_i project, $1 \le i \le r$. Suppose further that (HF) is satisfied for the flag manifold $F_{k_r,\cdots,k_1}(d)$ and $F_{l_r,\cdots,l_1}(d)$, where $k_i = d_i + \cdots + d_r, l_i = d_1 + \cdots + d_i, 1 \le i \le r$. Then the law of $(R_1, ..., R_r)$ possesses a C^∞-density with respect to Riemannian volume on $G_{d_1}(d) \times \cdots \times G_{d_r}(d)$.*

Proof:

Since the ideas of proof, apart from some technicalities, have been presented above, we can afford to be more sketchy here. Define

$$\begin{array}{rcl} f: \ F_{k_r,\cdots,k_1}(d) \times F_{l_r,\cdots,l_1}(d) & \to & G_{d_1}(d) \times \cdots \times G_{d_r}(d), \\ ((q_r, ..., q_1), (p_r, ..., p_1)) & \mapsto & (s_1, ..., s_r), \end{array}$$

where s_i is the projector on the intersection of the spaces onto which q_i and p_{r+1-i} project, $1 \le i \le r$. Let h_Q be the smooth density of $(Q_r, ..., Q_1)$, h_P the smooth density of $(P_r, ..., P_1)$. Then another application of the co-area formula of Federer implies that

$$h_R(s_1, .., s_r) = \alpha \int_{f^{-1}(s_1,..,s_r)} Jf(q,p)\, h_Q(q)\, h_P(p)\, H^k(d(q,p)),$$

writing q for $(q_r, .., q_1)$ and similarly for p, for $(s_1, ..., s_r) \in G_{d_1}(d) \times \cdots \times G_{d_r}(d)$, is a density of $(R_1, ..., R_r)$ with respect to $\sum_{i=1}^r d_i\,(d-d_i)$-dimensional Hausdorff measure on $G_{d_1}(d) \times \cdots \times G_{d_r}(d)$, and k an integer determined by the dimensions of the various Grassmann manifolds. Once again by rotational invariance of Hausdorff measure this density is C^∞. This completes the proof. □

In the following particular case all conditions of the preceding theorem are fulfilled.

Corollary 4.1 *Suppose that the Lie group generated by $A_0, \cdots, A_m$ is the general or special linear group. Then the law of $(R_1, \cdots, R_r)$ possesses a C^∞-density with respect to Riemannian volume.*

Proof:
The hypothesis (H) for any Grassmannian is a consequence of transitivity of the Lie group. Hence theorem 4.2 applies. □

Remark: 1. The conditional laws or joint laws of the random projectors on the Oseledets spaces can also be seen to be smooth. This follows by combining theorem 3.2 with the arguments of the preceding proofs.

2. If the ideal generated by the vector fields $A_1, \cdots, A_m$ in the Lie algebra of $A_0, \cdots, A_m$ coincides with the Lie algebra of the special or general linear group in $\mathbf{R}^d$, the simple fact that the mapping $A \mapsto h_A$ with respect to any $G_k(d), 1 \leq k \leq d$, is a homomorphism of Lie algebras, yields that (H) and thus also (HF) is satisfied in any dimension. Hence in this case the hypotheses of all our theorems are fulfilled. This is in particular true if the Lie group generated by $A_0, \cdots, A_m$ is the special or general linear group.

3. It appears that condition (H) for $k = 1, 2$, i.e. for projective space and planes implies the condition for any dimension $d \geq 3$. A reasoning for establishing this is based upon a complete check of Boothby's list of non-compact semisimple components of Lie groups for the conditions for $k = 1, 2$. Apart from a few trivial cases, this rules out all possibilities for the Lie group generated by $A_0, \cdots, A_m$ except to coincide with the general or special linear group (see San Martin [36]). We do not intend to discuss this question in the present paper.

4. Our results on the smoothness of the laws of the Oseledets spaces promise to be significant for the study of the asymptotic rotational behaviour of linear(ized) systems by concepts like *rotation numbers*. This will be made precise in a forthcoming paper.

5. The methods of Malliavin's calculus employed for smoothness properties of laws in this paper can be used to study finer properties of the laws like positivity, support and decay properties. For a recent account on the treatment of problems of this kind see Nualart [31]. We shall deal with these topics elsewhere.

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