CRITICAL DIMENSIONS FOR THE EXISTENCE OF SELF-INTERSECTION LOCAL TIMES OF THE N-PARAMETER BROWNIAN MOTION IN $\mathbf{R}^d$

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Abstract. Fix two rectangles A, B in $[0,1]^N$. Then the size of the random set of double points of the N-parameter Brownian motion $(W_t)_{t\in[0,1]^N}$ in $\mathbf{R}^d$, i.e. the set of pairs (s,t), where $s \in A$, $t \in B$, and $W_s = W_t$, can be measured as usual by a self-intersection local time. If $A = B$, we show that the critical dimension below which self-intersection local time does not explode, is given by $d = 2N$. If $A \cap B$ is a p-dimensional rectangle, it is $4N - 2p$ $(0 \le p \le N)$. If $A \cap B = \emptyset$, it is infinite. In all cases, we derive the rate of explosion of canonical approximations of self-intersection local time for dimensions above the critical one, and determine its smoothness in terms of the canonical Dirichlet structure on Wiener space.

Key words and phrases: N-parameter Brownian motion; self-intersection local time; multiple stochastic integrals; canonical Dirichlet structure.

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1. Introduction

Self intersection local times of Brownian motion in Euclidean space were originally introduced in order to construct certain continuous Euclidean quantum fields (see Varadhan [22], Szymanzik [21], Wolpert [26]). Since then they have been studied thoroughly in a purely mathematical framework by many authors, and for many stochastic fields. See for example Rosen [15], Yor [27], Le Gall [11], Dynkin [6], the latter also for the extensive bibliography.

Since self intersection local times of a field X measure the size of the random sets of double points (s,t) for which $X_s = X_t$, they may also be seen as an instrument for studying geometrical aspects of their level set structure. In this paper we consider the N-parameter Wiener process W with values in $\mathbf{R}^d$, and continue the study begun

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in [8] for the Brownian sheet in $\mathbf{R}^d$. It is well known that characteristics of the local behaviour of the Brownian sheet described for example by level sets, excursions and the Markov property possess properties much richer and more complex in structure than Brownian motion. This has been exhibited in a series of beautiful papers by Dalang and Walsh [4], [5], and Dalang and Mountford [3], Mountford [12].

If A and B are rectangles in the parameter set $\mathbf{R}_+^N$, we study the dependence of the size of the set of double points (s,t) such that $W_s = W_t$, as s is allowed to vary in A, t in B, on the size of $A \cap B$. Though the local property investigated this way is different from the ones studied in the papers cited, a similar richness of phenomena to observe is encountered, and may be related to the phenomenon of propagation of singularities along axial parallel lines in parameter space studied by Walsh (see [23]) for the Brownian sheet. To underline this, we briefly summarize the findings of [8]. We proved that - roughly - self intersection local times exist in all dimensions, if considered as Watanabe distributions with respect to the canonical Dirichlet structure on Wiener space generated by the Malliavin gradient. They are functions, i.e. distributions of positive order of smoothness below a certain dimension which only depends on N and d, and not on $A \cap B$. According to a well described pattern, above a certain *critical dimension* depending on the dimension of $A \cap B$, self intersection local time explodes in a very controlled way. After renormalizing the explosions in statements similar to the law of large numbers and the central limit theorem one obtains objects which are functions if $d < 8$, and distributions of some negative order for $d \geq 8$. If $A \cap B$ is 2-dimensional, the critical dimension is 4, if it is 1-dimensional, it is 6, and if it is 0-dimensional, 8.

In this paper we use the same technique to obtain similar results for the N-parameter Wiener process in $\mathbf{R}^d$. The technique is based on series expansions of the self intersection local time α and local functionals α_ϵ approximating it by formally replacing the integral of $\delta_0(W_s - W_t)$ with $p_\epsilon(W_s - W_t)$, where p_ϵ is the probability density of W_ϵ, in terms of the spectral decomposition of the Ornstein-Uhlenbeck operator of the underlying Gaussian Dirichlet structure. In fact, the results we find exhibit more clearly than in the special case of the Brownian sheet a pattern of relationships between the dimension N of the parameter space, the dimension p of the intersection of the rectangles A and B, and the critical dimension d below which self intersection local time does not explode. They also yield a clear description of the explosion rates $f(\epsilon)$ above the critical dimension with which α_ϵ has to be "renormalized", so that a strong law of large numbers type behaviour expressed as convergence of $\frac{\alpha_\epsilon}{f(\epsilon)}$ and of the type of the central limit theorem, expressing convergence of $\frac{\alpha_\epsilon - E(\alpha_\epsilon)}{\sqrt{f(\epsilon)}}$, become visible.

The limits - renormalized or not - exist as functions (distributions of positive order of smoothness) in case $d < 4N$, generally as Watanabe distributions of order $\rho < 2N - \frac{d}{2}$.

In Theorems 1 and 2 the case of $A = B$ (or equivalently $p = N$) is considered. We prove that the critical dimension is $2N$. The renormalizing functions for the strong law type result and the central limit type result are curiously related in the expected way, but with a dimension shift of 1. More precisely, for the strong law type the renormalizing function is given by $f(\epsilon) = \ln \frac{1}{\epsilon}$ if $d = 2N$ and by $f(\epsilon) = \epsilon^{(2N-d)/2}$ for $d > 2N$, for the central limit law type by 1 if $d = 2N$, $\ln \frac{1}{\epsilon}$ if $d = 2N + 1$, and by ϵ^{2N+1-d} if $d > 2N + 1$.

In Theorems 3, 4 and 5 we treat the case $A \cap B$ has dimension p with $0 \leq p < N$. Here the critical dimension turns out to be given by $4N - 2p$, and the curious dimension shift between behaviour of the strong law type and the central limit law type is not observed. The renormalizing function is as follows: in case $d = 4N - 2p$, it is given by $f(\epsilon) = \ln \frac{1}{\epsilon}$, in case $d > 4N - 2p$ by $\epsilon^{(4N-d-2p)/2}$ resp. $\epsilon^{4N-d-2p}$ in the two types of laws.

Finally, in Theorem 6 the case $A \cap B = \emptyset$ is considered. Here explosions are impossible in all dimensions, i.e. the critical dimension is infinite.

2. Preliminaries and notations

Throughout this paper we shall work with the canonical N-parameter Wiener process $W = (W^1, ..., W^d)$ indexed by $[0,1]^N$ with values in $\mathbf{R}^d$ on the canonical Wiener space $(\Omega, \mathcal{F}, P)$. P is the probability measure under which W_t possesses the probability density

$$p_{\bar{t}}^d(x) = \frac{1}{\sqrt{2\pi \bar{t}}^d} \exp(-\frac{|x|^2}{2\bar{t}}), \quad x \in \mathbf{R}^d, \quad t \in [0,1]^N,$$

where $\bar{t} = t_1 \cdots t_N$ for $t = (t_1, \ldots, t_N) \in [0,1]^N$. The ordering of the parameter space is supposed to be coordinatewise linear ordering on $\mathbf{R}_+$. Intervals with respect to this partial ordering are defined in the usual way, and $s < t$ means $s_i < t_i$, $1 \leq i \leq N$.

Suppose now $d = 1$. It is well known that $L^2(\Omega, \mathcal{F}, P)$ possesses an orthogonal decomposition by the eigenspaces of the Ornstein-Uhlenbeck operator on Wiener space, which are generated by the multiple Wiener-Ito integrals I_n, defined on $L^2(([0,1]^N)^n)$, $n \geq 0$ (see for example Bouleau, Hirsch [2], pp. 78-80). The multiple integrals possess the orthogonality property

$$E(I_n(f)I_m(g)) = \begin{cases} 0, & \text{if } n \neq m, \\ n! \int_{([0,1]^N)^n} fg \, d\lambda, & \text{if } n = m, \end{cases}$$

where λ denotes Lebesgue measure without reference to the dimension of the space on which it is defined. If H_n is the n^{th} Hermite polynomial defined by

$$H_n(x) = \frac{(-1)^n}{\sqrt{n!}} \exp(\frac{x^2}{2})(\frac{d}{dx})^n[\exp(-\frac{x^2}{2})],$$

$x \in \mathbf{R}$, $n \geq 0$, and if

$$W(h) = \int_{[0,1]^N} h \, dW$$

denotes the Gaussian stochastic integral of a function $h \in L^2([0,1]^N)$, the relation

$$H_n(W(h)) = \frac{1}{\sqrt{n!}} I_n(h^{\otimes_n})$$

holds true whenever $\|h\| = 1$. Here $h^{\otimes_n}$ denotes the n-fold tensor product of h with itself, while $\|\cdot\|$ is the norm in $L^2([0,1]^N)$. We write $W(D) = W(1_D)$ for $D \in \mathcal{B}(\mathbf{R}_+^N)$, so $W_t = W(R_t)$ for $R_t = [0,t]$. For $\rho \in \mathbf{R}$ we may define the *Sobolev space* of order ρ on Wiener space by introducing the norm

$$\|F\|_{2,\rho} = (\sum_{n=0}^{\infty} (1+n)^\rho \|I_n(f_n)\|_2^2)^{1/2}$$

on the space

$$\{F = \sum_{i=0}^{n} I_i(f_i) : f_i \in L^2(([0,1]^N)^i), \qquad 0 \leq i \leq n,\ n \in \mathbf{N}\}$$

which is dense in $L^2(\Omega, \mathcal{F}, P)$ and completing with respect to $\|\cdot\|_{2,\rho}$. We denote this space by $\mathbf{D}_{2,\rho}$. In case $\rho = 1$ we just recover the domain of the gradient operator of the canonical Dirichlet form on Wiener space, for $\rho < 0$ we obtain a space of distributions over Wiener space (see Watanabe [25], Bouleau, Hirsch [2], Nualart [13]).

To denote multiple Wiener-Ito integrals with respect to the independent components W^i of W in $\mathbf{R}^d$, we use the symbol I_n^i, $1 \leq i \leq d$, $n \geq 0$. Corresponding Sobolev spaces are defined for functionals of the N-parameter Wiener process with values in $\mathbf{R}^d$ (see Watanabe [25]).

We finally remark that $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of the sets A and B.

3. Two series representations and the characteristic integrals

We now fix two rectangles A and B off the boundary of $\mathbf{R}_+^N$ in which the time points s and t for which intersections are counted, are allowed to vary. We shall consider a canonical approximation $\alpha_\epsilon(x,\cdot)$, $\epsilon \to 0$, of self-intersection local time of the N-parameter Brownian motion in $\mathbf{R}^d$ corresponding to these rectangles.

To be more precise, let $A = [\alpha, \beta]$, $B = [\gamma, \delta] \subset [0,1]^N$ be such that $\alpha, \gamma > 0$. For $\epsilon > 0$, $x \in \mathbf{R}^d$ let

$$\alpha_\epsilon(x,\cdot) = \int_B \int_A p_\epsilon^d(W_t - W_s - x)\, ds\, dt$$

where

$$p_\epsilon^d(x) = \frac{1}{\sqrt{2\pi\epsilon}^d} \exp(-\frac{|x|^2}{2\epsilon}), \quad x \in \mathbf{R}^d, \quad \epsilon > 0,$$

is the density function of the d-dimensional Wiener process.

The derivation of the first representation of this approximation of self-intersection local time can be obtained as in the two-parameter case (cf. Imkeller, Weisz [8]).

Proposition 1. *For $x \in \mathbf{R}^d$, $\epsilon > 0$, we have*

$$\alpha_\epsilon(x,\cdot) = \sum_{n_i=0}^{\infty} \int_B \int_A \prod_{i=1}^{d} \frac{1}{\sqrt{n_i!}} I^i_{n_i}([\frac{1_{R_t} - 1_{R_s}}{\sqrt{\epsilon + \lambda(R_t \triangle R_s)}}]^{\otimes n_i})$$
$$H_{n_i}(\frac{x_i}{\sqrt{\epsilon + \lambda(R_t \triangle R_s)}}) p^d_{\epsilon+\lambda(R_t\triangle R_s)}(x)\, ds\, dt.$$

Analyzing just the representation of Proposition 1 would imply that we have to distinguish too many cases, due to the many different possible configurations R_t and R_s may take. The following decomposition of R_t provides an essential simplification of this task.

Choose $\eta > 0$ such that $\underline{\eta} = (\eta, \ldots, \eta) < \alpha, \gamma$. For $t \in [\underline{\eta}, 1]$ let

$$R_t = C_t \cup D_t,$$

where

$$\begin{aligned} C_t :&= \{x \in [0,1]^N : x \leq t,\ |\{i : x_i > \eta\}| \leq 1\} \\ &= \bigcup_{i=1}^{N} [0,\eta] \times \ldots \times [0,\eta] \times [0,t_i] \times [0,\eta] \times \ldots \times [0,\eta] \end{aligned}$$

and

$$D_t := \{x \in [0,1]^N : x \leq t,\ |\{i : x_i > \eta\}| > 1\}.$$

Then we have

$$W_t = W(C_t) + W(D_t).$$

For $s \in A$, $t \in B$, $x \in \mathbf{R}^d$, let now

$$\xi(s,t,x) = x - (W(D_t) - W(D_s)).$$

Due to the fact that the families

$$(\xi(s,t,x) : s \in A, t \in B, x \in \mathbf{R}^d) \quad \text{and} \quad (W(C_t) - W(C_s) : s \in A, t \in B)$$

are independent, an alternative representation of $\alpha_\epsilon(x,\cdot)$ can again be obtained as in the two-parameter case (see Imkeller, Weisz [8]).

Proposition 2. *For $x \in \mathbf{R}^d$, $\epsilon > 0$, we have*

$$\alpha_\epsilon(x,\cdot) = \sum_{n_i=0}^{\infty} \int_B \int_A \prod_{i=1}^{d} \frac{1}{\sqrt{n_i!}} I^i_{n_i}([\frac{1_{C_t} - 1_{C_s}}{\sqrt{\epsilon + \lambda(C_t \triangle C_s)}}]^{\otimes n_i})$$
$$H_{n_i}(\frac{\xi(s,t,x)_i}{\sqrt{\epsilon + \lambda(C_t \triangle C_s)}}) p^d_{\epsilon+\lambda(C_t\triangle C_s)}(x)\, ds\, dt.$$

Let us now fix $\rho \in \mathbf{R}$ and compute the norm of $\alpha_\epsilon(x,\cdot)$ in the two representations with respect to the quadratic Sobolev space of order ρ. We have

$$
\begin{aligned}
(1)\qquad & \|\alpha_\epsilon(x,\cdot)\|_{2,\rho}^2 \\
&= \sum_{k=0}^{\infty}(1+k)^\rho \int_B\int_A\int_B\int_A \frac{[\int_{[0,1]^N}(1_{R_t}-1_{R_s})(1_{R_v}-1_{R_u})\,d\lambda]^k}{[(\epsilon+\lambda(R_t\triangle R_s))(\epsilon+\lambda(R_v\triangle R_u))]^{\frac{k}{2}}} \\
&\qquad \sum_{n_1+\cdots+n_d=k}\prod_{i=1}^{d} H_{n_i}\left(\frac{x_i}{\sqrt{\epsilon+\lambda(R_t\triangle R_s)}}\right)H_{n_i}\left(\frac{x_i}{\sqrt{\epsilon+\lambda(R_v\triangle R_u)}}\right) \\
&\qquad p^d_{\epsilon+\lambda(R_t\triangle R_s)}(x)\,p^d_{\epsilon+\lambda(R_v\triangle R_u)}(x)\,ds\,dt\,du\,dv.
\end{aligned}
$$

As in Imkeller, Weisz [8], we can show that for $x \in \mathbf{R}^d$, $\epsilon > 0$,

$$
\begin{aligned}
& \|\alpha_\epsilon(x,\cdot)\|_{2,\rho}^2 \\
&\le c\sum_{k=0}^{\infty}(1+k)^\rho k^{d/2-1} \\
&\qquad \int_B\int_A\int_B\int_A \frac{\lambda((C_t\triangle C_s)\cap(C_v\triangle C_u))^k}{[(\epsilon+\lambda(C_t\triangle C_s))(\epsilon+\lambda(C_v\triangle C_u))]^{\frac{k+d}{2}}}\,ds\,dt\,du\,dv.
\end{aligned}
$$

In the following section, the convergence behaviour of $\alpha_\epsilon(x,\cdot)$ as $\epsilon \to 0$ will be investigated. The red thread though our arguments is the following. If we want to establish convergence of $\alpha_\epsilon(x,\cdot)$ to

$$
\begin{aligned}
(2)\qquad \alpha(x,\cdot) = \sum_{n_i=0}^{\infty}\int_B\int_A\prod_{i=1}^{d}\frac{1}{\sqrt{n_i!}}\,I^i_{n_i}\left(\left[\frac{1_{R_t}-1_{R_s}}{\sqrt{\lambda(R_t\triangle R_s)}}\right]^{\otimes n_i}\right) \\
H_{n_i}\left(\frac{x_i}{\sqrt{\lambda(R_t\triangle R_s)}}\right)p^d_{\lambda(R_t\triangle R_s)}(x)\,ds\,dt
\end{aligned}
$$

in $\mathbf{D}_{2,\rho}$, due to dominated convergence all we need to prove is

$$
(3)\qquad \tilde{I}(k,0) \le c_k \quad\text{and}\quad \sum_{k=0}^{\infty} c_k(1+k)^\rho k^{d/2-1} < \infty
$$

where

$$
\tilde{I}(k,\epsilon) = \int_B\int_A\int_B\int_A \frac{\lambda((C_t\triangle C_s)\cap(C_v\triangle C_u))^k}{[(\epsilon+\lambda(C_t\triangle C_s))(\epsilon+\lambda(C_v\triangle C_u))]^{\frac{k+d}{2}}}\,ds\,dt\,du\,dv,
$$

$k \in \mathbf{N}_0$, $\epsilon \ge 0$, denote the characteristic integrals.

If, on the other hand, in case $\alpha_\epsilon(x,\cdot)$ does not converge as $\epsilon \to 0$, we want to find its rate of explosion, we shall argue in two steps. We first determine a deterministic function $f(\epsilon)$ such that

$$\sup_{\epsilon>0} \frac{\tilde{I}(k,\epsilon)}{f(\epsilon)^2} \leq c_k \quad \text{and} \quad \sum_{k=0}^{\infty} c_k (1+k)^\rho k^{d/2-1} < \infty. \tag{4}$$

With (4) we will have proved that $\{\frac{\alpha_\epsilon(x,\cdot)}{f(\epsilon)} : \epsilon > 0\}$ is bounded in $\mathbf{D}_{2,\rho}$. To prove next that $f(\epsilon)$ gives the correct order of explosion, we then consider the other type of characteristic integrals given by

$$\tilde{J}(k,\epsilon) = \int_B \int_A \int_B \int_A \frac{[\int_{[0,1]^N} (1_{R_t} - 1_{R_s})(1_{R_v} - 1_{R_u})\, d\lambda]^k}{[(\epsilon + \lambda(R_t \triangle R_s))(\epsilon + \lambda(R_v \triangle R_u))]^{\frac{k+d}{2}}}\, ds\, dt\, du\, dv,$$

$k \in \mathbf{N}_0$, $\epsilon \geq 0$, which appear in (1). If we can establish

$$\lim_{\epsilon \to 0} \frac{\tilde{J}(k,\epsilon)}{f(\epsilon)^2} > 0$$

for some k, then this implies that

$$\lim_{\epsilon \to 0} \frac{\|\alpha_\epsilon(x,\cdot)\|_{2,\rho}}{f(\epsilon)} > 0$$

(cf. Imkeller, Weisz [8]). Consequently $f(\epsilon)$ is the correct rate of divergence.

Observe that by choice of $\underline{\eta}$, for $s \in A$, $t \in B$ we have

$$\lambda(C_t \triangle C_s) = \eta^{N-1} \sum_{i=1}^{N} |t_i - s_i|,$$

and for $u, s \in A$, $t, v \in B$

$$\lambda((C_t \triangle C_s) \cap (C_v \triangle C_u)) = \eta^{N-1} \sum_{i=1}^{N} \lambda([s_i \wedge t_i, s_i \vee t_i] \cap [u_i \wedge v_i, u_i \vee v_i]).$$

Symmetry gives

$$\begin{aligned} \tilde{I}(k,\epsilon) = & 2^{2N} \int_B \cdots \int_A 1_{\{s_i < t_i, u_i < v_i, 1 \leq i \leq N\}} \\ & \frac{[\sum_{i=1}^N \lambda([s_i,t_i] \cap [u_i,v_i])]^k}{[(\frac{\epsilon}{\eta^{N-1}} + \sum_{i=1}^N (t_i - s_i))(\frac{\epsilon}{\eta^{N-1}} + \sum_{i=1}^N +(v_i - u_i))]^{\frac{k+d}{2}}}\, ds \ldots dv, \end{aligned} \tag{5}$$

$\epsilon \geq 0$, $k \in \mathbf{N}_0$. Denote now

$$S_i = [s_i, t_i], \quad U_i = [u_i, v_i], \qquad 1 \leq i \leq N.$$

The following estimate has a similar proof as in Imkeller, Weisz [8].

Lemma 1. *We have*

$$[\epsilon+\sum_{i=1}^N \lambda(S_i\cap U_i)][\epsilon+\sum_{i=1}^N \lambda(S_i\cup U_i)] \le 2[\epsilon+\sum_{i=1}^N \lambda(S_i)][\epsilon+\sum_{i=1}^N \lambda(U_i)].$$

This lemma implies that instead of the ones appearing in (5) it is enough to estimate the integrals

$$I(k,\epsilon) = \int_B\int_A\int_B\int_A 1_{\{s_i<t_i,u_i<v_i,1\le i\le N\}} \frac{[\sum_{i=1}^N \lambda(S_i\cap U_i)]^k}{[(\epsilon+\sum_{i=1}^N \lambda(S_i\cap U_i))(\epsilon+\sum_{i=1}^N \lambda(S_i\cup U_i))]^{\frac{k+d}{2}}}\, ds\, dt\, du\, dv.$$

We remark at this place that constants appearing in the subsequent estimates of $\tilde{I}(k,\epsilon)$ and $I(k,\epsilon)$ may vary from line to line, but do not depend on ϵ and k.

4. Self-intersections of the N-parameter Wiener process in different rectangles

Essentially two cases of configurations of A and B will be investigated. In case 1, $A\cap B$ is a p-dimensional rectangle with $0\le p\le N$ fixed. Note that if $p=N$ then $A=B$ and if $p=0$ then A and B have a common edge, only. In case 2 we suppose that A and B are disjoint. We remark that this way we have indeed covered all interesting cases. By cutting of rectangles, switching the axes of $[0,1]^N$ or reversing time in one time direction we shall always end up with one of the configurations considered. We shall quantify how an increase of "disjointness" of A and B leads to an increase of the "independence" of W_s and W_t since s runs in A, t in B and consequently to a stepwise increase of the critical dimension below which self-intersection local times cannot explode.

In case 1 we can therefore suppose that for $A=[\alpha,\beta]$, $B=[\gamma,\delta]$ and for $0\le p\le N$ we have

$$\begin{aligned}\alpha_1=\gamma_1,\ldots,\alpha_p=\gamma_p\\ \beta_1=\delta_1,\ldots,\beta_p=\delta_p\\ \beta_{p+1}=\gamma_{p+1},\ldots,\beta_N=\gamma_N.\end{aligned}\tag{6}$$

To estimate $I(k,\epsilon)$ we investigate separately for every q the cases

$$|\{i: S_i\cap U_i=\emptyset\}|=q.$$

Note that $0 \le q \le p$ because $S_i \cap U_i \neq \emptyset$ for $i > p$ as a consequence of our hypotheses on the relative position of A and B. By symmetry we can suppose that

$$S_i \cap U_i = \emptyset, \quad i = 1, \ldots, q, \qquad 0 \le q \le p.$$

So for $0 \le q \le p$ we have to estimate

$$(7)\quad I(k,\epsilon,q) = \int_B\int_A\int_B\int_A 1_{\{a_i<b_i<c_i<d_i, 1\le i\le N\}} \frac{[\sum_{i=q+1}^N (c_i-b_i)]^k \prod_{i=1}^N da_i\, db_i\, dc_i\, dd_i}{[\epsilon + \sum_{i=q+1}^N (c_i-b_i)]^{\frac{k+d}{2}} [\epsilon + \sum_{i=1}^q (b_i - a_i + d_i - c_i) + \sum_{i=q+1}^N (d_i - a_i)]^{\frac{k+d}{2}}}.$$

In case $k = 0$ things are much simpler. So here we shall consider occasionally

$$(8)\qquad I(0,\epsilon) = \Big(\int_B\int_A 1_{\{s_i<t_i, 1\le i\le N\}} \frac{1}{[\epsilon + \sum_{i=1}^N (t_i - s_i)]^{\frac{d}{2}}}\, ds\, dt\Big)^2, \quad \epsilon \ge 0.$$

Obviously,

$$(9)\qquad I(k,\epsilon) \approx \sum_{q=0}^p I(k,\epsilon,q),$$

where $\approx$ means asymptotic equivalence.

Proposition 3.

(1) We have $I(0,0) < \infty$ for $d < 4N - 2p$ in case 1 and all $d \in \mathbf{N}$ in case 2.

(2) Moreover there is a constant c such that for $\epsilon > 0$

$$\frac{I(0,\epsilon)}{f(\epsilon)} \le c,$$

where $f(\epsilon) = (\ln \frac{1}{\epsilon})^2$ for $d = 4N - 2p$ and $f(\epsilon) = \epsilon^{4N-2p-d}$ for $d > 4N - 2p$.

Proof. We concentrate on the more difficult argument needed to prove (2). Indeed, whenever $d \ge 4N - 2p$,

$$\begin{aligned}
\sqrt{I(0,\epsilon)} &= \int_B\int_A 1_{\{s_i<t_i, 1\le i\le p; s_i<\beta_i<t_i, p+1\le i\le N\}} \frac{ds\, dt}{[\epsilon + \sum_{i=1}^N (t_i-s_i)]^{\frac{d}{2}}} \\
&\approx \int \ldots \int 1_{\{s_i<t_i, 1\le i\le p; s_i<\beta_i<t_i, p+1\le i\le N-1\}} \frac{dt_1 \ldots dt_{N-1}\, ds_1 \ldots ds_{N-1}}{[\epsilon + \sum_{i=1}^{N-1} (t_i-s_i)]^{\frac{d}{2}-2}} \\
&\approx \int \ldots \int 1_{\{s_i<t_i, 1\le i\le p\}} \frac{dt_1 \ldots dt_p\, ds_1 \ldots ds_p}{[\epsilon + \sum_{i=1}^p (t_i-s_i)]^{\frac{d}{2}-2(N-p)}} \\
&\approx \int \ldots \int 1_{\{s_i<t_i, 1\le i\le p-1\}} \frac{dt_1 \ldots dt_{p-1}\, ds_1 \ldots ds_{p-1}}{[\epsilon + \sum_{i=1}^{p-1} (t_i-s_i)]^{\frac{d}{2}-2(N-p)-1}} \\
&\approx \int\int 1_{\{s_1<t_1\}} \frac{dt_1\, ds_1}{[\epsilon + (t_1-s_1)]^{\frac{d}{2}-2(N-p)-(p-1)}}.
\end{aligned}$$

This is further asymptotically equivalent with $\ln \frac{1}{\epsilon}$ if $d = 4N - 2p$ and with $\epsilon^{2N-p-d/2}$ if $d > 4N - 2p$ as $\epsilon \to 0$. The proof of Proposition 3 is complete. •

We now consider the higher degree integrals $I(k, \epsilon)$, $k \in \mathbf{N}$.

Proposition 4.

(1) *There exists a constant c such that, for all $k \in \mathbf{N}$, $I(k, 0)k^{2N} \leq c$ for $d < 2N + 1$ if $p = N$, for $d < 4N - 2p$ if $0 \leq p < N$ and for all $d \in \mathbf{N}$ in case 2.*

(2) *Moreover, there exists a constant c such that for all $k \in \mathbf{N}$, $\epsilon > 0$*

$$\frac{I(k, \epsilon)k^{2N}}{f(\epsilon)} \leq c,$$

where $f(\epsilon) = \ln \frac{1}{\epsilon}$ for $d = 2N + 1$ if $p = N$, for $d = 4N - 2p$ if $0 \leq p < N$, $f(\epsilon) = \epsilon^{2N+1-d}$ for $d > 2N+1$ if $p = N$ and $f(\epsilon) = \epsilon^{4N-2p-d}$ for $d > 4N-2p$ if $0 \leq p < N$.

Proof. Again we concentrate on proving the more difficult second statement. We investigate $I(k, \epsilon, q)$ for each $0 \leq q \leq p$ and suppose that $\frac{k+d}{2} > 2N$. Remark that the remaining integrals are similar to deal with, except for the appearance of a logarithm in the process of integration. Note that if $q = N$ then $I(k, \epsilon, q) = 0$. So we can assume that $q < N$. We integrate first with respect to $a_1, \ldots, a_N$ and then with respect to $d_1, \ldots, d_N$. The result is

$$\begin{aligned}
I(k, \epsilon, q) \leq & \int_B \int_A \int_B \int_A 1_{\{a_i < b_i < c_i < d_i, 1 \leq i \leq p; a_i < b_i < \beta_i < c_i < d_i, p+1 \leq i \leq N\}} \\
& \qquad \frac{[\epsilon + \sum_{i=q+1}^N (c_i - b_i)]^{\frac{k-d}{2}} \, da_1 \ldots dd_N}{[\epsilon + \sum_{i=1}^q (b_i - a_i + d_i - c_i) + \sum_{i=q+1}^N (d_i - a_i)]^{\frac{k+d}{2}}} \\
\approx & \; ck^{-N} \int_B \int_A \int_B 1_{\{b_i < c_i < d_i, 1 \leq i \leq p; b_i < \beta_i < c_i < d_i, p+1 \leq i \leq N\}} \\
& \qquad \frac{[\epsilon + \sum_{i=q+1}^N (c_i - b_i)]^{\frac{k-d}{2}} \, db_1 \ldots dd_N}{[\epsilon + \sum_{i=1}^q (d_i - c_i) + \sum_{i=q+1}^N (d_i - b_i)]^{\frac{k+d}{2} - N}} \\
\approx & \; ck^{-2N} \int_B \int_A 1_{\{b_i < c_i, 1 \leq i \leq p; b_i < \beta_i < c_i, p+1 \leq i \leq N\}} \\
& \qquad \frac{[\epsilon + \sum_{i=q+1}^N (c_i - b_i)]^{\frac{k-d}{2}} \, db_1 \ldots dc_N}{[\epsilon + \sum_{i=q+1}^N (c_i - b_i)]^{\frac{k+d}{2} - 2N}} \\
= & \; ck^{-2N} \int_B \int_A 1_{\{b_i < c_i, 1 \leq i \leq p; b_i < \beta_i < c_i, p+1 \leq i \leq N\}} \\
& \qquad [\epsilon + \sum_{i=q+1}^N (c_i - b_i)]^{2N-d} \, db_1 \ldots dc_N.
\end{aligned}$$

We next integrate in $c_{p+1}, \dots, c_N$ to obtain

$$
\begin{aligned}
(10)\quad & I(k,\epsilon,q) \\
& \le ck^{-2N} \int \dots \int 1_{\{b_i<c_i,1\le i\le p; b_i<\beta_i, p+1\le i\le N\}} \\
& \qquad [\epsilon + \sum_{i=q+1}^{p} (c_i-b_i) + \sum_{i=p+1}^{N} (\beta_i - b_i)]^{2N-d+N-p} \, db_1 \dots db_N \, dc_1 \, \dots dc_p.
\end{aligned}
$$

For $q<p$ we can now integrate first in $b_{p+1}, \dots, b_N$ and then in $c_{q+2}, \dots, c_p$:

$$
\begin{aligned}
(11)\ I(k,\epsilon,q) &\le ck^{-2N} \int \dots \int 1_{\{b_i<c_i, q+1\le i\le p\}} \\
& \qquad [\epsilon + \sum_{i=q+1}^{p} (c_i-b_i)]^{2N-d+2(N-p)} \, db_{q+1} \, \dots db_p \, dc_{q+1} \, \dots dc_p \\
&\approx ck^{-2N} \int\int 1_{\{b_{q+1}<c_{q+1}\}} \\
& \qquad [\epsilon + (c_{q+1} - b_{q+1})]^{4N-d-2p+p-q-1} \, db_{q+1} \, dc_{q+1}
\end{aligned}
$$

and this is approximately equivalent with $\ln \frac{1}{\epsilon}$ if $d = 4N-p-q$ and with $\epsilon^{4N-p-q-d}$ if $d > 4N-p-q$.

For $q=p$ we integrate in (10) with respect to $b_{p+2}, \dots, b_N$ to get

$$
I(k,\epsilon,q) \le ck^{-2N} \int 1_{\{b_{p+1}<\beta_{p+1}\}} [\epsilon + (\beta_{p+1} - b_{p+1})]^{3N-p-d+N-p-1} \, db_{p+1}
$$

which is approximately equivalent with $\ln \frac{1}{\epsilon}$ if $d = 4N-2p$ and with $\epsilon^{4N-2p-d}$ if $d > 4N-2p$.

Suppose now that $p = N$. We proved in (11) that if $d = 2N+1$ then $I(k,\epsilon,q) < I(k,0,q) < \infty$ for $0 \le q < N-1$. Since $I(k,\epsilon,N) = 0$, we have $I(k,\epsilon) \le ck^{-2N} \ln \frac{1}{\epsilon}$. If $d > 2N+1$ then $I(k,0,q) < \infty$ for $0 \le q < 3N-d$. Hence

$$
I(k,\epsilon) \le c \sum_{q=3N-d}^{N-1} I(k,\epsilon,q) < ck^{-2N} \epsilon^{2N+1-d}.
$$

The cases $0 \le p < N$ can be treated similarly, using (9). $\bullet$

As mentioned earlier, we also have to verify that the orders of divergence obtained in the preceding propositions are correct. We therefore investigate the integrals $\tilde{J}(k,\epsilon)$ for particular k, and on particular domains D of integration contained in $B \times A \times B \times A$. If we call these integrals $J(k,\epsilon)$, then

$$
\lim_{\epsilon \to 0} \frac{J(k,\epsilon)}{f(\epsilon)} > 0
$$

will imply

$$\lim_{\epsilon\to 0}\frac{\tilde{J}(k,\epsilon)}{f(\epsilon)}>0,$$

and we will have proved that $\sqrt{f(\epsilon)}$ is the correct order of divergence.

Proposition 5. *For $\epsilon>0$ let*

$$J(0,\epsilon)=(\int_B\int_A 1_{\{s<t\}}\frac{1}{(\epsilon+\overline{t}-\overline{s})^{d/2}}\,ds\,dt)^2.$$

Then we have

$$\lim_{\epsilon\to 0}\frac{J(0,\epsilon)}{f(\epsilon)}>0,$$

where $f(\epsilon)=(\ln\frac{1}{\epsilon})^2$ for $d=4N-2p$ and $f(\epsilon)=\epsilon^{4N-2p-d}$ for $d>4N-2p$.

Proof. We have

$$\overline{t}-\overline{s}=\sum_{i=1}^{N}(t_i-s_i)s_1\cdots s_{i-1}t_{i+1}\cdots t_N\ \begin{cases}\le\sum_{i=1}^{N}(t_i-s_i),\\ \ge\eta^{N-1}\sum_{i=1}^{N}(t_i-s_i).\end{cases}\tag{12}$$

Hence $I(0,\epsilon)$ and $J(0,\epsilon)$ are asymptotically equivalent and the desired result follows from Proposition 3. •

Proposition 6. *Suppose that $p<N$. For $\epsilon>0$, $k\in\mathbf{N}$ let*

$$J(k,\epsilon)=\int_B\int_A\int_B\int_A 1_{\{s_i<t_i<u_i<v_i,1\le i\le p;s_i<u_i<v_i<t_i,p+1\le i\le N\}}\frac{[\int_{[0,1]^N}(1_{R_t}-1_{R_s})(1_{R_v}-1_{R_u})\,d\lambda]^k}{[(\epsilon+\lambda(R_t\triangle R_s))(\epsilon+\lambda(R_v\triangle R_u))]^{\frac{k+d}{2}}}\,ds\,dt\,du\,dv.$$

Then

$$\lim_{\epsilon\to 0}\frac{J(k,\epsilon)}{f(\epsilon)}>0$$

for $\frac{k+d}{2}>\max(2N-p,p)$, where $f(\epsilon)=\ln\frac{1}{\epsilon}$ for $d=4N-2p$ and $f(\epsilon)=\epsilon^{4N-2p-d}$ for $d>4N-2p$.

Proof. Using (12) we can see that

$$\begin{aligned}J(k,\epsilon)&=\int_B\int_A\int_B\int_A 1_{\{s_i<t_i<u_i<v_i,1\le i\le p;s_i<u_i<v_i<t_i,p+1\le i\le N\}}\\&\qquad\frac{[(\prod_{i=p+1}^{N}v_i-\prod_{i=p+1}^{N}u_i)\prod_{i=1}^{p}t_i]^k}{[(\epsilon+\overline{t}-\overline{s})(\epsilon+\overline{v}-\overline{u})]^{\frac{k+d}{2}}}\,ds\,dt\,du\,dv\\&\approx\int_B\int_A\int_B\int_A 1_{\{s_i<t_i<u_i<v_i,1\le i\le p;s_i<u_i<\beta_i<v_i<t_i,p+1\le i\le N\}}\\&\qquad\frac{[(\sum_{i=p+1}^{N}(v_i-u_i)]^k}{[(\epsilon+\sum_{i=1}^{N}(t_i-s_i)]^{\frac{k+d}{2}}[(\epsilon+\sum_{i=1}^{N}(v_i-u_i)]^{\frac{k+d}{2}}}\,ds\,dt\,du\,dv.\end{aligned}$$

Now integrate in s, t, $u_1, \ldots, u_p$ and $v_1, \ldots, v_p$ to obtain

$$(13)\quad J(k,\epsilon) \approx \int \ldots \int 1_{\{u_i<\beta_i<v_i, p+1\le i\le N\}} \frac{[(\sum_{i=p+1}^N (v_i-u_i)]^k \, du_{p+1} \ldots du_N \, dv_{p+1} \ldots dv_N}{[\epsilon+\sum_{i=p+1}^N (v_i-u_i)]^{\frac{k+d}{2}-2N+p}[\epsilon+\sum_{i=p+1}^N (v_i-u_i)]^{\frac{k+d}{2}-p}}$$

$$= \int \ldots \int 1_{\{u_i<\beta_i<v_i, p+1\le i\le N\}} \frac{[(\sum_{i=p+1}^N (v_i-u_i)]^k}{[\epsilon+\sum_{i=p+1}^N (v_i-u_i)]^{k+d-2N}} \, du_{p+1} \ldots du_N \, dv_{p+1} \ldots dv_N$$

$$= \sum_{l=0}^k (-\epsilon)^l \binom{k}{l} \int \ldots \int 1_{\{u_i<\beta_i<v_i, p+1\le i\le N\}} [\epsilon + \sum_{i=p+1}^N (v_i-u_i)]^{2N-l-d} \, du_{p+1} \ldots du_N \, dv_{p+1} \ldots dv_N.$$

Integrating finally in $u_{p+1}, \ldots, u_N$ and $v_{p+1}, \ldots, v_N$ we can conclude that this integral is asymptotically equivalent with $\ln \frac{1}{\epsilon}$ if $d = 4N-2p$ and with $\epsilon^{4N-2p-d}$ if $d > 4N-2p$. The proof of the proposition is complete. •

Proposition 7. *Suppose that $p = N$. For $\epsilon > 0$, $k \in \mathbf{N}$ let*

$$J(k,\epsilon) = \int_B \int_A \int_B \int_A 1_{\{s_i<t_i<u_i<v_i, 1\le i\le N-1; s_N<u_N<v_N<t_N\}} \frac{[\int_{[0,1]^N} (1_{R_t}-1_{R_s})(1_{R_v}-1_{R_u})\, d\lambda]^k}{[(\epsilon+\lambda(R_t \triangle R_s))(\epsilon+\lambda(R_v\triangle R_u))]^{\frac{k+d}{2}}} \, ds\, dt\, du\, dv.$$

Then

$$\lim_{\epsilon\to 0} \frac{J(k,\epsilon)}{f(\epsilon)} > 0$$

for $\frac{k+d}{2} > \max(2N-p, p)$, where $f(\epsilon) = \ln \frac{1}{\epsilon}$ for $d = 2N+1$ and $f(\epsilon) = \epsilon^{2N+1-d}$ for $d > 2N+1$.

Proof. It is easy to see that the estimate (13) may be replaced by

$$J(k,\epsilon) \approx \sum_{l=0}^k (-\epsilon)^l \binom{k}{l} \int\int 1_{\{u_N<v_N\}} [\epsilon + (v_N-u_N)]^{2N-l-d} \, du_N \, dv_N.$$

which is asymptotically equivalent with $\ln \frac{1}{\epsilon}$ if $d = 2N+1$ and with ϵ^{2N+1-d} if $d > 2N+1$. •

We are finally in a position to formulate our main results. In each case we first treat the existence of self-intersection local times, then the rates of explosion in case of non-existence.

4.1. The case $A = B$ (i.e. $p = N$).

Theorem 1.

(1) Let $d < 2N$. Then for any $x \in \mathbf{R}^d$

$$\alpha(x, \cdot) = \lim_{\epsilon \to 0} \alpha_\epsilon(x, \cdot)$$

exists in $\mathbf{D}_{2,\rho}$ for any $\rho < 2N - d/2$ and is given by (2). Hence it is a function if $d < 4N$.

(2) Let $d < 2N + 1$. Then for any $x \in \mathbf{R}^d$

$$\gamma(x, \cdot) = \lim_{\epsilon \to 0} [\alpha_\epsilon(x, \cdot) - E(\alpha_\epsilon(x, \cdot))]$$

exists in $\mathbf{D}_{2,\rho}$ for $\rho < 2N - d/2$, hence is a function if $d < 4N$.

Proof. Consult part (1) of Propositions 3 and 4 and remember that $E(\alpha_\epsilon(x, \cdot))$ is the first term in the development of $\alpha_\epsilon(x, \cdot)$. The order of smoothness follows from these propositions as well together with the fact that in (3) we can take $c_k = ck^{-2N}$, and

$$\sum_{k=1}^{\infty} (1 + k)^\rho k^{d/2-1} k^{-2N} < \infty \qquad \Longleftrightarrow \qquad \rho + \frac{d}{2} - 2N < 0,$$

i.e. iff $\rho < 2N - d/2$. $\bullet$

Theorem 2.

(1) Let $d \geq 2N$. Then for any $x \in \mathbf{R}^d$ there is a constant $c_x > 0$ such that

$$\lim_{\epsilon \to 0} \frac{\alpha_\epsilon(x, \cdot)}{f(\epsilon)} = c_x$$

in $\mathbf{D}_{2,\rho}$ for $\rho < 2N - d/2$, where

$$f(\epsilon) = \begin{cases} \ln 1/\epsilon & \text{for} \quad d = 2N, \\ \epsilon^{(2N-d)/2} & \text{for} \quad d > 2N. \end{cases}$$

(2) Let $d \geq 2N + 1$. Then for any $x \in \mathbf{R}^d$

$$\left\{ \frac{\alpha_\epsilon(x, \cdot) - E(\alpha_\epsilon(x, \cdot))}{f(\epsilon)} : \epsilon > 0 \right\}$$

is bounded in $\mathbf{D}_{2,\rho}$ for any $\rho < 2N - d/2$, where

$$f(\epsilon) = \begin{cases} \sqrt{\ln 1/\epsilon} & \text{for} \quad d = 2N + 1, \\ \epsilon^{(2N+1-d)/2} & \text{for} \quad d > 2N + 1. \end{cases}$$

Moreover, the limit of the $(2, \rho)$-norms of these random variables as $\epsilon \to 0$ is non-trivial.

Proof. This time, we make appeal to the second statements of Propositions 3 and 4. To see that the rates are sharp, consult Propositions 5 and 7. •

4.2. The case $A \cap B$ is a p-dimensional rectangle $(0 \le p < N)$.

The following theorems are reduced to the preceding propositions in much the same way.

Theorem 3. *Let $d < 4N - 2p$ and $0 \le p < N$. Then for any $x \in \mathbf{R}^d$*

$$\alpha(x, \cdot) = \lim_{\epsilon \to 0} \alpha_\epsilon(x, \cdot)$$

exists in $\mathbf{D}_{2,\rho}$ for any $\rho < 2N - d/2$ and is given by (2). Hence it is a function if $d < 4N$.

Theorem 4. *Let $d = 4N - 2p$ with $0 \le p < N$.*

(1) Then for any $x \in \mathbf{R}^d$ there exists a constant $c_x > 0$ such that

$$\lim_{\epsilon \to 0} \frac{\alpha_\epsilon(x, \cdot)}{\ln 1/\epsilon} = c_x$$

in $\mathbf{D}_{2,\rho}$ for $\rho < 2N - d/2$.

(2) For any $x \in \mathbf{R}^d$ the set

$$\left\{ \frac{\alpha_\epsilon(x, \cdot) - E(\alpha_\epsilon(x, \cdot)}{\sqrt{\ln 1/\epsilon}} : \epsilon > 0 \right\}$$

is bounded in $\mathbf{D}_{2,\rho}$ for $\rho < 2N - d/2$. Moreover, the limit of the $(2, \rho)$-norms of these random variables as $\epsilon \to 0$ is non-trivial.

Theorem 5. *Let $d > 4N - 2p$ with $0 \le p < N$. Then for any $x \in \mathbf{R}^d$ the set*

$$\{\epsilon^{(d+2p-4N)/2} \alpha_\epsilon(x, \cdot) : \epsilon > 0\}$$

is bounded in $\mathbf{D}_{2,\rho}$ for any $\rho < 2N - d/2$. The limit of the $(2, \rho)$-norms of these random variables as $\epsilon \to 0$ is non-trivial.

4.3. The case A and B are disjoint.

Theorem 6. *If A and B are disjoint then for any $x \in \mathbf{R}^d$ we have*

$$\alpha(x, \cdot) = \lim_{\epsilon \to 0} \alpha_\epsilon(x, \cdot)$$

exists in $\mathbf{D}_{2,\rho}$ for any $\rho < 2N - d/2$, hence is a function if $d < 4N$.

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