Supplement B: Stability of the eigenvalue problems

Jakub Chorowski

1 Compact, positive-definite, self adjoint operator

Theorem 1. *[3, Theorem 22] Consider T a compact, self-adjoint and positive-definite operator on some Hilbert space $\mathcal{H} = (H, \|\cdot\|)$. Denote its eigenpairs by $(\lambda_i, x_i)_{i=1,2,\dots}$, normalized so that $\|x_i\| = 1$ and ordered decreasingly with respect to the eigenvalues. Let $V \subset H$ be a finite dimensional subspace of H, and π the orthogonal projection on V. Assume that the biggest eigenvalue λ_1 is simple and that*

$$\|(I - \pi)\, x_1\| < \frac{\lambda_1 - \lambda_2}{6\lambda_1}.$$

Consider the projected operator $\pi T \pi$ and denote its normalized, ordered decreasingly, eigenpairs by $(\lambda_{V,i}, x_{V,i})_{i=1,2,\dots,dim(V_J)}$. Then $\lambda_{1,V} - \lambda_{2,V} \geq (\lambda_1 - \lambda_2)/2$ and

$$|\lambda_1 - \lambda_{V,1}| + \|x_1 - x_{V,1}\| \leq C\, \|(I - \pi)\, x_1\|\,,$$

where the constant C depends continuously only on the size of the spectral gap $\lambda_1 - \lambda_2$ and the value of the first eigenvalue λ_1.

2 Bilinear coercive form

Recall that H^1, H^2 denote the L^2−Sobolev spaces on $[0, 1]$ of order 1 and 2 respectively. For differentiable, strictly positive functions σ and μ consider the elliptic operator T on the Hilbert space $L^2([0, 1])$ with Neumann type domain $\mathrm{dom}(T) = \{v \in H^2 : v'(0) = v'(1) = 0\}$ and for $v \in \mathrm{dom}(T)$ given in the divergence form by

$$Tv(x) = -\frac{(\sigma^2(x)\mu(x)v'(x))'}{2\mu(x)}. \tag{1}$$

Note that the operator $-T$ is an infinitesimal generator of the diffusion process on $[0, 1]$ with instantaneous reflection at the boundaries, volatility function σ and invariant measure with density μ. We want to analyze the eigenvalue problem for T, i.e.

Eigenproblem 2. Find $(\lambda, w) \in \mathbb{R} \times \mathrm{dom}(T)$, with $w \neq 0$, such that

$$Tw = \lambda w.$$

Integrating by parts one can check that the eigenpairs of the Eigenproblem 2 solve

Eigenproblem 3. Find $(\lambda, w) \in \mathbb{R} \times H^1$, with $w \neq 0$, such that

$$\int_0^1 w'(x)v'(x)\sigma^2(x)\mu(x)dx = 2\lambda \int_0^1 w(x)v(x)\mu(x)dx \text{ for all } v \in H^1. \tag{2}$$

The Eigenproblem 3 is a weak formulation of the Eigenproblem 2, on the equivalent space $L^2(\mu)$, for the associated Dirichlet form $l(u,v) = \langle Tu, v\rangle_\mu$. The biggest advantage of the weak formulation is that the Eigenproblem 3 makes sense for any, not necessarily regular, functions μ. When μ is not differentiable, the Eigenproblem 2 has no longer probabilistic interpretation in terms of the infinitesimal generator. Nevertheless, such problems arise naturally when one considers spectral estimation method with fixed time horizon, when the role of the invariant measure is taken by the non differentiable occupation density.

In what follows, we want to generalize the results of [4] on the spectral properties of an infinitesimal generator to the solutions of the Eigenproblem 3 with Hölder regular function μ.

Definition 4. For any given $0 < d < D$ let

$$\Theta_\alpha := \Big\{(\sigma,\mu) \in C^1([0,1]) \times C^{0,\alpha}([0,1]) : \|\sigma\|_{C^1} \vee \|\mu\|_{C^{0,\alpha}} \le D, \\ \inf_{x\in[0,1]} (\sigma(x) \wedge \mu(x)) \ge d, \int_0^1 \mu(x)dx = 1\Big\}$$

Eigenproblem 3 is a conforming eigenvalue problem for a bilinear coercive form on the Hilbert space $L^2(\mu)$. [2] is a standard reference.

Proposition 5. *Let $(\sigma,\mu) \in \Theta_\alpha$. The Eigenproblem 3 has countably many solutions $(\lambda_i, w_i)_i$ with real nonnegative eigenvalues $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots$ and $\mu-$orthogonal eigenfunctions, satisfying Neumann boundary conditions $w_i'(0) = w_i'(1) = 0$. The smallest positive eigenvalue λ_1 is simple, the corresponding eigenfunction $w_1 \in C^{1,\alpha}$ and is strictly monotone.*

Proof. It is easy to check that for any (σ,μ) $\lambda_0 = 0$ and $w_0 \equiv 1$ form an eigenpair. Let $L_0^2(\mu) = \{v \in L^2(\mu) : \int_0^1 v(x)\mu(x)dx = 0\}$ and $H_0^1(\mu) = L_0^2(\mu) \cap H^1$. $L_0^2(\mu)$ with the $L^2(\mu)$ inner product and $H_0^1(\mu)$ with $\langle u,v\rangle_{H^1(\mu)} = \langle u,v\rangle_{L_0^2(\mu)} + \int_0^1 u'(x)v'(x)\mu(x)dx$ are Hilbert spaces. The identity embedding $I : H_0^1(\mu) \to L_0^2(\mu)$ is compact.

For $u,v \in H_0^1(\mu)$ let

$$l(u,v) = \int_0^1 u'(x)v'(x)\sigma^2(x)\mu(x)dx.$$

l is a symmetric positive-definite bilinear form on $H_0^1(\mu) \times H_0^1(\mu)$. Furthermore, for any $u \in H_0^1(\mu)$ holds

$$c\|u\|^2_{H_0^1(\mu)} \le l(u,u) \le C\|u\|^2_{H_0^1(\mu)} \tag{3}$$

for some constants $0 < c < C$ that depend only on d, D. Indeed, since σ and μ are uniformly bounded, we only have to show that $\int_0^1 u^2(x)dx \le \int_0^1 (u'(x))^2 dx$. Consider $u \in C^1([0,1]) \cap H_0^1(\mu)$. Since u is continuous and integrates to zero, there exists $x_0 \in [0,1]$ s.t. $u(x_0) = 0$. Since $u(x) = \int_{x_0}^x u'(y)dy$, the inequality $\|u\|_{L^2} \le \|u'\|_{L^2}$ easily follows from the Cauchy-Schwarz inequality. Since continuous functions are dense in H^1 we conclude that (3) holds.

l is the Dirichlet form of an unbounded operator T on $L_0^2(\mu)$. Define $D = \mathrm{dom}(T)$ as these $u \in H_0^1(\mu)$ that the functional $v \mapsto l(u,v)$ is continuous on $H_0^1(\mu)$ with norm $\|\cdot\|_{L^2(\mu)}$. By the definition of the weak differentiability $D = \{u : H_0^1(\mu) : u'\sigma^2\mu \in H^1\}$. Furthermore, D is dense in $L_0^2(\mu)$ (see [2, Exercise 4.51]). For $u \in D$ we define Tu via Riesz representation theorem by $l(u,v) = \langle Tu, v\rangle_{L^2(\mu)}$. Such defined T is an elliptic, densely defined, self-adjoint operator with compact resolvent (see [2, Proposition 4.17]). Consequently, T has a discrete spectrum $(\lambda_i)_{i=1,\dots}$, with all eigenvalues positive and corresponding eigenfunctions $\mu-$orthogonal.

Integrating by parts the right hand side of (2), we obtain

$$\int_0^1 w_i'(x)\sigma^2(x)\mu(x)v'(x)dx = -2\lambda_i \int_0^1 \int_0^x w_i(y)\mu(y)dyv'(x)dx \text{ for all } v \in H^1.$$

Since $\{v' : v \in H^1\}$ is dense in L^2, it follows that

$$w_i'(x) = \frac{2\lambda_i \int_0^x w_i(y)\mu(y)dy}{\sigma^2(x)\mu(x)} \in C^{0,\alpha}. \tag{4}$$

Since the eigenfunctions $\mu-$integrate to zero, we deduce $w_i'(0) = w_i'(1) = 0$.

Finally, we need to show that λ_1 is simple and the corresponding eigenfunction is strictly monotone. By the variational formula for the eigenpairs of a self-adjoint operator

$$2\lambda_1 = \inf_{u\in H_0^1(\mu)} \frac{\int_0^1 (u'(x))^2\sigma^2(x)\mu(x)dx}{\int_0^1 u^2(x)\mu(x)dx}. \tag{5}$$

Arguing as in [4, Lemma 6.1], we obtain that $\int_0^1 u^2(x)\mu(x)dx = \int_0^1\int_0^1 m(y,z)u'(y)u'(z)dydz$ with $m(y,z) = \int_0^{y\wedge z}\mu(x)dx \int_{y\vee z}^1 \mu(x)dx$. We deduce, that the eigenfunction w_1 must have the derivative of a constant sign, otherwise we could reduce the ratio in (5) by considering

$$\widetilde{w}_1 = w_1\mathbf{1}(w_1' \geq 0) - w_1\mathbf{1}(w_1' \leq 0).$$

Hence, the set $\{x : w_1'(x) = 0\}$ has zero Lebesgue measure. From (4) follows that $w_1'(x) = 0$ only for $x = 0, 1$, meaning that w_1 is strictly monotone on $(0, 1)$. Consequently. for any two eigenfunctions w_1 and $\bar{w}_1$, that correspond to λ_1, the scalar product

$$\int_0^1 w_1(x)\bar{w}_1(x)\mu(x)dx = \int_0^1\int_0^1 m(y,z)w_1'(y)\bar{w}_1'(z)dydz \neq 0,$$

hence the eigenspace corresponding to λ_1 is one dimensional. □

Proposition 6. *The eigenvalues λ_1, λ_2 and $\|w_1\|_{C^{1,\alpha}}/\|w_1\|_{L^2(\mu)}$ are uniformly bounded for all $(\sigma,\mu) \in \Theta_\alpha$. Furthermore, for every $0 < a < b < 1$, $\inf_{x\in[a,b]} |w_1'(x)|$ and the spectral gap $\lambda_2 - \lambda_1$ have uniform lower bounds on Θ_α.*

Proof. We adapt the notation from the proof of Proposition 5. Choose w_1 normalized $\|w_1\|_{L^2(\mu)} = 1$. We will first argue that λ_1, λ_2 and $\|w_1\|_{C^{1,\alpha}}$ are uniformly bounded on Θ_α. From (3), follows that

$$\lambda_1 = l(w_1, w_1) \geq c\|w_1\|^2_{H^1(\mu)} \geq c,$$

with $c > 0$ depending only on the bounds on σ and μ. It follows that the eigenvalues are uniformly separated from zero. By the variational formula

$$2\lambda_2 = \inf_{\substack{S\subset H^1\\ \dim(S)=3}} \sup_{u\in S} \frac{\int_0^1 (u'(x))^2\sigma^2(x)\mu(x)dx}{\int_0^1 u^2(x)\mu(x)dx} \leq \inf_{\substack{S\subset H^1\\ \dim(S)=3}} \sup_{u\in S} \frac{D^3\int_0^1 (u'(x))^2dx}{d\int_0^1 u^2(x)dx} \leq 4\pi^2\frac{D^3}{d},$$

since $4\pi^2$ is the third eigenvalue of the negative Laplace operator on $L^2([0,1])$ with Neumann boundary conditions. We conclude that the eigenvalues λ_1 and λ_2 are uniformly bounded. The uniform bound on $\|w_1\|_{C^{1,\alpha}}$ follows from the representation (4) and $\|\sigma\|_{C^1} \vee \|\mu\|_{C^{0,\alpha}} \leq D$.

We will now prove a uniform lower bound on the spectral gap $\lambda_2 - \lambda_1$. Assume by contradiction that for some sequence of coefficients $(\sigma_n, \mu_n) \in \Theta_\alpha$ the corresponding spectral gaps $(\lambda_{n,2} - \lambda_{n,1})$ converge to zero. Since Θ_α is compact in the uniform convergence metric, we can assume that (σ_n, μ_n) converges uniformly to some $(\sigma, \mu) \in \Theta_\alpha$. We will argue that the uniform convergence of coefficients leads to convergence of the eigenvalues, hence contradicts Proposition 5 (cf. [4, proof of Proposition 6.5]). However, since the function μ is embedded in the definition of spaces $L_0^2(\mu)$ and $H_0^1(\mu)$, we need first to reduce the Eigenproblem 3 to a universal function space.

Let $U(x) = \int_0^x \mu(y)dy$ be the antiderivative of μ. Since μ is a probability function, U is the distribution function, hence an increasing map of the interval $[0,1]$ into itself. Substituting $U(x) = y$, we find that the Eigenproblem 3 is equivalent to

$$\begin{aligned} \int_0^1 \widetilde{w}'(x)\widetilde{v}'(x)\widetilde{\sigma}^2 dx &= 2\lambda \int_0^1 \widetilde{w}'(x)\widetilde{v}'(x)dx \text{ for all } \widetilde{v} \in H^1 \\ \widetilde{w} &= w \circ U^{-1}, \end{aligned}$$

with $\widetilde{\sigma} = (\sigma\mu) \circ U^{-1}$. Consider $(\widetilde{\sigma}_n)_n$ and $\widetilde{\sigma}$ corresponding to (σ_n, μ_n) and (σ, μ) respectively. Note that $\widetilde{\sigma}_n$ converges to $\widetilde{\sigma}$ in $C^1[(0,1)]$. Denote $L_0^2 = L_0^2(1)$ and $H_0^1 := H_0^1(1)$. For $u, v \in H_0^1$ denote

$$\widetilde{l}_n(u,v) = \int_0^1 u'(x)v'(x)\widetilde{\sigma}_n(x)^2 dx$$

and by $\widetilde{T}_n$ the corresponding operators on L_0^2. Recall that the operators $\widetilde{T}_n$ are unbounded and self-adjoint on L_0^2, with dense domains $\widetilde{D}_n$. Since $\widetilde{D}_n$ do not have to possess a common core, which is needed to study convergence of the sequence $(\widetilde{T}_n)_n$, we introduce inverse operators $\widetilde{R}_n = \widetilde{T}_n^{-1}$. Using the divergence formula (1) for $\widetilde{T}_n$, we check that for $u \in L_0^2$

$$\widetilde{R}_n u(x) = -2 \int_0^x \widetilde{\sigma}_n^{-2}(y) \int_0^y u(z)dz + c_n(u), \tag{6}$$

where $c_n(u) \in \mathbb{R}$ is such that $\int_0^1 \widetilde{R}_n u(x)dx = 0$. The convergence $\widetilde{\sigma}_n \to \widetilde{\sigma}$ in $C^1[(0,1)]$ implies that operators $\widetilde{R}_n$ converge to $\widetilde{R}$ in the operator norm on L_0^2. By [2, Proposition 5.28] this entails regular convergence, which by [2, Theorem 5.20] is equivalent to the strongly stable convergence. Finally [2, Proposition 5.6] ensures the convergence of the eigenvalues with preservation of their multiplicities.

Set $0 < a < b < 1$. We finally have to prove the uniform lower bound on $\inf_{x\in[a,b]} |w_1'(x)|$. We will use the same indirect arguments as when bounding the spectral gap. Assume that for some sequence $(\sigma_n, \mu_n) \in \Theta_\alpha$, with (σ_n, μ_n) converging in the uniform norm to $(\sigma, \mu) \in \Theta_\alpha$, the corresponding eigenfunctions $w_{1,n}$ satisfy $\inf_n \inf_{x\in[a,b]} |w_{1,n}'(x)| = 0$. Arguing as for the spectral gap, we reduce the problem to bounded operators $(\widetilde{R}_n)_n$ and $\widetilde{R}$. From formula (6) we deduce that the uniform convergence of coefficients corresponds to convergence of $\widetilde{R}_n$ to $\widetilde{R}$ in the operator norm on $C([0,1])$. We conclude, that the eigenfunctions converge in the uniform norm, which contradicts Proposition 5 for the limit eigenproblem for (σ, μ). □

Eigenproblem 7. Let V_J be a finite dimensional subspace of L^2. Find $(\lambda_J, w_J) \in \mathbb{R} \times V_J$, with $w_J \neq 0$ such that

$$\int_0^1 w'(x)v'(x)\sigma^2(x)\mu(x)dx = \lambda \int_0^1 w(x)v(x)\mu(x)dx \text{ for any } v \in V_J.$$

Proposition 8. *Let $(V_J)_{J=1,\dots}$ be a sequence of approximation spaces satisfying the following Jackson's type inequality:*

$$\|(I-\pi_J)v\|_{H^1} \leq CJ^{-\alpha}\|v\|_{C^{1,\alpha}} \text{ for } v \in C^{1,\alpha},$$

where π_J is the L^2-orthogonal projection on V_J and $C > 0$ some universal constant . Furthermore, assume that every V_J contains constant functions.

For $(\sigma, \mu) \in \Theta_\alpha$ the Eigenproblem 7 has $\dim(V_J)$ solutions $(\lambda_{J,i}, w_{J,i})_i$ with real eigenvalues $0 = \lambda_{J,0} < \lambda_{J,1} < \lambda_{J,2} \leq \dots \leq \lambda_{J,\dim(V_J)-1}$. For J big enough, the eigenvalue $\lambda_{J,1}$ and the spectral gap $\lambda_{J,2} - \lambda_{J,1}$ are uniformly bounded on Θ_α.

Proof. We adapt the notation from the proof of Proposition 5. By the Lax-Milgram theorem, there exists an isomorphism $S_l : H_0^1(\mu) \to H_0^1(\mu)$ such that

$$l(S_l v, u) = \langle v, u\rangle_{H^1(\mu)}, \text{ for all } v, u \in H_0^1(\mu).$$

Note that since for any $v \in L_0^2(\mu)$ the functional $H_0^1(\mu) \ni u \longmapsto \langle v, u\rangle_{L^2(\mu)} \in \mathbb{R}$ is continuous on $H_0^1(\mu)$, by the Riesz representation theorem there exists a continuous operator $J : L_0^2(\mu) \to H_0^1(\mu)$ such that

$$\langle v, u\rangle_{L^2(\mu)} = \langle Jv, u\rangle_{H^1(\mu)}.$$

Define operator $B_l = S_l \circ J \circ I$, where I is the identity embedding of $H_0^1(\mu)$ into $L_0^2(\mu)$. By (3), the form l defines an equivalent norm on $H_0^1(\mu)$. Note that B_l is a self-adjoint and compact operator on the Hilbert space $H_0^1(\mu)$ with $l-$induced inner product. Consider (λ_i, w_i), a solution of the Eigenproblem 3. For any $v \in H_0^1(\mu)$ we have

$$l(w_i, v) = \lambda_i\langle w_i, v\rangle_{L^2(\mu)} = \lambda_i\langle Jw_i, v\rangle_{H^1(\mu)} = \lambda_i l(S_l J w_i, v) = l(\lambda_i B_l w_i, v),$$

hence (λ_i^{-1}, w_i) is an eigenpair of the operator B_l. In particular, Proposition 5 implies that the biggest eigenvalue λ_1^{-1} is simple.

Denote by π_J^l the $l-$orthogonal projection on the subspace V_J. Define the operator $B_{l,J} = \pi_J^l B_l \pi_J^l$. Since $B_{l,J}$ is a self-adjoint operator on V_J, with the $l-$induced inner product, it has $\dim(V_J) - 1$ solutions $(\lambda_{J,i}^{-1}, w_{J,i})_i$, with eigenvalues $\lambda_{J,1}^{-1} \geq \lambda_{J,2}^{-1} \geq \dots \geq \lambda_{J,\dim(V_J)-1}^{-1}$. Analogously as for the operator B_l, we check that $(\lambda_{J,i}, w_{J,i})$ are solutions of the finite dimensional Eigenproblem 7. From (3) and the uniform bound on μ, follows that

$$\|(I-\pi_n^l)w_1\|_l \leq \|(I-\pi_n^l)(I-\pi_J)w_1\|_l \leq 2\|(I-\pi_J)w_1\|_l \leq C\|(I-\pi_J)w_1\|_{H^1},$$

for some, uniform on Θ_α, constant C. Using Jackson's inequality, the uniform bound on $\|w_1\|_{C^{1,\alpha}}$ and uniform bounds on the eigenvalues λ_1, λ_2, we conclude that for J large enough

$$\|(I-\pi_n^l)w_1\|_l < \frac{\lambda_1^{-1} - \lambda_2^{-1}}{6\lambda_1^{-1}}.$$

The claim follows from Theorem 1. □

3 Generalized eigenvalue problem for positive definite symmetric matrix pairs

In this section, we briefly state the error bounds for the perturbed generalized eigenproblem for real symmetric matrices. For the general theory of generalized matrix eigenproblems we refer to [7, Chapter VI]. For an overview of the a posteriori methods for the matrix eigenvalue problems we refer to [2, Chapter 1] or [7, Chapter V].

Let $A, B \in \mathbb{R}^{n\times n}$ be real, symmetric matrices with B positive definite. Consider the following generalized symmetric eigenvalue problem:

Eigenproblem 9. Find $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$, with $x \neq 0$, such that

$$Ax = \lambda Bx.$$

Using Cholesky decomposition of the matrix $B = DD^*$, one can reduce the generalized Eigenproblem 9 to a standard eigenvalue problem for the matrix $D^{-1}AD^{-*}$. It follows, that the Eigenproblem 9 has n solutions $(\lambda_i, x_i)_{i=1,..,n}$, all its eigenvalues are real with $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and the corresponding eigenvectors $(x_i)_{i=1,..,n}$ are $B-$orthogonal.

Consider now the perturbed matrices $\widetilde{A}$, $\widetilde{B}$ with $\widetilde{B}$ positive definite, and the corresponding perturbed eigenproblem:

Eigenproblem 10. Find $(\widetilde{\lambda}, \widetilde{x}) \in \mathbb{R} \times \mathbb{R}^n$, with $x \neq 0$, such that

$$\widetilde{A}\widetilde{x} = \widetilde{\lambda}\widetilde{B}\widetilde{x}.$$

Let $\|\cdot\|_{l^2}$ denote the Euclidean norm on $\mathbb{R}^n$. First we state the standard a posteriori error bound for eigenvalues and the eigenvectors (cf. [1, Generalized Hermitian Eigenvalue Problems, Stability and Accuracy Assessments])

Theorem 11. *Let $(\lambda_i, x_i)_i$ and $(\widetilde{\lambda}_i, \widetilde{x}_i)_i$ be solutions of the Eigenproblems 9 and 10 respectively, ordered decreasingly with respect to the eigenvalues. Choose the eigenvectors normalized in $\|\cdot\|_{l^2}$. There exists an eigenpair (λ_{i_0}, x_{i_0}), with $1 \leq i_0 \leq n$ and $\|x_{i_0}\|_{l^2} = 1$ such that*

$$|\lambda_{i_0} - \widetilde{\lambda}_1| \leq \left\|B^{-1}\right\|_{l^2} \|(A - \widetilde{A})\widetilde{x}_1 + \widetilde{\lambda}_1(B - \widetilde{B})\widetilde{x}_1\|_{l^2},$$

$$\|x_{i_0} - \widetilde{x}_1\|_{l^2} \leq \frac{2\sqrt{2\kappa(B)}}{\delta(\lambda_{i_0})} \left\|B^{-1}\right\|_{l^2} \|(A - \widetilde{A})\widetilde{x}_1 + \widetilde{\lambda}_1(B - \widetilde{B})\widetilde{x}_1\|_{l^2},$$

where $\kappa(B) = \|B\|_{l^2}\|B^{-1}\|_{l^2}$ is the condition number of the matrix B and $\delta(\lambda_{i_0})$ is the so called localizing distance, i.e. $\delta(\lambda_{i_0}) = \min_{j\neq i_0} |\lambda_j - \widetilde{\lambda}_1|$.

The disadvantage of the above result is that it provides no information about the index i_0 of the best approximation of (λ_1, x_1). This is a typical downside for a posteriori methods that are supposed to provide information how far the calculated solution is from the nearest exact solution, but are not intended to compare ordered eigenpairs. A helpful result is the absolute Weyl theorem for generalized hermitian definite pairs, established by Y. Nakatsukasa [5]. For readers convenience, we state the theorem below in the form presented in [6, Theorem 8.3].

Theorem 12. *([5])Let $(\lambda_i, x_i)_i$ and $(\widetilde{\lambda}_i, \widetilde{x}_i)_i$ be solutions of the Eigenproblems 9 and 10 respectively, ordered decreasingly with respect to the eigenvalues. Denote $\Delta A = A - \widetilde{A}$ and $\Delta B = B - \widetilde{B}$. Then*

$$|\lambda_i - \widetilde{\lambda}_i| \quad \leq \quad \|\widetilde{B}^{-1}\|_{l^2}\|\Delta A - \lambda_i \Delta B\|_{l^2},$$

$$|\lambda_i - \widetilde{\lambda}_i| \quad \leq \quad \|B^{-1}\|_{l^2} \|\Delta A - \widetilde{\lambda}_i \Delta B\|_{l^2},$$

for all $i = 1, ..., n$.

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