L^2 error bounds for the Florens-Zmirou estimator. (Statistik stochastischer Prozesse)

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Definition of the Florens-Zmirou estimator.

Set $0 < m < M$ and define $\Theta(m, M) = \{\sigma \in C^1(\mathbb{R}) : m \le \inf_{x\in\mathbb{R}} \sigma(x) \le \sup_{x\in\mathbb{R}} \sigma(x) \le M, \sup_{x\in\mathbb{R}} |\sigma'(x)| \le M\}$. Note that each $\sigma \in \Theta$ satisfies the global Lipschitz and linear growth conditions, hence the corresponding equation

$$\begin{aligned} dX_t &= \sigma(X_t)dW_t, \\ X_0 &= X^{(0)} \in L^2, \end{aligned}$$

has a unique strong solution. For $\Delta > 0$ we observe a path $t \to X_t$ at equidistant times $0, \Delta, 2\Delta, ..., N\Delta = 1$. When $x \in \mathbb{R}$ is visited by the observed path (i.e. $X_t = x$ for some $t \in (0, 1)$) we define the Florens-Zmirou ([Florens-Zmirou, 1993]) estimator of the diffusion coefficient σ by

$$\sigma^2_{FZ}(x, h_\Delta) = \frac{\sum_{n=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}} \frac{1}{\Delta}(X_{(n+1)\Delta} - X_{n\Delta})^2}{\sum_{n=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}}}.$$

Local Time.

For any Borel set A define the occupation measure of the set A as $\mu(A) = \int_0^1 \mathbf{1}_A(X_s)ds$, i.e. the amount of time the path $(X_t)_{0\le t\le 1}$ stayed in A. Then measure μ has Lebesgue density L ([Revuz and Yor, 1999], [Björk, 2013]) called the local time (chronological local time) of X at time one . For every positive Borel measurable function f the occupation formula: $\int_0^1 f(X_s)ds = \int_{\mathbb{R}} f(x)L(x)dx$ holds.

Lemma. *For every $p > 2$ holds $\sup_{(\sigma,b)\in\Theta} \mathbb{E}_\sigma[L^p(x)] < C_p$.*

Proof. By the Tanaka formula $L(x) = |X_1 - x| - |X_0 - x| - \int_0^1 \operatorname{sgn}(X_s - x)dX_s \le |X_1 - X_0| + |\int_0^1 \operatorname{sgn}(X_s - x)dX_s|$. Using the Burkholder-Davis-Gundy inequality we obtain

- $\mathbb{E}_\sigma[|X_1 - X_0|^p] \le \mathbb{E}_\sigma\big[|\int_0^1 \sigma(X_s)dW_s|^p\big] \le \widetilde{C}_p\mathbb{E}_\sigma\big[|\int_0^1 \sigma^2(X_s)ds|^{\frac{p}{2}}\big] \le \widetilde{C}_p M^p$.
- $\mathbb{E}_\sigma\big[|\int_0^1 \operatorname{sgn}(X_s - x)dX_s|^p\big] \le \widetilde{C}_p\mathbb{E}_\sigma\big[|\int_0^1 \operatorname{sgn}^2(X_s - x)\sigma^2(X_s)ds|^{\frac{p}{2}}\big] \le \widetilde{C}_p M^p$.

□

Error of the Florens-Zmirou estimator.

Theorem 1. *Consider interval K, some positive $\nu > 0$ and let $\mathcal{L} = \{\inf_{x\in K} L_T(x) \ge \nu\}$. Let $h_\Delta \sim \Delta^{\frac{1}{3}}$. Then for every $x \in int(K)$ holds*

$$\sup_{\sigma\in\Theta} \mathbb{E}_\sigma\big[\mathbf{1}_{\mathcal{L}} \cdot |\sigma^2_{FZ}(x, h_\Delta) \wedge M^2 - \sigma^2(x)|^2\big] \le C\Delta^{\frac{2}{3}},$$

where the constant C depends only on the set K and level ν.

Notation. We will write $f_\sigma \lesssim g_\sigma$ (resp. $g_\sigma \gtrsim f_\sigma$) if for every $\sigma \in \Theta$ holds $f_\sigma \le C \cdot g_\sigma$ with constant $C > 0$ depending only on K and ν.

Proof.

1 (Bias and martingale part) For $n = 0, ..., N-1$ define

$$\eta_n = \frac{1}{\Delta}\Big(\int_{n\Delta}^{(n+1)\Delta} \sigma(X_s)dW_s\Big)^2 - \frac{1}{\Delta}\int_{n\Delta}^{(n+1)\Delta} \sigma^2(X_s)ds.$$

- $\mathbb{E}_\sigma[\eta_n|\mathcal{F}_n] = 0$ and in particular $\mathbb{E}_\sigma[\eta_n\eta_m] = 0$ for $n \neq m$.
- $\mathbb{E}_\sigma[\eta_n^2|\mathcal{F}_n] \lesssim 1$. Indeed, by the Burkholder-Davies-Gundy inequality:

$$\begin{aligned}\Delta^2\mathbb{E}_\sigma[\eta_n^2|\mathcal{F}_n] &\lesssim \mathbb{E}_\sigma\Big[\Big(\int_{n\Delta}^{(n+1)\Delta} \sigma(X_s)dW_s\Big)^4|\mathcal{F}_n\Big] + \mathbb{E}_\sigma\Big[\Big(\int_{n\Delta}^{(n+1)\Delta} \sigma^2(X_s)ds\Big)^2|\mathcal{F}_n\Big] \\ &\lesssim \mathbb{E}_\sigma\Big[\Big(\int_{n\Delta}^{(n+1)\Delta} \sigma^2(X_s)ds\Big)^2|\mathcal{F}_n\Big] + \Delta^2 \lesssim \Delta^2.\end{aligned}$$

We decompose the estimation error into martingale and bias parts:

$$\begin{aligned}&|\sigma^2_{FZ}(x, h_\Delta) - \sigma^2(x)| = \\ &= \Big|\frac{\sum_{n=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}} \frac{1}{\Delta}\big(\int_{n\Delta}^{(n+1)\Delta} \sigma(X_s)dW_s\big)^2 - \sigma^2(x)}{\sum_{n=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}}}\Big| \\ &\lesssim \underbrace{\Big|\frac{\sum_{n=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}}\eta_n}{\sum_{n=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}}}\Big|}_{M_{x,\Delta}} + \underbrace{\Big|\frac{\sum_{n=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}}\big(\frac{1}{\Delta}\int_{n\Delta}^{(n+1)\Delta} \sigma^2(X_s)ds - \sigma^2(x)\big)}{\sum_{n=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}}}\Big|}_{B_{x,\Delta}}.\end{aligned}$$

2 (The "good" high-probability set) Denote by $\omega(\Delta)$ the modulus of continuity of the path $(X_t)_{t\in(0,1)}$, i.e.

$$\omega(\Delta) = \sup_{\substack{0 \le s,t \le 1 \\ |t-s| < \Delta}} |X_t - X_s|.$$

Set $0 < \epsilon < 1/6$ and let $\alpha = 3/2 - 3\epsilon > 1$. Define the event $\mathcal{R} = \{\omega(\Delta) < h_\Delta^\alpha\}$. Then for every $p > 1$ holds

$$\mathbb{P}_\sigma(\mathcal{R}^c) \lesssim h_\Delta^{-p\alpha}\Big(\Delta \log\big(2\Delta^{-1}\big)\Big)^{\frac{p}{2}} \lesssim \Delta^{\epsilon p} \log\big(2\Delta^{-1}\big)^{\frac{p}{2}}. \tag{1}$$

In particular $\mathbb{P}_\sigma(\mathcal{R}^c) \lesssim \Delta^{2/3}$ for p big enough.

Proof. (Proof of (1))

Set $p > 0$. By Markov's inequality we just have to show that there exists a constant C_p depending only on p and the upper bound of σ, such that

$$\mathbb{E}_\sigma[\omega(\Delta)^p] \le C_p\Big(\Delta \log\big(\frac{2T}{\Delta}\big)\Big)^{\frac{p}{2}}. \tag{2}$$

- (2) holds for Brownian motion - [Fischer and Nappo, 2010].
- Let $dX_t = \sigma(X_t)dW_t$. By Dambis, Dubins-Schwarz theorem $X_t = B_{\int_0^t \sigma^2(X_s)ds}$ for some Brownian motion B. Consequently

$$|X_t - X_s| = \big|B_{\int_0^t \sigma^2(X_s)ds} - B_{\int_0^s \sigma^2(X_s)ds}\big| \le \omega^B(|t-s|M^2)$$

□

3 (Bias part error) When $|X_{n\Delta} - x| < h_\Delta$ we have

$$\begin{aligned} \frac{1}{\Delta}\int_{n\Delta}^{(n+1)\Delta} |\sigma^2(X_s) - \sigma^2(x)| ds &\lesssim \frac{1}{\Delta}\int_{n\Delta}^{(n+1)\Delta} |X_s - x| ds \\ &\leq \frac{1}{\Delta}\int_{n\Delta}^{(n+1)\Delta} |X_s - X_{n\Delta}| ds + |X_{n\Delta} - x| \\ &\lesssim \omega(\Delta) + h_\Delta. \end{aligned}$$

Consequently $\mathbf{1}_{\mathcal{R}} \cdot B_{x,\Delta} \lesssim h_\Delta$.

4 (Martingale part error) Denote $\sum_{n=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}} = N(x, h_\Delta)$. Then, on the event $\mathcal{R}$ we have

$$\Big|\frac{N(x,h_\Delta)}{Nh_\Delta} - \frac{1}{h_\Delta}\int_{x-h_\Delta}^{x+h_\Delta} L(z)dz\Big| \lesssim \frac{1}{h_\Delta}\int_{\{h_\Delta - h_\Delta^\alpha \leq |z-x| < h_\Delta + h_\Delta^\alpha\}} L(z)dz. \qquad (3)$$

Indeed by the triangle inequality

$$\begin{aligned} &\Big|\frac{1}{N}\sum_{n=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}} - \int_0^1 \mathbf{1}_{\{|X_s-x|<h_\Delta\}} ds\Big| \leq \\ &\leq \sum_{n=0}^{N-1}\int_{n\Delta}^{(n+1)\Delta} \big|\mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}} - \mathbf{1}_{\{|X_s-x|<h_\Delta\}}\big| ds \\ &= \sum_{n=0}^{N-1}\int_{n\Delta}^{(n+1)\Delta} \mathbf{1}_{\{h_\Delta \leq |X_s-x| < h_\Delta+\omega(\Delta)\}} ds \\ &\quad + \sum_{n=0}^{N-1}\int_{n\Delta}^{(n+1)\Delta} \mathbf{1}_{\{h_\Delta-\omega(\Delta) \leq |X_s-x| < h_\Delta\}} ds \\ &= \int_0^1 \mathbf{1}_{\{h_\Delta - h_\Delta^\alpha \leq |X_s-x| < h_\Delta + h_\Delta^\alpha\}} ds \\ &= \int_{\{h_\Delta - h_\Delta^\alpha \leq |z-x| < h_\Delta + h_\Delta^\alpha\}} L(z)dz. \end{aligned}$$

Denote for simplicity $\{z:\ h_\Delta - h_\Delta^\alpha \leq |z-x| < h_\Delta + h_\Delta^\alpha\} = A$ and observe that the Lebesgue measure of A is $4h_\Delta^\alpha$. Using first Markov's and next Hölder's inequalities we obtain

$$\mathbb{P}_\sigma\Big(\frac{1}{h_\Delta}\int_A L(z)dz \geq c\Big) \lesssim \mathbb{E}_\sigma\Big[\frac{1}{h_\Delta^p}\big(\int_A L(z)dz\big)^p\Big] \lesssim \frac{h_\Delta^{\alpha(p-1)}}{h_\Delta^p}\int_A \mathbb{E}_\sigma[L^p(z)]dz \lesssim h_\Delta^{(\alpha-1)p} \lesssim \Delta^{\frac{2}{3}}$$

for p big enough. Consequently there exists a high probability event $Q \subset \mathcal{R}$, $\mathbb{P}_\sigma(Q^c) \lesssim \Delta^{2/3}$, such that $\frac{N(x,h_\Delta)}{Nh_\Delta}$ is bounded from below on $Q \cap \mathcal{L}$. Now using martingale properties of η_n we obtain:

$$\begin{aligned} \mathbb{E}_\sigma\Big[\mathbf{1}_{Q\cap\mathcal{L}} \cdot M_{x,\Delta}^2\Big] &= \mathbb{E}_\sigma\Big[\Big(\frac{1}{N(x,h_\Delta)}\sum_{n=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}}\eta_n\Big)^2 \cdot \mathbf{1}_{Q\cap\mathcal{L}}\Big] \\ &\lesssim \frac{1}{N^2h_\Delta^2}\mathbb{E}_\sigma\Big[\Big(\sum_{n=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}}\eta_n\Big)^2 \mathbf{1}_{Q\cap\mathcal{L}}\Big] \\ &\lesssim \frac{1}{N^2h_\Delta^2}\mathbb{E}_\sigma\Big[\sum_{n,m=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}}\mathbf{1}_{\{|X_{m\Delta}-x|<h_\Delta\}}\eta_n\eta_m\Big] \\ &= \frac{1}{N^2h_\Delta^2}\mathbb{E}_\sigma\Big[\sum_{n=0}^{N-1} \mathbf{1}_{\{|X_{n\Delta}-x|<h_\Delta\}}\mathbb{E}_\sigma[\eta_n^2|\mathcal{F}_n]\Big] \end{aligned}$$

$$\lesssim \frac{1}{N^2 h_\Delta^2}\mathbb{E}_\sigma\Big[N(x,h_\Delta)\Big].$$

Finally

$$\begin{aligned}
\frac{1}{Nh_\Delta}\mathbb{E}_\sigma\Big[N(x,h_\Delta)\Big] &\lesssim \frac{1}{Nh_\Delta}\mathbb{E}_\sigma\Big[N(x,h_\Delta)\mathbf{1}_{\mathcal{R}}\Big] + \frac{1}{Nh_\Delta}\mathbb{E}_\sigma\Big[N(x,h_\Delta)\mathbf{1}_{\mathcal{R}^c}\Big] \\
&\lesssim \mathbb{E}_\sigma\Big[\frac{1}{h_\Delta}\int_{x-h_\Delta}^{x+h_\Delta} L(z)dz + \frac{1}{h_\Delta}\int_A L(z)dz\Big] + h_\Delta^{-1}\mathbb{P}_\sigma(\mathcal{R}^c) \\
&\lesssim \frac{1}{h_\Delta}\int_{(x-h_\Delta,x+h_\Delta)\cup A} \mathbb{E}_\sigma[L(z)]dz + h_\Delta^{-1}\Delta^{\frac{2}{3}} \\
&\lesssim 1.
\end{aligned}$$

5 (Conclusion) We have shown

$$\mathbb{E}_\sigma[\mathbf{1}_{\mathcal{L}\cap Q}\cdot|\sigma^2_{FZ}(x,h_\Delta)-\sigma^2(x)|^2] \lesssim \mathbb{E}_\sigma[\mathbf{1}_{\mathcal{L}\cap Q}\cdot M^2_{x,\Delta} + \mathbf{1}_{\mathcal{R}}\cdot B^2_{x,\Delta})] \lesssim \frac{1}{Nh_\Delta} + h_\Delta^2 \sim \Delta^{\frac{2}{3}}.$$

Furthermore

$$\mathbb{E}_\sigma[\mathbf{1}_{\mathcal{L}\cap Q^c}\cdot|\sigma^2_{FZ}(x,h_\Delta)\wedge M^2-\sigma^2(x)|^2] \lesssim \mathbb{P}_\sigma(Q^c) \lesssim \Delta^{\frac{2}{3}}.$$

Corollary. *Let $\Theta^* = \Theta(m,M)\times\{b\in C(\mathbb{R}) : b$ is Lipschitz and $\sup_{x\in\mathbb{R}} b(x)\le M\}$. For $(\sigma,b)\in\Theta^*$ consider diffusion Y defined by the SDE: $dY_t = b(Y_t)dt+\sigma(Y_t)dW_t$, $Y_0 = x_0$. Then for the event $\mathcal{L}$ and x defined as before, given that $h_\Delta\sim\Delta^{\frac{1}{3}}$, we have*

$$\sup_{(\sigma,b)\in\Theta^*} \mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{L}}\cdot|\sigma^2_{FZ}(x,h_\Delta)\wedge M^2-\sigma^2(x)|\big] \le C(\mathcal{L})\Delta^{\frac{1}{3}}.$$

Proof. Using boundedness of coefficients b and σ one can easily verify the assumptions of Girsanov theorem. The laws of diffusions X and Y on $C([0,1])$ are equivalent and

$$\frac{dP_Y}{dP_X}(X) = \exp\Big(\int_0^1 \frac{b(X_s)}{\sigma^2(X_s)}dX_s - \frac{1}{2}\int_0^1 \frac{b^2(X_s)}{\sigma^2(X_s)}ds\Big) = \exp\Big(\int_0^1 \frac{b(X_s)}{\sigma(X_s)}dW_s - \frac{1}{2}\int_0^1 \frac{b^2(X_s)}{\sigma^2(X_s)}ds\Big).$$

Denote $\mathbf{1}_{\mathcal{L}}\cdot|\sigma^2_{FZ}(x,h_\Delta)\wedge M^2-\sigma^2(x)| = \mathcal{E}_{x,\Delta}$. By Cauchy-Schwarz we obtain

$$\begin{aligned}
\mathbb{E}_{\sigma,b}\big[\mathcal{E}_{x,\Delta}\big] &= \mathbb{E}_\sigma\Big[\mathcal{E}_{x,\Delta}\frac{dP_Y}{dP_X}(X)\Big] \\
&= \mathbb{E}_\sigma\Big[\mathcal{E}_{x,\Delta}\exp\Big(\int_0^1 \frac{b(X_s)}{\sigma(X_s)}dW_s - \frac{1}{2}\int_0^1\frac{b^2(X_s)}{\sigma^2(X_s)}ds\Big)\Big] \\
&\le \mathbb{E}_\sigma\Big[\mathcal{E}_{x,\Delta}\exp\Big(\int_0^1 \frac{b(X_s)}{\sigma(X_s)}dW_s\Big)\Big] \\
&\le \mathbb{E}_\sigma[\mathcal{E}^2_{x,\Delta}]^{\frac{1}{2}}\mathbb{E}_\sigma\Big[\exp\Big(2\int_0^1 \frac{b(X_s)}{\sigma(X_s)}dW_s\Big)\Big]^{\frac{1}{2}}.
\end{aligned}$$

We just have to argue that $\mathbb{E}_\sigma\Big[\exp\Big(\int_0^1 \frac{2b(X_s)}{\sigma(X_s)}dW_s\Big)\Big]$ is uniformly bounded. Since

$$\mathbb{E}_\sigma\Big[\exp\Big(\int_0^1 2(b\sigma^{-1})^2(X_s)ds\Big)\Big] < \infty$$

by the Novikov's condition the process $M_t = \exp\Big(\int_0^t 2(b\sigma^{-1})(X_s)dW_s - \int_0^t 2(b\sigma^{-1})^2(X_s)ds\Big)$ is a martingale and consequently

$$\mathbb{E}_\sigma\Big[\exp\Big(\int_0^1 2(b\sigma^{-1})(X_s)dW_s\Big)\Big] = \mathbb{E}_\sigma\Big[\exp\Big(\int_0^1 2(b\sigma^{-1})^2(X_s)ds\Big)\Big].$$

□

Theorem. *(Florens-Zmirou,1993)*
Let X satisfy

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad t\in[0,1],$$

where b is a bounded function, with two bounded derivatives, σ has three continuous and bounded derivatives and furthermore $m < \sigma < M$ for some positive $0 < m < M$. If Nh_Δ^3 tends to zero, then

$$\sqrt{Nh_\Delta}\Big(\frac{\sigma_{FZ}(x,h_\Delta)}{\sigma^2(x)} - 1\Big) \xrightarrow{D} L(x)^{-1/2}Z,$$

where Z is a standard normal variable independent of $L(x)$.

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