Supplement A: Construction and properties of a reflected diffusion

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1 Construction

Assumption 1. *For given constants $0 < d < D$ let the pair $(\sigma, b) \in \Theta$, where*

$$\Theta := \Theta(d, D) = \{(\sigma, b) \in C^1([0,1]) \times C^1([0,1]) : \|b\|_\infty \vee \|\sigma^2\|_\infty \vee \|\sigma'\|_\infty < D, \inf_{x \in [0,1]} \sigma^2(x) \geq d\}.$$

For $(\sigma, b) \in \Theta$ consider the following Skorokhod type stochastic differential equation:

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t + dK_t, \\ X_0 &= x_0 \in [0,1] \text{ and } X_t \in [0,1] \text{ for every } t \geq 0, \end{aligned} \tag{1}$$

where $(W_t, t \geq 0)$ is a standard Brownian motion and $(K_t, t \geq 0)$ is some adapted continuous process with finite variation, starting form 0, and such that for every $t \geq 0$ holds $\int_0^t \mathbf{1}_{(0,1)}(X_s)dK_s = 0$. By the Engelbert-Schmidt theorem the SDE (1) has a weak solution, see [13, Thm. 4.1]. In this section, we will present an explicit construction of a strong solution. To that end we extend the coefficients b, σ to the whole real line.

Definition 2. Define $f : \mathbb{R} \to [0,1]$ by

$$f(x) = \begin{cases} x - 2n & : 2n \leq x < 2n+1 \\ 2(n+1) - x & : 2n+1 \leq x < 2n+2 \end{cases}$$

and $\widetilde{b}, \widetilde{\sigma} : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} \widetilde{b}(x) &= b(f(x))f'_-(x) \\ \widetilde{\sigma}(x) &= \sigma(f(x)). \end{aligned}$$

Theorem 3. *For every initial condition $x_0 \in [0,1]$, independent of the driving Brownian motion W, the SDE*

$$\begin{aligned} dY_t &= \widetilde{b}(Y_t)dt + \widetilde{\sigma}(Y_t)dW_t, \\ Y_0 &= x_0, \end{aligned} \tag{2}$$

has a non-exploding unique strong solution. Define

$$X_t = f(Y_t).$$

The process $(X_t, t \geq 0)$ is a strong solution of the SDE (1).

Proof. The existence of a non-exploding unique strong solution $(Y_t, t \geq 0)$ of the SDE (2) follows from [8, Proposition 5.17]. Process Y is a continuous semimartingale, hence by [12, Chapter VI Theorem 1.2] admits a local time process $(L_t^Y, t \geq 0)$. By the Itô-Tanaka formula ([12, Chapter VI Theorem 1.5]) process X satisfies

$$\begin{aligned} X_t &= x_0 + \int_0^t \widetilde{b}(Y_s) f'_-(Y_s) ds + \int_0^t \widetilde{\sigma}(Y_s) f'_-(Y_s) dW_s + \sum_{n \in \mathbb{Z}} L_t^Y(2n) - \sum_{n \in \mathbb{Z}} L_t^Y(2n+1) \\ &= x_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s + K_t, \end{aligned}$$

where $B_t = \int_0^t f'_-(Y_s) dW_s$ and $K_t = \sum_{n \in \mathbb{Z}} L_t^Y(2n) - \sum_{n \in \mathbb{Z}} L_t^Y(2n+1)$. Note that for any $T > 0$ the path $(X_t, 0 \leq t \leq T)$ is bounded, hence K is well defined. Using Lévy's characterization theorem we find that B is a standard Brownian motion. From the properties of the local time L_t^Y follows that K is an adapted continuous process with finite variation, starting from zero and varying on the set $\bigcup_{n \in \mathbb{Z}} \{Y_t = 2n\} \cup \{Y_t = 2n+1\} \subseteq \{X_t \in \{0,1\}\}$. Consequently, X is a strong solution of the SDE (1). □

Notation. We will write $f \lesssim g$ (resp. $g \gtrsim f$) when $f \leq C \cdot g$ for some universal constant $C > 0$. $f \sim g$ is equivalent to $f \lesssim g$ and $g \lesssim f$.

From now on we take the Assumption (1) as granted. We denote by $\mathbb{P}_{\sigma,b}$ the law of the diffusion X on the canonical space Ω of continuous functions over the positive axis with values in $[0,1]$, equipped with the topology of the uniform convergence on compact sets and endowed with its $\sigma-$field $\mathcal{F}$. We denote by $\mathbb{E}_{\sigma,b}$ the corresponding expectation operator.

2 Modulus of continuity

In this Section we want to prove a uniform upper bound on the moments of the modulus of continuity of the reflected diffusion X.

Definition 4. Denote by ω_T the modulus of continuity of the path $(X_t, 0 \leq t \leq T)$, i.e.

$$\omega_T(\delta) = \sup_{\substack{0 \leq s,t \leq T \\ |t-s| \leq \delta}} |X_t - X_s|.$$

Theorem 5. *For every $p \geq 1$ there exists constant $C_p > 0$ s.t.*

$$\sup_{(\sigma,b) \in \Theta} \mathbb{E}_{\sigma,b}[\omega_T^p(\Delta)] \leq C_p \Delta^{p/2} (1 \vee \ln(2T/\Delta))^p \tag{3}$$

Proof. Fischer and Nappo [3] proved the above bound for the standard Brownian motion. We will now generalize their result to diffusions with boundary reflection.

Step 1. Consider a martingale M with $dM_t = \sigma(X_t) dW_t$. By Dambis, Dubins-Schwarz theorem $M_t = B_{\int_0^t \sigma^2(X_u) du}$ for some Brownian motion B. Consequently

$$|M_t - M_s| = \left| B_{\int_0^t \sigma^2(X_u) du} - B_{\int_0^s \sigma^2(X_u) du} \right| \leq \omega^B(|t-s| \|\sigma^2\|_\infty),$$

where ω^B is the modulus of continuity of B. Thus (3) holds for the martingale M, with a constant that depends only on the upper bound on the volatility σ.

Step 2. Consider a semimartingale X with $dX_t = b(X_t)dt + dM_t$. Then

$$|X_t - X_s| \leq \Big| \int_0^t b(X_u)du - \int_0^s b(X_u)du \Big| + |M_t - M_s| \leq |t-s| \|b\|_\infty + \omega^M(|t-s|).$$

Consequently (3) holds for semimartingales with a constant that depends only on the upper bounds on σ and b.

Step 3. For $(\sigma, b) \in \Theta$ consider the reflected diffusion process X satisfying the SDE (1). Let

$$\begin{aligned} dY_t &= \widetilde{b}(Y_t)dt + \widetilde{\sigma}(Y_t)dW_t, \\ X_t &= f(Y_t), \end{aligned}$$

where $\widetilde{b}, \widetilde{\sigma}$ and f are as in Definition 2. From Step 2 follows that (3) holds for the semimartingale Y with a uniform constant on Θ. Since $\omega^X \leq \omega^Y$, we conclude that the claim (3) holds for the reflected diffusion X. □

3 Local time

In this section we introduce some preliminary results regarding the local time of the reflected diffusion X. A standard reference is [12, Chapter VI].

Definition 6. Set $t > 0$. For any Borel set $A \subseteq [0,1]$ we define the occupation measure $T_t(A)$ of the path $(X_s, 0 \leq s \leq t)$, with respect to the quadratic variation of X, by

$$T_t(A) = \int_0^t \mathbf{1}_A(X_s)\sigma^2(X_s)ds.$$

When $T_t(A)$ is absolutely continuous with respect to the Lebesgue measure dx on the interval $[0,1]$, we define the local time by the Radon-Nikodym derivative:

$$L_t(x) = \frac{dT_t}{dx}.$$

Theorem 7 (Itô-Tanaka formula)**.** *Let X be the solution of the SDE (1) with $(\sigma, b) \in \Theta$. Then, the local time L exists and has a continuous version in both $t > 0$ and $x \in (0,1)$. For every $x \in [0,1]$ the process $(L_t(x), t \geq 0)$ is non-decreasing and increases only when $X_t = x$. Furthermore, if f is the difference of two convex functions, we have*

$$\begin{aligned} f(X_t) = f(X_0) + \int_0^t f'_-(X_s)\sigma(X_s)dW_s + \int_0^t f'_-(X_s)b(X_s)ds + \frac{1}{2}\int_0^1 L_t(x)f''(dx) + \\ + \int_0^t f'_-(X_s)dK_s. \end{aligned} \tag{4}$$

Proof. Reflected diffusion X is a continuous semimartingale. By [12, Chapter VI, Theorem 1.2 and Theorem 1.5] there exists a process $(L_t(x) : x \in (0,1), t \geq 0)$, continuous and non-decreasing in t, cadlag in x and such that (4) holds. Furthermore, by [12, Chapter VI, Theorem 1.7] for every $x \in (0,1)$

$$L_t(x) - L_t(x_-) = 2\int_0^t \mathbf{1}_{\{X_s = x\}} b(X_s)ds + 2\int_0^t \mathbf{1}_{\{X_s = x\}} dK_s = 0.$$

The concentration of the associated measure dL_t on the set $\{X_t = x\}$ follows from [12, Chapter VI Proposition 1.3]. □

Lemma 8. *For every $T>0$ and $p\geq 1$ we have*

$$\sup_{(\sigma,b)\in\Theta}\sup_{x\in(0,1)}\mathbb{E}_{\sigma,b}[\sup_{t\leq T}L_t^p(x)]<\infty.$$

Proof. The usual way to bound the moments of the local time is to use the Itô-Tanaka formula for function $f_x(y)=(y-x)^+$, see e.g. [12, Chapter VI Theorem 1.7]. Because of the additional reflection term dK_t, we make a less intuitive choice of the function f that guarantees $f'(0)=f'(1)=0$.

Set $T>0$, $p\geq 1$ and $x\in(0,1/2]$. Let $f_x(y)=\mathbf{1}(x\leq y\leq 3/4)(3y-2y^2)$. By (4)

$$\begin{aligned}\frac{3-4x}{2}L_t(x) &= f_x(X_t)-f_x(X_0)-\int_0^t\mathbf{1}(x<X_s\leq\tfrac{3}{4})(3-4X_s)\sigma(X_s)dW_s\\ &\quad -\int_0^t\mathbf{1}(x<X_s\leq\tfrac{3}{4})(3-4X_s)b(X_s)ds+2\int_0^t\mathbf{1}(x<X_s\leq\tfrac{3}{4})\sigma^2(X_s)ds.\end{aligned}$$

Applying the uniform (on Θ) bounds on b and σ, together with the Burkholder-Davies-Gundy inequality, we conclude that for $t\leq T$

$$\sup_{(\sigma,b)\in\Theta}\sup_{x\in(0,1/2)}\mathbb{E}_{\sigma,b}[L_t(x)^p]\leq C_{p,T},$$

holds with some positive constant $C_{p,T}$. For $x\in(1/2,1)$ we consider function $f_x(y)=\mathbf{1}(1/4\leq y\leq x)(y-2y^2)$ and proceed similarly. □

Theorem 9. *For any $T>0$, $p\geq 1$ and $x,y\in(0,1)$ we have*

$$\sup_{(\sigma,b)\in\Theta}\mathbb{E}_{\sigma,b}\big[\sup_{t\leq T}|L_t(x)-L_t(y)|^{2p}\big]\leq C_{p,T}|x-y|^p. \tag{5}$$

In particular, the family L of the local times can be chosen such that functions $x\longmapsto L_t(x)$ are almost surely Hölder continuous of order α for every $\alpha<1/2$ and uniformly in $t\leq T$. Moreover, for every $p\geq 1$ and $t\leq T$ we have

$$\sup_{(\sigma,b)\in\Theta}\mathbb{E}_{\sigma,b}\big[\sup_{x\in[0,1]}L_t^p(x)\big]<\infty. \tag{6}$$

Proof. The proof goes along the same lines as [12, Chapter VI Theorem 1.7]. We will first show the inequality (5). For $x\in(0,1)$, by the Itô-Tanaka formula (4)

$$\begin{aligned}\frac{1}{2}L_t(x)=(X_t-x)^+-(X_0-x)^+-\int_0^t\mathbf{1}(X_s>x)\sigma(X_s)dW_s+\\ -\int_0^t\mathbf{1}(X_s>x)b(X_s)ds-\int_0^t\mathbf{1}(X_s=1)dK_s.\end{aligned}$$

Since the function $x\longmapsto(X_t-x)^+-(X_0-x)^+-\int_0^t\mathbf{1}(X_s=1)dK_s$ is uniformly Lipschitz on Θ, we need only to consider the martingale term $M_t^x=\int_0^t\mathbf{1}(X_s>x)\sigma(X_s)dW_s$ and the finite variation term $D_t^x=\int_0^t\mathbf{1}(X_s>x)b(X_s)ds$. For $x,y\in(0,1)$ Hölder's inequality and Lemma 8 yield

$$\mathbb{E}_{\sigma,b}\big[\sup_{t\leq T}|D_t^x-D_t^y|^{2p}\big]\lesssim\mathbb{E}_{\sigma,b}\Big[\Big(\int_x^y L_T(z)dz\Big)^{2p}\Big]$$

$$\lesssim |y-x|^{2p-1} \Big| \int_x^y \mathbb{E}_{\sigma,b}[L_T^{2p}(z)]dz \Big| \lesssim C_{p,T}|y-x|^{2p},$$

for some constant $\widetilde{C}_{p,T} > 0$. To bound the increments of the martingale M^x we use the Burkholder-Davies-Gundy inequality together with Hölder's inequality, obtaining

$$\mathbb{E}_{\sigma,b}\big[\sup_{t\leq T} |M_t^x - M_t^y|^{2p}\big] \lesssim C_p \mathbb{E}_{\sigma,b}\Big[\Big(\int_x^y L_T(z)dz\Big)^p\Big] \leq \widetilde{C}_{p,T}|y-x|^p.$$

We finished the proof of the bound (5). From the Kolmogorov continuity criterion (see [12, Chapter I, Theorem 2.1]) follows that there exists a modification $\widetilde{L}$ of the family of local times L, such that functions $x \longmapsto \widetilde{L}_t(x)$ are almost surely Hölder continuous of order α for every $\alpha < 1/2$ and uniformly in $t \leq T$. Furthermore, for any $\alpha < 1/2$ and $p \geq 2$ we have

$$\sup_{(\sigma,b)\in\Theta} \mathbb{E}_{\sigma,b}\Big[\Big(\sup_{x\neq y} \frac{|\widetilde{L}_t(x) - \widetilde{L}_t(y)|}{|x-y|^\alpha}\Big)^p\Big] < \infty.$$

Fix $x_0 \in (0,1)$. By the bound above and Lemma 8 we conclude that

$$\sup_{(\sigma,b)\in\Theta} \mathbb{E}_{\sigma,b}[\sup_{x\in[0,1]} \widetilde{L}_t^p(x)] \leq \sup_{(\sigma,b)\in\Theta} \mathbb{E}_{\sigma,b}\Big[\Big(\sup_{x\neq x_0} \frac{|\widetilde{L}_t(x) - \widetilde{L}_t(x_0)|}{|x-x_0|^\alpha} + \widetilde{L}_t(x_0)\Big)^p\Big] < \infty. \qquad \square$$

Theorem 10. *Set $T > 0$ and define the (chronological) occupation density μ_T by*

$$\mu_T(x) = \frac{L_T(x)}{T\sigma^2(x)}.$$

Then, for any bounded Borel measurable function f, the following occupation formula holds:

$$\frac{1}{T}\int_0^T f(X_s)ds = \int_0^1 f(x)\mu_T(x)dx.$$

Furthermore, the occupation density μ_T inherits the regularity properties of the local time L_T. In particular, for every $p \geq 1$ and $T > 0$ we have

$$\sup_{(\sigma,b)\in\Theta} \mathbb{E}_{\sigma,b}\big[\sup_{x\in[0,1]} \mu_T^p(x)\big] \;<\; \infty. \tag{7}$$

$$\sup_{(\sigma,b)\in\Theta} \mathbb{E}_{\sigma,b}\big[|\mu_T(x) - \mu_T(y)|^{2p}\big] \;\leq\; C_{p,T}|x-y|^p. \tag{8}$$

Proof. The existence and form of the occupation density follow from Theorem 7 and Definition 6. Given that $\sigma^2 \gtrsim 1$ inequality (7) follows directly from (6). Finally, (5) and the uniform Lipschitz property of σ^2 imply (8). $\square$

4 Estimation of the occupation time of an interval

Estimation of the local time from finite data observations has been extensively studied and is nowadays well established, see e.g. [6, 7]. Nevertheless, until recently, much less was known about the estimation of the occupation time of a given Borel set. In the breakthrough paper Ngo and Ogawa [10] authors considered Riemann sum approximations of the occupation time of a half-line. [10, Theorem 2.2] stated a convergence rate $\Delta^{\frac{3}{4}}$ for diffusion processes with bounded coefficients, which was defined as normalization required for tightness of the estimation errors. It is important to note that $\Delta^{\frac{3}{4}}$ is a better rate than could be obtained using only the regularity properties of the local time. In the special case of the Brownian occupation time of the positive half-line, $\Delta^{\frac{3}{4}}$ was shown to be the upper bound of the root mean squared error (see [10, Theorem 2.3]) and to be optimal. Further development was done in Kohatsu-Higa et al. [9]. By means of the Malliavin calculus [9, Theorem 2.3] proved that for any sufficiently regular scalar diffusion X and an exponentially bounded function h inequality

$$\mathbb{E}_{\sigma,b}\Big[\Big|\frac{1}{T}\int_0^T h(X_s)ds - \frac{1}{N}\sum_{n=0}^{N-1} h(X_{n\Delta})\Big|^p\Big] \le C_{T,X,h}\Delta^{p+1/2} \tag{9}$$

holds with constant $C_{T,X,h} > 0$ depending on the time horizon T, diffusion X and function h. In what follows, we extend (9), for h being characteristic function, to reflected diffusions with coefficients in Θ.

Theorem 11. *For any $T > 0$ and $\alpha \in (0,1)$ we have*

$$\sup_{(\sigma,b)\in\Theta} \mathbb{E}_{\sigma,b}\Big[\Big|\frac{1}{N}\sum_{n=0}^{N-1} \mathbf{1}(X_{n\Delta} < \alpha) - \frac{1}{T}\int_0^T \mathbf{1}(X_s < \alpha)ds\Big|^2\Big]^{\frac{1}{2}} \le C_T\Delta^{\frac{2}{3}},$$

with some positive constant C_T.

Remark 12. As the right hand side of (9) does not scale linearly in p, it is not optimal to use the Girsanov theorem to generalize the Brownian bound to diffusions with bounded coefficients. Nevertheless, we proceed with this approach, as the suboptimal rate $\Delta^{2/3}$ is sufficient for our purposes.

Proof. Fix $T > 0$. The proof is divided in several steps, generalizing the result from a standard Brownian motion process to reflected diffusions.

Step 1. Let W_t be a standard Brownian motion. We will show that there exists a constant $C_T > 0$ such that for any $\alpha \in \mathbb{R}$ we have

$$\mathbb{E}\Big[\Big|\frac{1}{N}\sum_{n=0}^{N-1} \mathbf{1}(W_{n\Delta} < \alpha) - \frac{1}{T}\int_0^T \mathbf{1}(W_s < \alpha)ds\Big|^{2p}\Big] \le C_T\Delta^{p+1/2}. \tag{10}$$

Set $\alpha \in \mathbb{R}$ and $h_\alpha(x) = \mathbf{1}(x < \alpha)$. Following [9, proof of Proposition 2.1], for $M \in \mathbb{N}$, denote

$$\begin{aligned} \rho_M(x) &= c_\rho M e^{\frac{1}{(Mx)^2-1}}\mathbf{1}_{(-1,1)}(x), \\ h_{\alpha,M}(x) &= \int_{\mathbb{R}} h_\alpha(x-y)\rho_M(y)dy, \end{aligned}$$

where the constant $c_\rho = \Big(\int_{-1}^1 e^{\frac{1}{y^2-1}}dy\Big)^{-1}$ is such that ρ_M integrates to 1. Direct calculations show that

$$\begin{cases} \mathcal{A}(i): & h_{\alpha,M} \to h_\alpha \text{ in } L^1 \\ \mathcal{A}(ii): & \sup_M \sup_{x\in\mathbb{R}} |h_\alpha(x)| + |h_{\alpha,M}(x)| \le 2 \\ \mathcal{A}(iii): & \sup_M \sup_{u\ge 0} \int |h'_{\alpha,M}(x)| e^{-\frac{x^2}{u}}dx = c_\rho \int_{-1}^1 e^{\frac{1}{y^2-1}} e^{-\frac{(\alpha+y/M)^2}{u}}dy \le 1. \end{cases}$$

Let

$$S_{N,\alpha} = \frac{1}{N}\sum_{n=0}^{N-1} h_\alpha(W_{n\Delta}) - \frac{1}{T}\int_0^T h_\alpha(W_s)ds,$$
$$S_{N,\alpha,M} = \frac{1}{N}\sum_{n=0}^{N-1} h_{\alpha,M}(W_{n\Delta}) - \frac{1}{T}\int_0^T h_{\alpha,M}(W_s)ds.$$

Arguing as in [9, proof of Eq. 3.10], we obtain that for any α

$$\lim_{M\to\infty} \mathbb{E}_{\sigma,b}[S_{N,\alpha,M}^{2p}] = \mathbb{E}_{\sigma,b}[S_{N,\alpha}^{2p}].$$

Hence, it is sufficient to show that

$$\mathbb{E}_{\sigma,b}[S_{N,\alpha,M}^{2p}] \le C_T N^{-(p+1/2)}, \tag{11}$$

for some constant C_T independent of M and α. But, given uniform bounds $\mathcal{A}(ii)$ and $\mathcal{A}(iii)$, inequality (11) follows by the same calculations as in [9, proof of Theorem 3.2].

Step 2. Consider diffusion Y satisfying $dY_t = h(Y_t)dt + dW_t$, with uniformly bounded drift h. We will show that

$$\mathbb{E}\Big[\Big|\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}(Y_{n\Delta}<\alpha) - \frac{1}{T}\int_0^T \mathbf{1}(Y_s<\alpha)ds\Big|^2\Big]^{\frac{1}{2}} \le C_{T,\|h\|_\infty}\Delta^{17/24}. \tag{12}$$

Denote

$$Z_t = \exp\Big(-\int_0^t h(Y_s)dW_s - \frac{1}{2}\int_0^t h^2(Y_s)ds\Big).$$

Since h is bounded, by Novikov's condition Z is a martingale. Define the probability measure $\mathbb{Q}$ by the Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\,|_{\mathcal{F}_t} = Z_t.$$

By Girsanov's theorem the process Y is a standard Brownian motion under the probability measure $\mathbb{Q}$. From Hölder's inequality, for $\frac{1}{p}+\frac{1}{q}=1$, follows that

$$\mathbb{E}\Big[\Big|\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}(Y_{n\Delta}<\alpha) - \frac{1}{T}\int_0^T \mathbf{1}(Y_s<\alpha)ds\Big|^2\Big]^{\frac{1}{2}} =$$
$$= \mathbb{E}_{\mathbb{Q}}\Big[\Big|\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}(Y_{n\Delta}<\alpha) - \frac{1}{T}\int_0^T \mathbf{1}(Y_s<\alpha)ds\Big|^2 Z_T^{-1}\Big]^{\frac{1}{2}}.$$
$$\le \mathbb{E}_{\mathbb{Q}}\Big[\Big|\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}(Y_{n\Delta}<\alpha) - \frac{1}{T}\int_0^T \mathbf{1}(Y_s<\alpha)ds\Big|^{2p}\Big]^{\frac{1}{2p}} \mathbb{E}_{\mathbb{Q}}\big[Z_T^{-q}\big]^{\frac{1}{2q}}.$$

Since the drift function h is uniformly bounded,

$$\mathbb{E}_{\mathbb{Q}}\big[Z_T^{-q}\big] = \mathbb{E}\big[Z_T^{-(q-1)}\big] = \mathbb{E}\Big[\exp\Big((q-1)\int_0^T h(Y_s)dW_s + \frac{q-1}{2}\int_0^T h^2(Y_s)ds\Big)\Big]$$

$$\leq \exp\Big(\frac{q(q-1)}{2}T\|h\|_\infty^2\Big) = C^{2q}_{q,\|h\|_\infty}.$$

Hence, by (10), with $p = 6/5$, inequality (12) holds.

Step 3. Consider diffusion Y satisfying $dY_t = \widetilde{b}(Y_t)dt + \widetilde{\sigma}(Y_t)dW_t$, with bounded drift $\widetilde{b}$ and positive, Lipschitz continuous $\widetilde{\sigma}$. Let $S(x) = \int_0^x \widetilde{\sigma}^{-1}(y)dy$, $dZ_t = S(Y_t)$. It follows from Itô's formula that

$$dZ_t = g(Y_t)dt + dW_t,$$

where $g(x) = \frac{\widetilde{b}(x)}{\widetilde{\sigma}(x)} - \frac{1}{2}\widetilde{\sigma}'(x)$. Since $\widetilde{\sigma}$ is a strictly positive function, S is increasing and invertible. Denote $h(x) = g(S^{-1}(x))$. We have

$$dZ_t = h(Z_t)dt + dW_t,$$

with

$$\|h\|_\infty \leq \frac{\|\widetilde{b}\|_\infty}{\inf \widetilde{\sigma}} + \frac{1}{2}\|\widetilde{\sigma}'\|_\infty.$$

From (12) follows, that there exists a constant C_T, depending only on the bounds on the diffusion coefficients, such that

$$\mathbb{E}\Big[\Big|\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}(Y_{n\Delta} < \alpha) - \frac{1}{T}\int_0^T \mathbf{1}(Y_s < \alpha)ds\Big|^2\Big]^{\frac{1}{2}} =$$

$$= \mathbb{E}\Big[\Big|\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}(Z_{n\Delta} < S(\alpha)) - \frac{1}{T}\int_0^T \mathbf{1}(Z_s < S(\alpha))ds\Big|^2\Big]^{\frac{1}{2}} \leq C_T\Delta^{\frac{17}{24}}. \qquad (13)$$

Step 4. Fix $(\sigma, b) \in \Theta$. Let

$$\begin{aligned} dY_t &= \widetilde{b}(Y_t)dt + \widetilde{\sigma}(Y_t)dW_t, \\ X_t &= f(Y_t), \end{aligned}$$

where $\widetilde{b}, \widetilde{\sigma}$ and f are as in Definition 2. Recall that by Theorem 3 process X is the reflected diffusion with coefficients (σ, b). By definition of the function f, for any $\alpha \in (0,1)$ and $s > 0$, we have

$$\{X_s < \alpha\} = \bigcup_{m\in\mathbb{Z}} \{Y_s \in (2m-\alpha, 2m+\alpha)\}. \qquad (14)$$

Denote

$$\Gamma_N(m) = \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}(Y_{n\Delta} \in (2m-\alpha, 2m+\alpha)) - \frac{1}{T}\int_0^T \mathbf{1}(Y_s \in (2m-\alpha, 2m+\alpha))ds.$$

By (13) there exists a uniform on Θ constant $C_T > 0$, such that for any $m \in \mathbb{Z}$, we have

$$\mathbb{E}_{\sigma,b}\Big[\Gamma_N^2(m)\Big]^{\frac{1}{2}} \leq C_T\Delta^{\frac{17}{24}}.$$

Let $Y_t^* = \sup_{s\leq t}|Y_s|$. From (14) follows that

$$\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}(X_{n\Delta} < \alpha) - \int_0^T \mathbf{1}(X_s < \alpha)ds = \sum_{m\in\mathbb{Z}}\Gamma_N(m) = \sum_{m\in\mathbb{Z}}\Gamma_N(m)\mathbf{1}(Y_T^* \geq 2|m| - 1).$$

Since $\Gamma_N(m) \le 2$, for any M, using (14) we obtain

$$\mathbb{E}_{\sigma,b}\Big[\Big|\sum_{m\in\mathbb{Z}}\Gamma_N(m)\mathbf{1}(Y_T^* \ge 2|m|+1)\Big|^2\Big] \lesssim$$

$$\lesssim M\mathbb{E}_{\sigma,b}\Big[\sum_{|m|\le M}\Gamma_N^2(m)\Big] + \mathbb{E}_{\sigma,b}\Big[\Big|\sum_{m>M}\mathbf{1}(Y_T^* \ge 2m-1)\Big|^2\Big]$$

$$\lesssim C_T M^2\Delta^{\frac{17}{12}} + \sum_{m,k>M}\mathbb{P}_{\sigma,b}(Y_T^* \ge 2(m\vee k)-1).$$

By the Burkholder-Davies-Gundy inequality, together with uniform on Θ bounds on diffusion coefficients, for any $p \ge 1$,

$$\mathbb{E}_{\sigma,b}\big[(Y_T^*)^p\big] \lesssim \mathbb{E}_{\sigma,b}\Big[\Big(\int_0^T \widetilde{\sigma}^2(Y_s)ds\Big)^{\frac{p}{2}}\Big] + (T\|\widetilde{b}\|_\infty)^p \lesssim C_{p,T}.$$

Consequently, by the Markov inequality

$$\sum_{m,k>M}\mathbb{P}_{\sigma,b}(Y_T^* \ge 2(m\vee k)-1) \le 2\sum_{M<k\le m} C_{p,T}(2m-1)^{-p}$$

$$\le 2C_{p,T}\sum_{m>M}(2m-1)^{-(p-1)} \lesssim M^{-(p-2)}.$$

We conclude that

$$\mathbb{E}_{\sigma,b}\Big[\Big|\sum_{m\in\mathbb{Z}}\Gamma_N(m)\mathbf{1}(Y_s^* \ge 2|m|+1)\Big|^2\Big] \lesssim M^2\Delta^{\frac{17}{12}} + M^{-(p-2)}.$$

Hence the claim follows for $M \sim \Delta^{-\frac{3}{8}}$ and any $p \ge 4$. □

5 Upper bounds on the transition kernel

In this section we prove a uniform Gaussian upper bound on the transition kernel of the reflected diffusion X with coefficients in Θ. Under the assumption of smooth coefficients, existence of Gaussian off-diagonal bounds follows from the general theory of partial differential equations, see [4, 9] c.f. [5, Chapter 9]. Sharp upper bounds are also established for diffusion processes in the divergence form [2, Chapter VII] and more recently were derived for multivariate diffusions with the infinitesimal generator satisfying Neumann boundary conditions, see [14]. Nevertheless, as demonstrated by [11, Theorem 2] Gaussian upper bounds do not hold in general for scalar diffusions with bounded measurable drift.

Theorem 13. *The transition kernel p_t of the reflected diffusion X satisfies*

$$\sup_{(\sigma,b)\in\Theta} p_t(x,y) \lesssim \frac{C_T}{\sqrt{t}}e^{-\frac{(x-y)^2}{ct}},$$

for all $x, y \in [0,1]$, $0 \le t \le T$ and $c, C_T > 0$.

Proof. We will generalize the bound on the transition kernel from diffusions with bounded drift and unit volatility to reflected processes with coefficients $(\sigma, b) \in \Theta$.

Step 1. Consider diffusion Z satisfying $dZ_t = g(Z_t)dt + dW_t$, where g is a bounded measurable function. Then by [11, Theorem 1] the transition kernel p^Z of the process Z satisfies

$$p_t^Z(x,y) \lesssim \frac{1}{\sqrt{t}} \int_{|x-y|/\sqrt{t}}^{\infty} z e^{-(z-\|g\|_\infty \sqrt{t})^2/2} dz.$$

Using the inequality [1, Formula 7.1.13]:

$$\int_x^\infty e^{-z^2} dz \leq \frac{e^{-x^2}}{x + \sqrt{x^2 + 4/\pi}} \leq \frac{\sqrt{\pi}}{2} e^{-x^2}, \tag{15}$$

we obtain that

$$\int_{a/\sqrt{t}}^{\infty} z e^{-(z-b\sqrt{t})^2/2} dz = \int_{\frac{a}{\sqrt{t}}-b\sqrt{t}}^{\infty} (w + b\sqrt{t}) e^{-w^2/2} dw \leq e^{-\frac{(a-bt)^2}{2t}} \Big(1 + b\frac{\sqrt{\pi t}}{\sqrt{2}}\Big).$$

Thus

$$p_t^Z(x,y) \leq C_{T,\|g\|_\infty} \frac{1}{\sqrt{t}} e^{-\frac{(x-y)^2}{2t} + \|g\|_\infty |x-y|}. \tag{16}$$

Step 2. Consider diffusion Y satisfying $dY_t = \widetilde{b}(Y_t)dt + \widetilde{\sigma}(Y_t)dW_t$. Let $S(x) = \int_0^x \widetilde{\sigma}^{-1}(y)dy$ and $Z_t = S(Y_t)$. It follows from Itô's formula that

$$dZ_t = g(Y_t)dt + dW_t,$$

where $g(x) = \frac{\widetilde{b}(x)}{\widetilde{\sigma}(x)} - \frac{1}{2}\widetilde{\sigma}'(x)$. For any x, y we have

$$\|\widetilde{\sigma}^{-1}\|_\infty^{-1} |x-y| \leq |S^{-1}(x) - S^{-1}(y)| \leq \|\widetilde{\sigma}\|_\infty |x-y|.$$

Hence, from (16) follows that

$$\begin{aligned} p_t^Y(x,y) = p_t^Z(S^{-1}(x), S^{-1}(y)) &\lesssim \frac{C_{T,\|g\|_\infty}}{\sqrt{t}} \exp\Big(-\frac{(S^{-1}(x) - S^{-1}(y))^2}{2t} + \|g\|_\infty |S^{-1}(x) - S^{-1}(y)|\Big) \\ &\lesssim \frac{C_{T,\|g\|_\infty}}{\sqrt{t}} \exp\Big(-\frac{(x-y)^2}{2t\|\widetilde{\sigma}^{-1}\|_\infty} + \|g\|_\infty \|\widetilde{\sigma}\|_\infty |x-y|\Big). \end{aligned} \tag{17}$$

Step 3. For $(\sigma, b) \in \Theta$ let $\widetilde{b}, \widetilde{\sigma}$ and f be as in Definition 2 and Y as in Step 2. Note first, that by (17) there exist uniform on Θ constants $c_T, c_1, c_2 > 0$ such that

$$p_t^Y(x,y) \leq \frac{c_T}{\sqrt{t}} \exp\Big(-\frac{(x-y)^2}{c_1 t} + c_2|x-y|\Big). \tag{18}$$

By Theorem 3 $X_t = f(Y_t)$ is the reflected diffusion process corresponding to the coefficients (σ, b). For $y \in [0,1]$ let $(y_m)_{m\in\mathbb{Z}}$ be such that $y_m \in [m, m+1]$ and $f(y_m) = y$. Then

$$p_t^X(x,y) = \sum_{m\in\mathbb{Z}} p_t^Y(x, y_m).$$

Since for any $m \in \mathbb{Z}$ we have $|x-y| + (|m|-2)_+ \leq |x - y_m| \leq |m| + 2$, from (18) follows that

$$\begin{aligned} p_t^X(x,y) &\leq \frac{c_T}{\sqrt{t}} \sum_{m\in\mathbb{Z}} \exp\Big(-\frac{(x-y_m)^2}{c_1 t} + c_2|x-y_m|\Big) \\ &\leq \frac{c_T}{\sqrt{t}} e^{-\frac{(x-y)^2}{c_1 t}} \sum_{m\in\mathbb{Z}} \exp\Big(-\frac{[(|m|-2)_+]^2}{c_1 T} + c_2(|m|+2)\Big). \end{aligned}$$

□

6 Mean crossings bounds

Throughout this section we consider fixed time horizon, for simplicity we assume $T = 1$.

Definition 14. For $\alpha \in (0, 1)$ and $n = 0, ..., N - 1$ denote

$$\chi(n, \alpha) = \mathbf{1}_{[0,\alpha)}(X_{(n+1)\Delta}) - \mathbf{1}_{[0,\alpha)}(X_{n\Delta}).$$

Theorem 15. *For every $\alpha \in (0, 1)$ we have*

$$\mathbb{E}_{\sigma,b}\Big[\Big(\sum_{n=0}^{N-1} |\chi(n, \alpha)|(X_{(n+1)\Delta} - X_{n\Delta})^2\Big)^2\Big]^{\frac{1}{2}} \lesssim \Delta^{1/2}.$$

Proof. Fix $\alpha \in (0, 1)$. Since$|\chi(n, \alpha)| = 1$ if and only if the increment $(X_{n\Delta}, X_{(n+1)\Delta})$ crosses the level α, the claim is equivalent to the inequalities:

$$\begin{aligned}
\mathbb{E}_{\sigma,b}\Big[\Big(\sum_{n=0}^{N-1} \mathbf{1}(X_{n\Delta} < \alpha)\mathbf{1}(X_{(n+1)\Delta} > \alpha)(X_{(n+1)\Delta} - X_{n\Delta})^2\Big)^2\Big]^{\frac{1}{2}} &\lesssim \Delta^{1/2}, \\
\mathbb{E}_{\sigma,b}\Big[\Big(\sum_{n=0}^{N-1} \mathbf{1}(X_{n\Delta} > \alpha)\mathbf{1}(X_{(n+1)\Delta} < \alpha)(X_{(n+1)\Delta} - X_{n\Delta})^2\Big)^2\Big]^{\frac{1}{2}} &\lesssim \Delta^{1/2}.
\end{aligned}$$

Below, we only prove the first inequality. The second one can be obtained in a similar way or by a time reversal argument. Denote

$$\eta_n = \mathbf{1}(X_{n\Delta} < \alpha)\mathbf{1}(X_{(n+1)\Delta} > \alpha)(X_{(n+1)\Delta} - X_{n\Delta})^2.$$

We have

$$\mathbb{E}_{\sigma,b}\Big[\Big(\sum_{n=0}^{N-1} \mathbf{1}(X_{n\Delta} < \alpha)\mathbf{1}(X_{(n+1)\Delta} > \alpha)(X_{(n+1)\Delta} - X_{n\Delta})^2\Big)^2\Big] = \sum_{n=0}^{N-1} \mathbb{E}_{\sigma,b}[\eta_n^2] + 2 \sum_{0 \le n < m}^{N-1} \mathbb{E}_{\sigma,b}[\eta_n \eta_m].$$

Denote by p_t the transition kernel of the diffusion X. From Theorem 13 and the inequality (15) follows that

$$\begin{aligned}
\int_0^\alpha \int_\alpha^1 p_\Delta(x, y)(y - x)^4 dy dx &\lesssim \int_0^\alpha \int_\alpha^1 \frac{1}{\sqrt{\Delta}} e^{-\frac{(y-x)^2}{c\Delta}} (y - x)^4 dy dx \lesssim \Delta^2 \int_0^\alpha \int_{\frac{\alpha - x}{\sqrt{\Delta c}}}^{\frac{1-x}{\sqrt{\Delta c}}} e^{-z^2} z^4 dz dx \\
&\lesssim \Delta^2 \int_0^\alpha \int_{\frac{\alpha - x}{\sqrt{\Delta c}}}^{\frac{1-x}{\sqrt{\Delta c}}} e^{-\frac{z^2}{2}} dz dx \lesssim \Delta^2 \int_0^\alpha e^{-\frac{(\alpha - x)^2}{2c\Delta}} dx \lesssim \Delta^{5/2}. \qquad (19)
\end{aligned}$$

Similarly

$$\int_0^\alpha \int_\alpha^1 p_\Delta(x, y)(y - x)^2 dy dx \lesssim \Delta^{3/2}. \qquad (20)$$

For simplicity we will use the stationarity of X, which is granted by the assumption $x_0 \stackrel{d}{=} \mu$.Using more elaborated arguments the result could be obtained for an arbitrary initial condition. By stationarity, for any t, the one dimensional margin X_t is distributed with respect to the invariant measure $\mu(x)dx$. Conditioning on $X_{n\Delta}$, from (19) and uniform bounds on the density μ follows

$$\mathbb{E}_{\sigma,b}\big[\eta_n^2\big] = \int_0^\alpha \int_\alpha^1 p_\Delta(x,y)(y-x)^4 dy\mu(x)dx \lesssim \Delta^{5/2}.$$

Hence

$$\sum_{n=0}^{N-1} \mathbb{E}_{\sigma,b}[\eta_n^2] \lesssim N\Delta^{\frac{5}{2}} = \Delta^{\frac{3}{2}}.$$

The Cauchy-Schwarz inequality implies

$$\sum_{n=0}^{N-2} \mathbb{E}_{\sigma,b}[\eta_n\eta_{n+1}] \lesssim \sum_{n=0}^{N-2} \mathbb{E}_{\sigma,b}[\eta_n^2]^{\frac{1}{2}}\mathbb{E}_{\sigma,b}[\eta_{n+1}^2]^{\frac{1}{2}} \lesssim N\Delta^{\frac{5}{2}} \lesssim \Delta^{\frac{3}{2}}.$$

Finally, using (20), for $m > n+1$, we obtain

$$\begin{aligned}
\mathbb{E}_{\sigma,b}[\eta_n\eta_m] &= \int_0^\alpha \int_\alpha^1 \int_0^\alpha \int_\alpha^1 p_\Delta(x,y)(y-x)^2 p_{(m-n-1)\Delta}(z,x)(z-w)^2 p_\Delta(w,z)\mu(w)dydxdzdw \\
&\lesssim \int_0^\alpha \int_\alpha^1 p_\Delta(x,y)(y-x)^2 dydx \frac{1}{\sqrt{(m-n-1)\Delta}} \int_0^\alpha \int_\alpha^1 (z-w)^2 p_\Delta(w,z)dzdw \\
&\lesssim \Delta^{5/2}\frac{1}{\sqrt{m-n-1}}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\sum_{n=0}^{N-3}\sum_{m=n+2}^{N-1} \mathbb{E}_{\sigma,b}[\eta_n\eta_m] &\lesssim \Delta^{5/2}\sum_{n=0}^{N-3}\sum_{k=1}^{N-n-2}\frac{1}{\sqrt{k}} \lesssim \Delta^{5/2}\sum_{n=0}^{N-3}\sqrt{N-n-2} \\
&= \Delta^{5/2}\sum_{n=1}^{N-2}\sqrt{n} \lesssim \Delta^{5/2}N^{3/2} = \Delta.
\end{aligned}$$

□

Definition 16. Define the event

$$\mathcal{R}_1 := \{\omega_1(\Delta)\|\mu_1\|_\infty \le \Delta^{5/11}v\}.$$

Using Markov's inequality together with Theorem 5 and (6) we obtain that

$$\mathbb{P}_{\sigma,b}(\Omega\setminus\mathcal{R}_1) \lesssim \Delta^{2/3}.$$

Theorem 17. *For any $\alpha \in [\frac{1}{J}, 1-\frac{1}{J}]$ holds*

$$\mathbb{E}_{\sigma,b}\Big[\mathbf{1}_{\mathcal{R}_1}\cdot\Big|\sum_{n=0}^{N-1}\big(\mathbf{1}_{[0,\alpha)}(X_{(n+1)\Delta}) - \mathbf{1}_{[0,\alpha)}(X_{n\Delta})\big)\big((X_{(n+1)\Delta}-\alpha)^2 - (X_{n\Delta}-\alpha)^2\big)\Big|^2\Big]^{\frac{1}{2}} \lesssim \Delta^{2/3}.$$

Proof. Fix $\alpha \in [\frac{1}{J}, 1-\frac{1}{J}]$. On the event $\mathcal{R}_1$, whenever $\mathbf{1}_{[0,\alpha)}(X_{(n+1)\Delta}) - \mathbf{1}_{[0,\alpha)}(X_{n\Delta}) \neq 0$ we must have $|X_{n\Delta}-\alpha|, |X_{(n+1)\Delta}-\alpha| \le \omega(\Delta) < \Delta^{4/9}$. Consider function $d : [0,1] \to \mathbb{R}$ given by

$$d(x) = (x-\alpha)^2\mathbf{1}(|x-\alpha| \le \Delta^{4/9}).$$

We have

$$\big(\mathbf{1}_{[0,\alpha)}(X_{(n+1)\Delta}) - \mathbf{1}_{[0,\alpha)}(X_{n\Delta})\big)\big((X_{(n+1)\Delta}-\alpha)^2 - (X_{n\Delta}-\alpha)^2\big) = \\ = \big(\mathbf{1}_{[0,\alpha)}(X_{(n+1)\Delta}) - \mathbf{1}_{[0,\alpha)}(X_{n\Delta})\big)\big(d(X_{(n+1)\Delta}) - d(X_{n\Delta})\big).$$

Step 1. We will first show that

$$\mathbb{E}_{\sigma,b}\Big[\mathbf{1}_{\mathcal{R}_1}\cdot\Big|\sum_{n=0}^{N-1}\mathbf{1}_{[0,\alpha)}(X_{n\Delta})\big(d(X_{(n+1)\Delta})-d(X_{n\Delta})\big)\Big|^2\Big]^{\frac{1}{2}} \lesssim \Delta^{2/3}. \tag{21}$$

Note that

$$\begin{aligned} d'(x) &= 2(x-\alpha)\mathbf{1}(|x-\alpha|\le\Delta^{4/9}),\\ \frac{1}{2}d''(x) &= -\Delta^{4/9}\delta_{\{\alpha-\Delta^{4/9}\}} + \mathbf{1}(|x-\alpha|\le\Delta^{4/9}) - \Delta^{4/9}\delta_{\{\alpha+\Delta^{4/9}\}}, \end{aligned}$$

where the second derivative must be understood in the distributional sense. Since we fixed α separated from the boundaries, $d'(0)=d'(1)=0$ for Δ small enough. Denote by

$$L_{s,t}(x) := L_t(x) - L_s(x),$$

the local time of the path fragment $(X_u, s\le u\le t)$. From the Itô-Tanaka formula (4) follows that

$$\begin{aligned} d(X_{(n+1)\Delta}) - d(X_{n\Delta}) &= \int_{n\Delta}^{(n+1)\Delta} d'(X_s)\sigma(X_s)dW_t + \int_{n\Delta}^{(n+1)\Delta} d'(X_s)b(X_s)ds + \\ &+ \int_{n\Delta}^{(n+1)\Delta}\sigma^2(X_s)\mathbf{1}(|X_s-\alpha|\le\Delta^{4/9})ds - \Delta^{4/9}L_{n\Delta,(n+1)\Delta}(\alpha-\Delta^{4/9})\\ &- \Delta^{4/9}L_{n\Delta,(n+1)\Delta}(\alpha+\Delta^{4/9}) := \int_{n\Delta}^{(n+1)\Delta} d'(X_s)\sigma(X_s)dW_t + D_n. \end{aligned}$$

First, we will bound the sum of the martingale terms. Since martingale increments are uncorrelated, using Itô isometry, we obtain that

$$\begin{aligned} &\mathbb{E}_{\sigma,b}\Big[\Big|\sum_{n=0}^{N-1}\mathbf{1}_{[0,\alpha)}(X_{n\Delta})\int_{n\Delta}^{(n+1)\Delta} d'(X_s)\sigma(X_s)dW_t\Big|^2\Big] = \\ &= \sum_{n=0}^{N-1}\mathbb{E}_{\sigma,b}\Big[\mathbf{1}_{[0,\alpha)}(X_{n\Delta})\int_{n\Delta}^{(n+1)\Delta}(d'(X_s))^2\sigma^2(X_s)ds\Big] \lesssim \Delta^{\frac{8}{9}}\mathbb{E}_{\sigma,b}\Big[\int_0^1\mathbf{1}(|X_s-\alpha|\le\Delta^{\frac{4}{9}})ds\Big]\\ &= \Delta^{\frac{8}{9}}\int_{\alpha-\Delta^{\frac{4}{9}}}^{\alpha+\Delta^{\frac{4}{9}}}\mathbb{E}_{\sigma,b}[\mu_1(x)]dx \lesssim \Delta^{\frac{4}{3}}, \end{aligned}$$

where the last inequality follows from Theorem 10. Now, we will bound the sum of the finite variation terms: $\sum_{n=0}^{N-1}\mathbf{1}_{[0,\alpha)}(X_{n\Delta})D_n$. Note first, that since b is uniformly bounded, we have

$$\sum_{n=0}^{N-1}\mathbf{1}_{[0,\alpha)}(X_{n\Delta})\Big|\int_{n\Delta}^{(n+1)\Delta}d'(X_s)b(X_s)ds\Big| \lesssim \Delta^{4/9}\int_0^1\mathbf{1}(|x-\alpha|\le\Delta^{4/9})\mu_1(x)dx \lesssim \Delta^{8/9}\|\mu_1\|_\infty.$$

Since by the inequality (7) $\|\mu_1\|_\infty$ has all moments finite, the root mean squared value of this sum is of smaller order than $\Delta^{2/3}$. Now, note that since on the event $\mathcal{R}_1$ $\omega(\Delta) < \Delta^{4/9}$, condition $X_{n\Delta} < \alpha$ implies $L_{n\Delta,(n+1)\Delta}(\alpha + \Delta^{4/9}) = 0$. On the other hand, whenever $L_{n\Delta,(n+1)\Delta}(\alpha - \Delta^{4/9}) \neq 0$ we must have $X_{n\Delta} < \alpha$. Hence

$$\sum_{n=0}^{N-1} \mathbf{1}_{[0,\alpha)}(X_{n\Delta})(\Delta^{4/9} L_{n\Delta,(n+1)\Delta}(\alpha - \Delta^{4/9}) + \Delta^{4/9} L_{n\Delta,(n+1)\Delta}(\alpha + \Delta^{4/9})) = \Delta^{4/9} L_1(\alpha - \Delta^{4/9}).$$

Using first the Cauchy-Schwarz inequality and then Theorem 9 we obtain

$$\mathbb{E}_{\sigma,b}\Big[\big|\Delta^{4/9} L_1(\alpha - \Delta^{4/9}) - \int_{\alpha-\Delta^{4/9}}^{\alpha} L_1(x)dx\big|^2\Big] \lesssim \Delta^{4/9} \int_{\alpha-\Delta^{4/9}}^{\alpha} \mathbb{E}_{\sigma,b}[|L_1(x) - L_1(\alpha - \Delta^{4/9})|^2]dx$$
$$\lesssim \Delta^{4/3}.$$

Consequently, to prove (21) we just have to argue that the root mean squared error of

$$\begin{aligned}
&\int_{\alpha-\Delta^{4/9}}^{\alpha} L_1(x)dx - \sum_{n=0}^{N-1} \mathbf{1}_{[0,\alpha)}(X_{n\Delta}) \int_{n\Delta}^{(n+1)\Delta} \sigma^2(X_s)\mathbf{1}(|X_s - \alpha| \leq \Delta^{4/9})ds \\
&\quad = \sum_{n=0}^{N-1} \int_{n\Delta}^{(n+1)\Delta} (\mathbf{1}(X_s < \alpha) - \mathbf{1}(X_{n\Delta} < \alpha))\sigma^2(X_s)\mathbf{1}(|X_s - \alpha| \leq \Delta^{4/9})ds \\
&\quad = \sum_{n=0}^{N-1} \int_{n\Delta}^{(n+1)\Delta} (\mathbf{1}(X_s < \alpha) - \mathbf{1}(X_{n\Delta} < \alpha))\sigma^2(X_s)ds \qquad (22)
\end{aligned}$$

is of order $\Delta^{2/3}$. From the Lipschitz property of σ^2 follows that

$$\Big|\sum_{n=0}^{N-1} \int_{n\Delta}^{(n+1)\Delta} (\mathbf{1}(X_s < \alpha) - \mathbf{1}(X_{n\Delta} < \alpha))(\sigma^2(X_s) - \sigma^2(\alpha))ds\Big| \lesssim$$
$$\lesssim \sum_{n=0}^{N-1} \int_{n\Delta}^{(n+1)\Delta} \mathbf{1}(|X_s - \alpha| \leq \Delta^{\frac{4}{9}})\Delta^{\frac{4}{9}} ds \lesssim \Delta^{\frac{4}{9}} \int_{\alpha-\Delta^{4/9}}^{\alpha+\Delta^{4/9}} \mu_1(dx) \lesssim \Delta^{\frac{8}{9}} \|\mu_1\|_\infty.$$

Thus, by (7), we reduced (22) to

$$\int_0^1 \mathbf{1}(X_s < \alpha)ds - \frac{1}{N}\sum_{n=0}^{N-1} (\mathbf{1}(X_{n\Delta} < \alpha),$$

which is of the right order by Theorem 11. We conclude that (21) holds.

Step 2. Consider the time reversed process $Y_t = X_{1-t}$. Since X is reversible, the process Y, under the measure $\mathbb{P}_{\sigma,b}$, has the same law as X. Furthermore, the occupation density and the modulus of continuity of processes Y and X are identical, hence $\mathcal{R}_1$ is a "good" event also for Y. Inequality (21) is equivalent to

$$\mathbb{E}_{\sigma,b}\Big[\mathbf{1}_{\mathcal{R}_1} \cdot \Big|\sum_{m=0}^{N-1} \mathbf{1}_{[0,\alpha)}(Y_{m\Delta})(d(Y_{(m+1)\Delta}) - d(Y_{m\Delta}))\Big|^2\Big]^{\frac{1}{2}} \lesssim \Delta^{\frac{2}{3}}.$$

Substituting $n = N - m$ we obtain

$$\sum_{m=0}^{N-1} \mathbf{1}_{[0,\alpha)}(Y_{m\Delta})(d(Y_{(m+1)\Delta}) - d(Y_{m\Delta})) = -\sum_{n=0}^{N-1} \mathbf{1}_{[0,\alpha)}(X_{(n+1)\Delta})(d(X_{(n+1)\Delta}) - d(X_{n\Delta})). \qquad \square$$

References

[1] Abramowitz, M. and Stegun, I. A. (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York, 9th edition.

[2] Bass, R. (1997). *Diffusions and Elliptic Operators*. Springer-Verlag, New York.

[3] Fischer, M. and Nappo, G. (2010). On the moments of the modulus of continuity of Itô processes. *Stochastic Analysis and Applications*, 28(1).

[4] Florens-Zmirou, D. (1993). On estimating the diffusion coefficient from discrete observations. *J. Appl. Prob.*, 30.

[5] Friedman, A. (1964). *Partial differential equations of parabolic type*. Prentice-Hall Inc., Englewood Cliffs, N.J., 2nd edition.

[6] J. M. Azaïs (1989). Approximation des trajectoires et temps local des diffusions. *Ann. Inst. Henri Poincaré Probab. Statist.*, 25(2):175–194.

[7] Jacod, J. (1998). Rates of convergence of the local time of a diffusion. *Anns. Inst. Henri Poincaré Probab. Statist.*, 34(4):505–544.

[8] Karatzas, I. and Shreve, S. E. (1991). *Brownian motion and stochastic calculus*. Springer-Verlag, New York.

[9] Kohatsu-Higa, A., Makhlouf, A., and Ngo, H. L. (2014). Approximations of non-smooth integral type functionals of one dimensional diffusion processes. *Stochastic Processes and their Applications*, 124:1881 – 1909.

[10] Ngo, H.-L. and Ogawa, S. (2011). On the discrete approximation of occupation time of diffusion processes. *Electronic Journal of Statistics*, 5:1374–1393.

[11] Qian, Z. and Zheng, W. (2002). Sharp bounds for transition probability densities of a class of diffusions. *C. R. Acad. Sci. Paris*, I 335:953–957.

[12] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion*, volume I, II. Springer, New York, 3 edition.

[13] Rozkosz, A. and Słomiński, L. (1997). On stability and existence of solutions of SDEs with reflection at the boundary. *Stochastic processes and their applications*, 68:285–302.

[14] Yang, X. and Zhang, T. (2013). Estimates of Heat Kernels with Neumann Boundary Conditions. *Potential Analysis*, 38:549–572.