Exercise sheet, Theme “Convex Optimization”

Aufgabe 1 (Properties of convex functions)
Let f be a real-valued function defined on a convex set $\mathcal{X} \subset \mathbb{R}^d$.

a) Prove: a convex function f is always continuous on $\mathrm{Int}(\mathcal{X})$ (and upper semi-continuous on $\partial\mathcal{X}$)

b) For this question only, f is not assumed convex. Prove: if for all $x \in \mathcal{X}$, $\partial f(x) \neq \emptyset$, then f is a convex function.

c) Prove: if f is convex, then for all $x \in \mathrm{Int}(\mathcal{X})$, $\partial f(x) \neq \emptyset$. *Hint:* one can use without justification the “supporting hyperplane” theorem: if $\mathcal{C}$ is a convex set and $x_0 \in \partial C$, then there exists a vector $w \neq 0$ such that for all $x \in \mathcal{C}$, $\langle x - x_0, w\rangle \geq 0$. Use the fact that the epigraph of f is convex.

d) Prove: if f is convex and differentiable at x, then $\nabla f(x) \in \partial f(x)$ and is even its unique element if $x \in \mathrm{Int}(\mathcal{X})$.

e) Prove: $f(x^*)$ is a local minimimum of f iff $f(x^*)$ is a global minimum, iff $0 \in \partial f(x^*)$.

f) Prove: if f is convex and differentiable, then $f(x^*)$ is minimum of f iff for all $y \in \mathcal{X}$ it holds $\langle \nabla f(x^*), y - x^*\rangle \geq 0$.

Aufgabe 2 (Brunn-Minkowski inequality)
In this exercise we will prove a fundamental theorem which will be used in the next one. Let the ambient space be $\mathbb{R}^d$ and denote for to measurable sets A, B the “Minkowski sum” of A and B as

$$A + B = \{a + b, a \in A, b \in B\}\,.$$

The Brunn-Minkowski inequality states the following: for any measurable A, B of finite volume and $\lambda \in [0, 1]$ such that $(1-\lambda)A + \lambda B$ is measurable, it holds

$$\mathcal{V}_d((1-\lambda)A + \lambda B)^{\frac{1}{d}} \geq (1-\lambda)\mathcal{V}_d(A)^{\frac{1}{d}} + \lambda\mathcal{V}_d(B)^{\frac{1}{d}},$$

where $\mathcal{V}_d$ denotes volume (d-dimensional Lebesgue measure). In this sense the function $A \mapsto \mathcal{V}_d(A)^{\frac{1}{d}}$ is “concave”.
Below we call “cuboid” a d-dimensional rectangular (axis-aligned) box (=cartesian product of intervals along each coordinate).

a) Prove the inequality when both A and B are cuboids. *Hint: assume w.l.o.g. that* $\mathcal{V}_d((1-\lambda)A + \lambda B) = 1$ *and use the inequality between geometric and arithmetic mean.*

b) We now establish the inequality when A and B are disjoint finite unions of cuboids, by recursion on the total number of cuboids.

- Assume w.l.o.g. that A is a disjoint union of at least two cuboids and prove that there exists at least an axis-aligned hyperplane H that separates two cuboids of A.
- Denote H^+ one of the half-spaces defined by H and $A_+ := H_+ \cap A$ and similarly for other sets. Justify that w.l.o.g. we can by translation of B assume that $\frac{\mathcal{V}_d(A)}{\mathcal{V}_d(B)} = \frac{\mathcal{V}_d(A_+)}{\mathcal{V}_d(B_+)} = \frac{\mathcal{V}_d(A_-)}{\mathcal{V}_d(B_-)}$.
- Justify the inequality $\mathcal{V}_d((1-\lambda)A + \lambda B) \leq \mathcal{V}_d((1-\lambda)A_+ + \lambda B_+) + \mathcal{V}_d((1-\lambda)A_- + \lambda B_-)$.

- Apply the induction hypothesis on the quantities on the right-hand-side of the above inequality, and use the assumption on the volume ratios to conclude.

c) Conclude by approximating arbitrary measurable sets by finite union of cuboids.

Aufgabe 3 (Grünbaum's Lemma)
We proceed to proving the following property which was used in the lecture: if K is a compact convex set of $\mathbb{R}^d$, and H a hyperplane going through the center of gravity of K, then the intersection of K with either half-space defined by H has volume at least $\frac{1}{e}\mathcal{V}_d(K)$.
Without loss of generality, we assume K has center of gravity at the origin and $H = \{x : x_1 = 0\}$. For any set A, we denote $A_t := A \cap \{x : x_1 = t\}$; $A_+ := A \cap \{x : x_1 \geq 0\}$ and $A_- := A \cap \{x : x_1 \leq 0\}$.

a) Construct a "symmetrized" version K' of K ("Schwarzsche Abrundung") as follows. For any $t \in \mathbb{R}$, K'_t is the $(d-1)$-dimensional ball $B(0, r_t)$ with r_t chosen so that $\mathcal{V}_{d-1}(B(0, r_t)) = \mathcal{V}_{d-1}(K_t)$.

   Prove that K' is convex. (*Hint:* use the Brunn-Minkowski inequality to establish that $t \mapsto r_t$ is concave.) Prove that $\mathcal{V}_d(K'_+) = \mathcal{V}_d(K_+)$ and $\mathcal{V}_d(K'_-) = \mathcal{V}_d(K_-)$.

b) Now consider a second transformation by "conification" of K'. Consider a cone C defined as follows: C_t is the $(d-1)$-dimensional ball $B(0, r'_t)$ with $r'_t = (r_0 - \alpha t)\mathbf{1}\{r_0\alpha^{-1} \geq t \geq t_-\}$, with α chosen so that $\mathcal{V}_d(C_+) = \mathcal{V}_d(K'_+)$ and t_- chosen so that $\mathcal{V}_d(C_-) = \mathcal{V}_d(K'_-)$ (note that $C_0 = K'_0$ by construction).

   Prove that the center of gravity of C must have nonnegative first coordinate g_C (and zero other coordinates)

   Hint: let $F(t) = \mathcal{V}_d(K'_{[0,t]})$ and $G(t) = \mathcal{V}_d(C_{[0,t]})$. Then $F(0) = G(0) = 0$ and $F(\infty) = G(\infty)$. Furthermore $F'(t) = C_d r_t^{d-1}$ and $G'(t) = C_d(r'_t)^{d-1}$. By concavity of r_t and linearity of r'_t, we have $r'_t \geq r_t$ for $t \in [0, T_0]$ then $r'_t \leq r_t$ for $t \in [T_0, \infty]$ for some T_0. Hence $H = F - G$ is such that $H(0) = H(\infty) = 0$, and H nondecreasing on $[0, T_0]$ then nonincreasing on $[T_0, \infty]$, hence $H(t) \geq 0$ for all $t \geq 0$.

c) Deduce from the previous question that the cone is a worst-case situation. Compute the position of g_C of the center of gravity of a cone of height h and its volume, and conclude.