Themen des statistischen maschinellen Lernens
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Sheet 1

1. Let X be a Rademacher random variable, i.e. $\mathbb{P}(X=-1)=\mathbb{P}(X=1)=1/2$. Show that $X \in SG(1)$.

2. Let X be a real-valued random variable.
   (a) Show that if $X \in SG(\sigma^2)$, then $X^2 \in SE(2\sigma^2)$.
   (b) Show that if $X \in SG(\sigma^2)$, then $X - \mathbb{E}X \in SG(2\sigma^2)$. (*Hint*: use Jensen's inequality.)
   (c) Show that if $\mathbb{E}[X]=0$ and $X^2 \in SE(\mu)$, then there exists an absolute constant $C>0$ such that $X \in SG(C\mu)$.
   (d) Show that if $X \in SE(\mu)$, then there exists an absolute constant $C>0$ such that $X - \mathbb{E}X \in SE(C\mu)$.

3. (Hoeffding's lemma). Let X be a random variable with $\mathbb{E}[X]=0$, taking values in the bounded interval $[-K,K]$, where $K>0$.
   (a) Use the convexity of $x \mapsto e^{ux}$ to show that
   $$e^{u\delta} \le \frac{K-\delta}{2K}e^{-uK} + \frac{K+\delta}{2K}e^{uK} \text{ for every } |\delta| \le K.$$
   (b) Deduce that
   $$\mathbb{E}\big[e^{uX}\big] \le (e^{-uK}+e^{uK})/2 \le e^{u^2K^2/2} \text{ for every } u \in \mathbb{R}.$$
   Thus $X \in SG(K^2)$.

4. Suppose that $X \in SG(\sigma^2)$. Show that
   $$\mathbb{E}\big[|X|^k\big] \le (2\sigma^2)^{k/2} k\Gamma(k/2)$$
   for all integers $k \ge 1$, where $\Gamma(\cdot)$ denotes the Gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, $x>0$. (*Hint*: apply the formula $\mathbb{E}[|X|^k] = \int_0^\infty kt^{k-1}\mathbb{P}(|X| \ge t)dt$.) Deduce that there exists an absolute constant $C>0$ such that $(\mathbb{E}[|X|^k])^{1/k} \le C\sqrt{k}\sigma$ for all integers $k \ge 1$.

   Conversely, suppose that X is a centered random variable such that for some constant $\sigma > 0$,
   $$(\mathbb{E}[|X|^k])^{1/k} \le \sqrt{k}\sigma$$
   for all integers $k \ge 1$. Show that there exists an absolute constant $C>0$ such that $X \in SG(C\sigma^2)$. (*Hint*: By the dominated convergence theorem, we have $\mathbb{E}\big[e^{uX}\big] = 1 + \sum_{k=2}^\infty u^k\mathbb{E}[X^k]/k!$.)