# SPDE Gallery 1

Stochastic convection-diffusion equation with spatial transport

$$\partial_t X(t,x) = (a\Delta + \vartheta \cdot \nabla + c)X(t,x))dt + \dot{W}(t,x), \quad t, x \in [0,1].$$

To estimate the spatially varying transport coefficient of a stochastic convection-diffusion equation, the contribution of different local measurements must be weighted and controlled by a bandwidth to account for bias reduction. Heatmap with $a = 0.02$, $c = 0.3$ and $\vartheta(x) = -0.5 - 3x^2$ for $x \in [0,1]$.

Strauch, C., & Tiepner, A. (2024). Nonparametric velocity estimation in stochastic convection-diffusion equations from multiple local measurements. *arXiv preprint* arXiv:2402.08353.
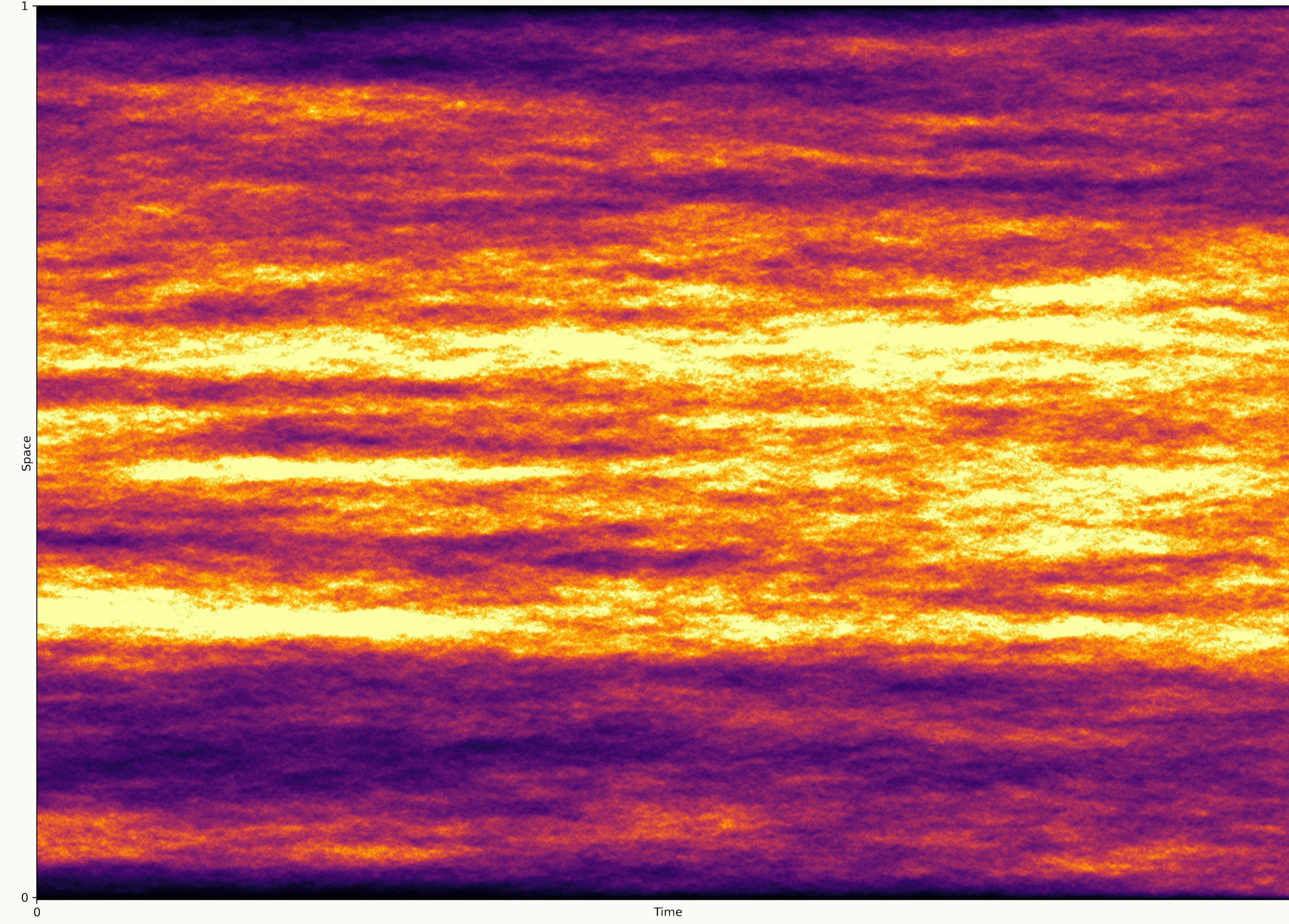
Stochastic heat equation with multiplicative noise

$$X(t,x) = \vartheta\Delta X(t,x) + \sigma(X(t,x))\dot{W}(t,x), \quad t, x \in [0,1].$$

Under multiplicative noise, the standard local estimator of ϑ is asymptotically mixed normal as the resolution becomes finer. This is based on Brownian martingale representation in Hilbert spaces and stable convergence. Weighting by quadratic variation improves the estimator and enjoys asymptotic normality. Heatmap with $\vartheta = 0.20$, $\sigma(x) = 0.20 \times |x|^{0.8} + 0.01$.

Janák, J., & Reiß, M. (2023). Parameter estimation for the stochastic heat equation with multiplicative noise from local measurements. *arXiv preprint* arXiv:2303.00074.
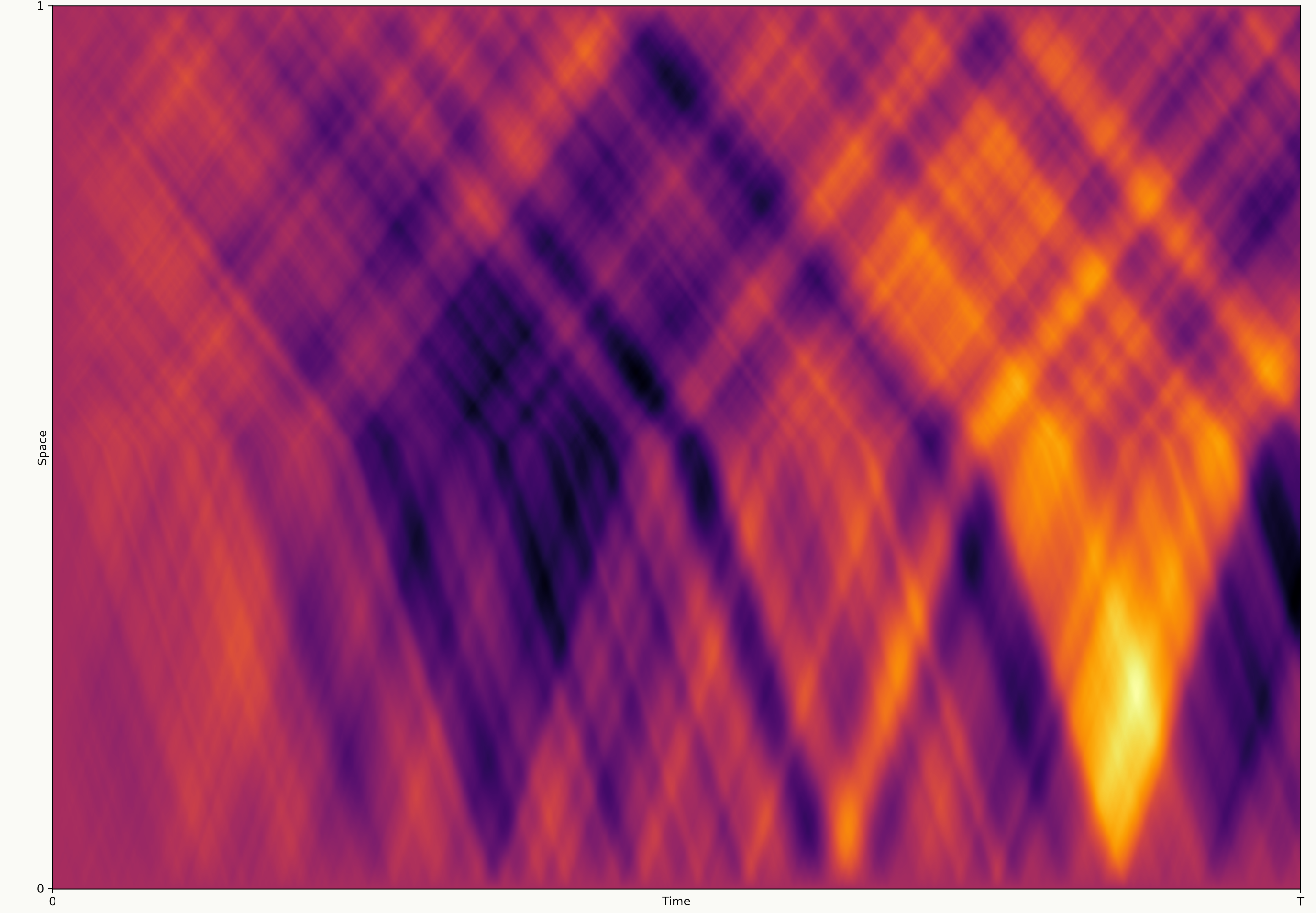
Stochastic wave equation with spatially dependent speed

$$\partial_t^2 u(t,x) = \partial_x(\vartheta\partial_x)u(t,x) + \dot{W}(t,x), \quad t, x \in [0,1].$$

The estimation of the wave speed ϑ using local measurements is intrinsically related to the energetic behaviour of the wave equation and involves the theory of Riemann-Lebesgue operators. Heatmap with $\vartheta(x) = \vartheta_0 \mathbb{1}_{[0,0.5]}(x) + \vartheta_1 \mathbb{1}_{(0.5,1]}(x)$, $x \in [0,1]$.

Ziebell, E. (2024). Non-parametric estimation for the stochastic wave equation. Coming soon. *arXiv preprint* arXiv:2404.18823
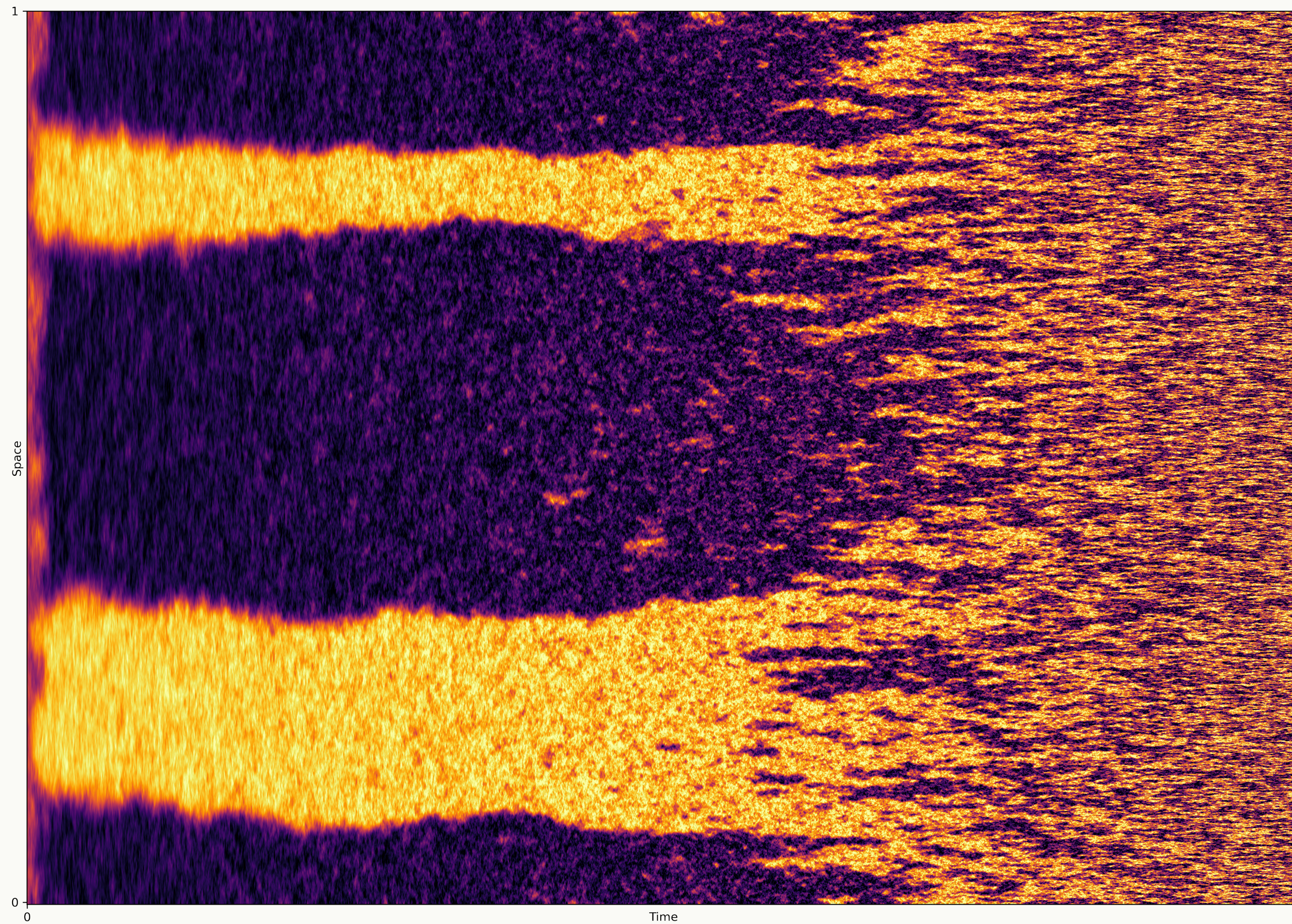
Stochastic Allen-Cahn equation with time-varying diffusivity

$$\partial_t X(t,x) = \nu(t)\Delta X(t,x) + \vartheta(X(t,x)) + \sigma\dot{W}(t,x), \quad t \in [0,5], x \in [0,1].$$

The estimation of the reaction function ϑ uses the asymptotic spatial ergodicity of the system as the diffusivity tends to zero and requires Malliavin calculus. Heatmap with $\vartheta(z) = -4(z^3 - 16z)$ for $z \in \mathbb{R}$, $\sigma = 1$ and $\nu(t) = 10^{(-0.6t-2)}$.

Gaudlitz, S. (2023). Non-parametric estimation of the reaction term in semi-linear SPDEs with spatial ergodicity. *arXiv preprint* arXiv:2307.05457.
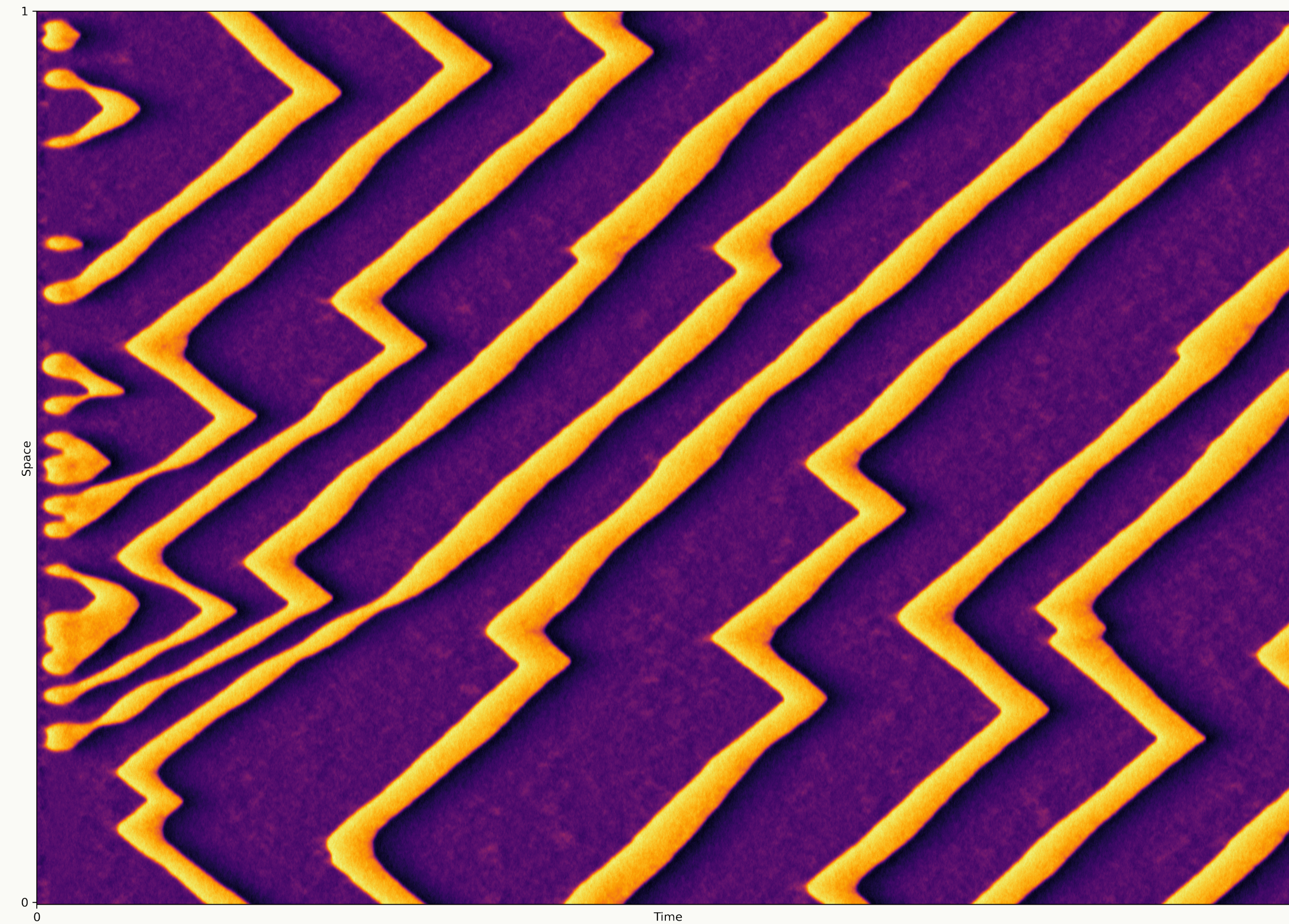
Activator component of a stochastic FitzHugh-Nagumo system with mass stabilisation

$$\partial_t U(t,x) = D_U\Delta U(t,x) + k_1 U(t,x)(u_0 - U(t,x))(U(t,x) - u_0 a[U(t,\cdot)]) - k_2 V(t,x) + \dot{W}(t,x)$$
$$\partial_t V(t,x) = D_V\Delta V(t,x) + \varepsilon(bU(t,x) - V(t,x)), \quad t, x \in [0,1].$$

The estimation of the activator diffusivity turns out to be asymptotically robust under misspecification in the reaction model in the generating equations. This can even encompass the presence or absence of different dynamical features in the observed trajectory.

Pasemann, G., Flemming, S., Alonso, S., Beta, C., & Stannat, W. (2021). Diffusivity estimation for activator–inhibitor models: Theory and application to intracellular dynamics of the actin cytoskeleton. Journal of nonlinear science, 31(3), 59.
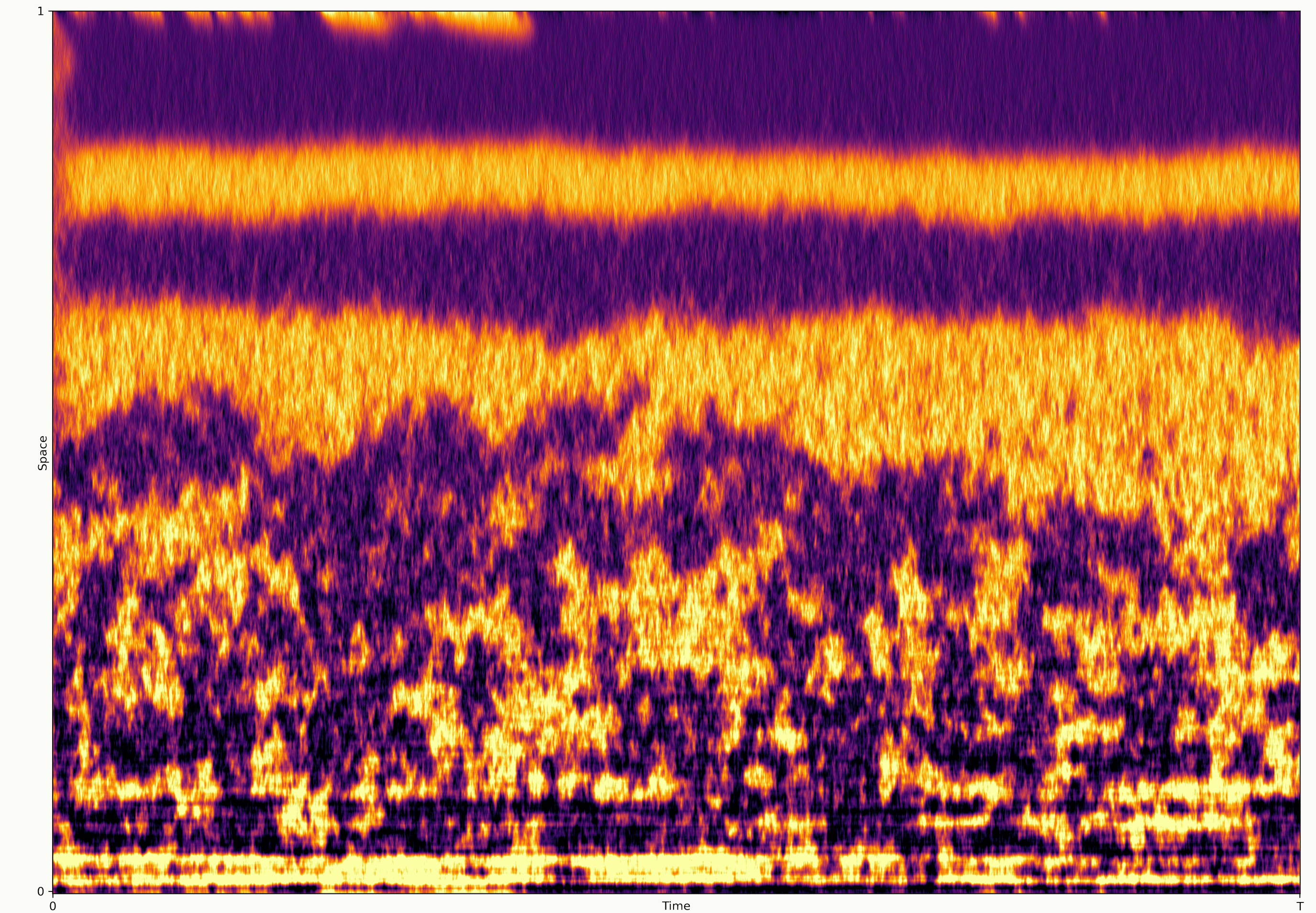
Stochastic Allen-Cahn equation with noise and space-dependent Hurst parameter

$$(\partial_t - \nu\Delta)X(t,x) + \vartheta(X(t,x)) = \sigma\dot{W}^{H(x)}(t,x), \quad t, x \in [0,1].$$

Estimation of the Hurst parameter H in parabolic SPDEs can be based on exploiting the local effect of parabolic spatio-temporal scaling (corresponding to LHS) on the probabilistic distribution of the space-time noise (RHS) via its self-similarity. Heatmap with $H(x) = x$ and $\vartheta(y) = 40y(1-y)(y-1/2)$.

Kříž P., Self-similarity and Hurst index estimation for parabolic SPDEs. Manuscript.