Stochastic Analysis (Stochastic Processes II)
course notes
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1 Construction and properties of Brownian motion

1.1 Motivation

1.1 Definition. A process $(B_t,\, t \geqslant 0)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ is called Brownian motion (Brownsche Bewegung) if

(a) $B_0 = 0$ and $B_t \sim N(0,t)$, $t > 0$, holds;

(b) the increments are stationary and independent: for $0 \leqslant t_0 < t_1 < \cdots < t_m$ we have

$$(B_{t_1} - B_{t_0}, \ldots, B_{t_m} - B_{t_{m-1}}) \sim N\big(0, \operatorname{diag}(t_1 - t_0, \ldots, t_m - t_{m-1})\big).$$

(c) B has continuous sample paths, i.e. $t \mapsto B_t(\omega)$ is continuous (for $\mathbb{P}$-almost all $\omega \in \Omega$).

1.2 Definition. Brownian motion $(B_t, t \in [0,T])$ induces an image measure (law) $\mathbb{P}^W := \mathbb{P}^{(B_t, t\in[0,T])}$ on the path space $(C([0,T]), \mathfrak{B}_{C([0,T])})$, called Wiener measure.

1.3 Remark. Recall the construction of Brownian motion as a limit of rescaled, interpolated random walks via Donsker's invariance principle.

1.4 Lemma. *Let $(B_t, t \geqslant 0)$ be a Brownian motion. Then the following processes are also Brownian motions:*

(a) $(-B_t, t \geqslant 0)$*;*

(b) $(a^{-1/2} B_{at}, t \geqslant 0)$ *for any* $a > 0$ *('time change');*

(c) $(X_t, t \geqslant 0)$ *with* $X_t = tB_{1/t}$ *for* $t > 0$ *and* $X_0 = 0$ *('time inversion').*

1.2 Construction of Brownian motion

1.5 Lemma. *Brownian motion* $(B_t,\, t \geqslant 0)$ *is a centred Gaussian process with covariance function* $\operatorname{Cov}(B_t, B_s) = t \wedge s$, $t, s \geqslant 0$. *Conversely, a continuous Gaussian process* $(X_t,\, t \geqslant 0)$ *with* $\mathbb{E}[X_t] = 0$, $\operatorname{Cov}(X_t, X_s) = t \wedge s$, $t, s \geqslant 0$, *is a Brownian motion.*

1.6 Definition. Two processes $(X_t,\, t \in T)$, $(Y_t,\, t \in T)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ are called

(a) indistinguishable (ununterscheidbar) if $\mathbb{P}(\forall\, t \in T: \; X_t = Y_t) = 1$;

(b) versions or modifications (Versionen, Modifikationen) of each other if we have $\forall\, t \in T: \; \mathbb{P}(X_t = Y_t) = 1$.

1.7 Theorem. *(Kolmogorov, Centsov, 1956) Let* $(X_t)_{t \in [0,T]}$ *be a stochastic process on* $(\Omega, \mathscr{F}, \mathbb{P})$*. If there are constants* $C > 0$, $\alpha, \beta > 0$ *such that*

$$\forall s, t \in [0,T]: \; \mathbb{E}[|X_t - X_s|^\alpha] \leqslant C|t-s|^{1+\beta},$$

then X *has a continuous version* $\tilde{X}$*, which has even* γ*-Hölder continuous paths for any* $\gamma \in (0, \beta/\alpha)$*, i.e.*

$$\forall \omega \in \Omega\, \exists L(\omega) > 0\, \forall t, s \in [0,T]: \; |\tilde{X}_t(\omega) - \tilde{X}_s(\omega)| \leqslant L(\omega)|t-s|^\gamma.$$

1.8 Corollary. *A Brownian motion with γ-Hölder-continuous sample paths exists for any $\gamma \in (0, 1/2)$.*