Stochastic Analysis
(Stochastic Processes II)
course notes
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Contents

1 Construction and properties of Brownian motion

1.1 Motivation

1.1 Definition. A process $(B_t,\, t \geqslant 0)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ is called Brownian motion (Brownsche Bewegung) if

(a) $B_0 = 0$ and $B_t \sim N(0,t)$, $t > 0$, holds;

(b) the increments are stationary and independent: for $0 \leqslant t_0 < t_1 < \cdots < t_m$ we have

$$(B_{t_1} - B_{t_0}, \dots, B_{t_m} - B_{t_{m-1}}) \sim N\big(0, \operatorname{diag}(t_1 - t_0, \dots, t_m - t_{m-1})\big).$$

(c) B has continuous sample paths, i.e. $t \mapsto B_t(\omega)$ is continuous (for $\mathbb{P}$-almost all $\omega \in \Omega$).

1.2 Definition. Brownian motion $(B_t, t \in [0,T])$ induces an image measure (law) $\mathbb{P}^W := \mathbb{P}^{(B_t, t \in [0,T])}$ on the path space $(C([0,T]), \mathfrak{B}_{C([0,T])})$, called Wiener measure.

1.3 Remark. Recall the construction of Brownian motion as a limit of rescaled, interpolated random walks via Donsker's invariance principle.

1.4 Lemma. *Let $(B_t, t \geqslant 0)$ be a Brownian motion. Then the following processes are also Brownian motions:*

(a) $(-B_t, t \geqslant 0)$*;*

(b) $(a^{-1/2} B_{at}, t \geqslant 0)$ *for any* $a > 0$ *('time change');*

(c) $(X_t, t \geqslant 0)$ *with* $X_t = t B_{1/t}$ *for* $t > 0$ *and* $X_0 = 0$ *('time inversion').*

1.2 Construction of Brownian motion

1.5 Lemma. *Brownian motion* $(B_t,\, t \geqslant 0)$ *is a centred Gaussian process with covariance function* $\operatorname{Cov}(B_t, B_s) = t \wedge s$, $t, s \geqslant 0$. *Conversely, a continuous Gaussian process* $(X_t,\, t \geqslant 0)$ *with* $\mathbb{E}[X_t] = 0$, $\operatorname{Cov}(X_t, X_s) = t \wedge s$, $t, s \geqslant 0$, *is a Brownian motion.*

1.6 Definition. Two processes $(X_t,\, t \in T)$, $(Y_t,\, t \in T)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ are called

(a) indistinguishable (ununterscheidbar) if $\mathbb{P}(\forall\, t \in T :\ X_t = Y_t) = 1$;

(b) versions or modifications (Versionen, Modifikationen) of each other if we have $\forall\, t \in T :\ \mathbb{P}(X_t = Y_t) = 1$.

1.7 Theorem. *(Kolmogorov, Centsov, 1956) Let* $(X_t)_{t \in [0,T]}$ *be a stochastic process on* $(\Omega, \mathscr{F}, \mathbb{P})$*. If there are constants* $C > 0$, $\alpha, \beta > 0$ *such that*

$$\forall s, t \in [0,T] :\ \mathbb{E}[|X_t - X_s|^\alpha] \leqslant C|t-s|^{1+\beta},$$

then X *has a continuous version* $\tilde{X}$*, which has even* γ*-Hölder continuous paths for any* $\gamma \in (0, \beta/\alpha)$*, i.e.*

$$\forall \omega \in \Omega\, \exists L(\omega) > 0\, \forall t, s \in [0,T] :\ |\tilde{X}_t(\omega) - \tilde{X}_s(\omega)| \leqslant L(\omega)|t-s|^\gamma.$$

1.8 Corollary. *Brownian motion exists and has a.s.* γ*-Hölder-continuous sample paths for any* $\gamma \in (0, 1/2)$*.*

1.3 Properties of Brownian sample paths

1.9 Theorem. *(Quadratic variation, Lévy) Let* $\tau_n = \{t_0^{(n)}, \dots, t_{m_n}^{(n)}\}$ *with* $0 = t_0^{(n)} < \dots < t_{m_n}^{(n)} = 1$, $n \geqslant 1$, *be partitions of* $[0,1]$ *with* $\tau_n \subseteq \tau_{n+1}$ *(refinement) and* $\max_{i=1,\dots m_n} |t_i^{(n)} - t_{i-1}^{(n)}| \to 0$ *as* $n \to \infty$ *(asymptotically dense). Then for a Brownian motion* B

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} (B_{t_i^{(n)} \wedge t} - B_{t_{i-1}^{(n)} \wedge t})^2 = t$$

holds in L^2 *and almost surely.*

1.10 Corollary. *Brownian motion is a.s. not of bounded variation on any interval* $[0,t]$ *and in particular not continuously differentiable.*

1.11 Theorem. *(Law of the iterated logarithm, Khinchine 1933) For Brownian motion* B *we have almost surely:*

(a) $\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{2t\log(\log(t^{-1}))}} = 1$;

(b) $\liminf_{t\downarrow 0} \frac{B_t}{\sqrt{2t\log(\log(t^{-1}))}} = -1$;

(c) $\limsup_{t\uparrow \infty} \frac{B_t}{\sqrt{2t\log(\log(t))}} = 1$;

(d) $\liminf_{t\uparrow \infty} \frac{B_t}{\sqrt{2t\log(\log(t))}} = -1$.

1.12 Lemma. *For* $Z \sim N(0,1)$ *and* $a > 0$ *we have the bounds*

$$\frac{1}{\sqrt{2\pi}} \frac{1}{a+1/a} e^{-a^2/2} \leqslant \mathbb{P}(Z \geqslant a) \leqslant \frac{1}{\sqrt{2\pi}} \frac{1}{a} e^{-a^2/2}.$$

1.4 Brownian motion as a martingale and Markov process

1.13 Definition. A process $(X_t, t \geqslant 0)$ is called

(a) adapted to a filtration $(\mathscr{F}_t)_{t\geqslant 0}$ if X_t is $\mathscr{F}_t$-measurable for all $t \geqslant 0$;

(b) $(\mathscr{F}_t)$-martingale (sub-/super-martingale) if it is adapted, $X_t \in L^1(\mathbb{P})$ and $\mathbb{E}[X_t \,|\, \mathscr{F}_s] = X_s$ ($\mathbb{E}[X_t \,|\, \mathscr{F}_s] \geqslant X_s$, $\mathbb{E}[X_t \,|\, \mathscr{F}_s] \leqslant X_s$) for all $0 \leqslant s \leqslant t$;

(c) $(\mathscr{F}_t)$-Brownian motion if it is adapted, continuous, $X_0 = 0$, the increments $X_t - X_s$ are independent of $\mathscr{F}_s$ and $X_t - X_s \sim N(0, t-s)$ for all $0 \leqslant s < t$.

1.14 Proposition. *If* B *is an* $(\mathscr{F}_t)$*-Brownian motion, then the following processes are* $(\mathscr{F}_t)$*-martingales:*

$$B_t; \quad B_t^2 - t; \quad \exp(\lambda B_t - \tfrac{1}{2}\lambda^2 t) \text{ for any } \lambda \in \mathbb{R}\,.$$

1.15 Proposition. *If* B *is a Brownian motion with respect to a filtration* $(\mathscr{F}_t^0)_{t\geqslant 0}$*, then also with respect to its right-continuous extension* $\mathscr{F}_t = \mathscr{F}_{t+}^0 := \bigcap_{s>t} \mathscr{F}_s^0$.

1.16 Definition. A random variable τ with values in $[0, +\infty]$ is called $(\mathscr{F}_t)$-stopping time if $\{\tau \leqslant t\} \in \mathscr{F}_t$ holds for all $t \geqslant 0$. The σ-algebra of τ-history is given by $\mathscr{F}_\tau := \{A \in \mathscr{F} \mid A \cap \{\tau \leqslant t\} \in \mathscr{F}_t \text{ for all } t \geqslant 0\}$.

From now on we always assume a right-continuous filtration $(\mathscr{F}_t)_{t \geqslant 0}$.

1.17 Lemma. *For an adapted right-continuous process X and a finite stopping time τ, the map $\omega \mapsto X_{\tau(\omega)}(\omega)$ is $\mathscr{F}_\tau$-measurable.*

1.18 Lemma. *Let τ be an $(\mathscr{F}_t)_{t \geqslant 0}$-stopping time and $t \geqslant 0$. Then:*

(a) $\mathscr{F}_t \cap \mathscr{F}_\tau \subseteq \mathscr{F}_{t \wedge \tau}$;

(b) $A \in \mathscr{F}_\tau, A' \in \mathscr{F}_t \Rightarrow A \cap \{\tau \leqslant t\}, A' \cap \{\tau > t\} \in \mathscr{F}_{t \wedge \tau}$;

(c) For an $\mathscr{F}_\tau$-measurable random variable X and an $\mathscr{F}_t$-measurable random variable X', the random variables $X\mathbf{1}(\tau \leqslant t), X'\mathbf{1}(\tau > t)$ are $\mathscr{F}_{t \wedge \tau}$-measurable.

1.19 Theorem. *Let $(X_t, t \geqslant 0)$ be an $(\mathscr{F}_t)$-adapted right-continuous process with $X_t \in L^1(\mathbb{P})$ for all $t \geqslant 0$. Then the following are equivalent:*

(a) X is a martingale;

(b) for any bounded stopping time τ we have $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$;

(c) for all bounded stopping times $\sigma \leqslant \tau$ we have $\mathbb{E}[X_\tau \mid \mathscr{F}_\sigma] = X_\sigma$ (optional sampling);

(d) for all stopping times τ the process $(X_{t \wedge \tau}, t \geqslant 0)$ is an $(\mathscr{F}_t)$-martingale (optional stopping).

1.20 Corollary. *For a right-continuous martingale $(M_t, t \geqslant 0)$ and a finite stopping time τ we have $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$ provided $(M_{t \wedge \tau}, t \geqslant 0)$ is uniformly integrable (e.g. dominated or bounded).*

1.21 Proposition. *For a Brownian motion $(B_t, t \geqslant 0)$ and the stopping time $\tau_{a,b} := \inf\{t \geqslant 0 \mid X_t \notin (a, b)\}$ of first hitting $a < 0$ or $b > 0$ we have*

$$\mathbb{P}(B_{\tau_{a,b}} = b) = \frac{|a|}{|a| + b}, \quad \mathbb{P}(B_{\tau_{a,b}} = a) = \frac{b}{|a| + b}, \quad \mathbb{E}[\tau_{a,b}] = |a|b.$$

1.22 Proposition. *For a Brownian motion $(B_t, t \geqslant 0)$ and the passage time $\tau_b := \inf\{t \geqslant 0 \mid X_t = b\}$ at $b > 0$ we have*

$$\mathbb{E}[e^{-\lambda \tau_b}] = e^{-b\sqrt{2\lambda}}, \quad \lambda \geqslant 0,$$

which yields (using inverse Laplace transfrom) that τ_b has the density

$$f_b(t) = \frac{b}{\sqrt{2\pi t^3}} e^{-b^2/(2t)}, \quad t > 0.$$

1.23 Theorem. *Brownian motion B is a strong Markov process in the sense that for any finite stopping time τ the process $\tilde{B}_t := B_{\tau+t} - B_\tau$, $t \geqslant 0$, is again a Brownian motion, independent of $\mathscr{F}_\tau$.*

1.24 Corollary. *(Reflection principle) We have* $\mathbb{P}(\max_{0\leqslant s\leqslant t} B_s \geqslant b) = 2\,\mathbb{P}(B_t \geqslant b)$ *for a Brownian motion* B *and* $t, b \geqslant 0$.

1.25 Corollary. *The random variables* $M_t = \max_{0\leqslant s\leqslant t} B_t$, $|B_t|$ *and* $M_t - B_t$ *have the same distribution for a Brownian motion* B *and* $t \geqslant 0$.

2 Continuous martingales and stochastic integration

2.1 Local martingales and simple stochastic integrals

2.1 Definition. An $(\mathscr{F}_t)$-adapted continuous process $(M_t, t \geqslant 0)$ is called continuous local martingale if there are $(\mathscr{F}_t)$-stopping times $\tau_1 \leqslant \tau_2 \leqslant \cdots$ with $\tau_n \uparrow +\infty$ a.s. such that $M_t^{\tau_n} := M_{t\wedge\tau_n}$, $t \geqslant 0$, are $(\mathscr{F}_t)$-martingales for all $n \geqslant 1$. The sequence (τ_n) is called localising sequence of stopping times for M.

2.2 Definition. A piecewise constant process $(X_t, t \geqslant 0)$ of the form

$$X_t(\omega) = \sum_{k=0}^{\infty} \xi_k(\omega)\mathbf{1}_{(\tau_k(\omega),\tau_{k+1}(\omega)]}(t), \quad t \geqslant 0,\ \omega \in \Omega,$$

is a simple process if $0 = \tau_0 \leqslant \tau_1 \leqslant \cdots$ are $(\mathscr{F}_t)$-stopping times with $\tau_k \uparrow \infty$ a.s. and each ξ_k is an $(\mathscr{F}_{\tau_k})$-measurable (real) random variable. For any other adapted process $(Y_t, t \geqslant 0)$ we call

$$\int_0^t X_s dY_s := \sum_{k=0}^{\infty} \xi_k (Y_{t\wedge\tau_{k+1}} - Y_{t\wedge\tau_k}), \quad t \geqslant 0,$$

the stochastic integral of X with respect to Y.

2.3 Proposition. *If* X *is a simple bounded process and* M *a continuous* L^2-*martingale, then* $(\int_0^t X_s dM_s, t \geqslant 0)$ *is a continuous* L^2-*martingale as well.*

If X *is a simple process and* M *is a continuous martingale, then* $(\int_0^t X_s dM_s, t \geqslant 0)$ *is a continuous local martingale.*

2.4 Proposition. *Let* $X_t = \sum_{k=0}^{\infty} \xi_k \mathbf{1}_{(\tau_k,\tau_{k+1}]}(t)$ *be a bounded simple process with* $|X_t| \leqslant C$ *for all* $t \geqslant 0$ *and a deterministic constant* $C \geqslant 0$. *If* M *is a continuous* L^2-*martingale, then*

$$\mathbb{E}\Big[\Big(\int_0^t X_s dM_s\Big)^2\Big] = \sum_{k=0}^{\infty} \mathbb{E}\Big[\xi_k^2\, \mathbb{E}[M_{t\wedge\tau_{k+1}}^2 - M_{t\wedge\tau_k}^2 \mid \mathscr{F}_{t\wedge\tau_k}]\Big] \leqslant C^2\, \mathbb{E}[M_t^2].$$

2.2 Quadratic variation

2.5 Definition. From now on we always suppose that the filtration $(\mathscr{F}_t)$ contains all null sets (it is completed). Together with right-continuity it fulfills the '*usual conditions*'.

$\mathscr{M}_T^2$ denotes the space of all continuous $(\mathscr{F}_t)$-martingales $(M_t, 0 \leqslant t \leqslant T)$ with $M_0 = 0$, $M_T \in L^2(\mathbb{P})$. For $M, N \in \mathscr{M}_T^2$ we set $\|M\|_{\mathscr{M}_T^2} := \|M_T\|_{L^2(\mathbb{P})}$, $\langle M, N\rangle_{\mathscr{M}_T^2} := \langle M_T, N_T\rangle_{L^2(\mathbb{P})} = \mathbb{E}[M_T N_T]$.

2.6 Proposition. *$(\mathscr{M}_T^2, \langle \bullet, \bullet \rangle_{\mathscr{M}_T^2})$ is a Hilbert space, i.e. a complete space with scalar product, if indistinguishable $M, M' \in \mathscr{M}_T^2$ are identified.*

2.7 Theorem. *Suppose the martingale $M \in \mathscr{M}_T^2$ has finite variation, i.e. $V_T(M) = \sup_{n \geqslant 1} \sum_{i=1}^{m_n} |M_{t_i^{(n)}} - M_{t_{i-1}^{(n)}}| < \infty$ a.s. for partitions $= 0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{m_n}^{(n)} = T$ with $\max_i |t_i^{(n)} - t_{i-1}^{(n)}| \to 0$. Then $M = 0$ a.s.*

2.8 Theorem. *For every bounded continuous martingale $(M_t, t \geqslant 0)$ there exists a unique (up to indistinguishability) quadratic variation process $(\langle M \rangle_t, t \geqslant 0)$, which is adapted, continuous, increasing with $\langle M \rangle_0 = 0$ and $(M_t^2 - \langle M \rangle_t, t \geqslant 0)$ is a martingale.*

2.9 Corollary. *For every continuous local martingale $(M_t, t \geqslant 0)$ there exists a unique (up to indistinguishability) quadratic variation process $(\langle M \rangle_t, t \geqslant 0)$, which is adapted, continuous, increasing with $\langle M \rangle_0 = 0$ and $(M_t^2 - \langle M \rangle_t, t \geqslant 0)$ is a local martingale.*

2.3 Stochastic integration

2.10 Lemma. *For a bounded simple process $(X_t, t \in [0,T])$ and $M \in \mathscr{M}_T^2$ the stochastic integral is a continuous L^2-martingale with*

(a) $\langle \int_0^\bullet X_s dM_s \rangle_t = \int_0^t X_s^2 d\langle M \rangle_s$, $t \in [0,T]$;

(b) $\mathbb{E}[(\int_0^t X_s dM_s)^2] = \mathbb{E}[\int_0^t X_s^2 d\langle M \rangle_s]$ (Itô isometry), $t \in [0,T]$.

Proof. We have for $t, h \geqslant 0$ (with $\int_t^{t+h} := \int_0^{t+h} - \int_0^t$)

$$\Big(\int_0^{t+h} X_s dM_s\Big)^2 - \Big(\int_0^t X_s dM_s\Big)^2 = \Big(\int_t^{t+h} X_s dM_s\Big)^2 + 2\Big(\int_t^{t+h} X_s dM_s\Big)\Big(\int_0^t X_s dM_s\Big).$$

We have

$$\mathbb{E}\Big[\int_t^{t+h} X_s dM_s \,\Big|\, \mathscr{F}_t\Big] = \mathbb{E}\Big[\sum_{k \geqslant 0} \xi_k \mathbf{1}(\tau_k \leqslant t+h)(M_{\tau_{k+1} \wedge (t+h) \vee t} - M_{\tau_k \wedge (t+h) \vee t}) \,\Big|\, \mathscr{F}_t\Big]$$
$$= \sum_{k \geqslant 0} \mathbb{E}\Big[\xi_k \mathbf{1}(\tau_k \leqslant t+h)\, \mathbb{E}\Big[M_{\tau_{k+1} \wedge (t+h) \vee t} - M_{\tau_k \wedge (t+h) \vee t} \,\Big|\, \mathscr{F}_{\tau_k \wedge (t+h) \vee t}\Big] \,\Big|\, \mathscr{F}_t\Big] = 0,$$

where we used that $\xi_k \mathbf{1}(\tau_k \leqslant t+h)$ is $\mathscr{F}_{\tau_k \wedge (t+h) \vee t}$-measurable by Lemma 1.18 (and $\mathscr{F}_{\tau_k \wedge (t+h)} \subseteq \mathscr{F}_{\tau_k \wedge (t+h) \vee t}$) as well as that the inner conditional expectation vanishes by optional sampling. Since $\int_0^t X_s dM_s$ is $\mathscr{F}_t$-measurable, we obtain

$$\mathbb{E}\Big[2\Big(\int_t^{t+h} X_s dM_s\Big)\Big(\int_0^t X_s dM_s\Big) \,\Big|\, \mathscr{F}_t\Big] = 0.$$

Exactly as in the proof of Proposition 2.4

$$\begin{aligned}
\mathbb{E}\Big[\Big(\int_t^{t+h} X_s dM_s\Big)^2 \,\Big|\, \mathscr{F}_t\Big] &= \sum_{k\geqslant 0} \mathbb{E}\Big[\xi_k^2\, \mathbb{E}[M^2_{\tau_{k+1}\wedge(t+h)\vee t} - M^2_{\tau_k\wedge(t+h)\vee t} \,|\, \mathscr{F}_{\tau_k\wedge(t+h)\vee t}] \,\Big|\, \mathscr{F}_t\Big] \\
&= \sum_{k\geqslant 0} \mathbb{E}\Big[\xi_k^2\, \mathbb{E}[\langle M\rangle_{\tau_{k+1}\wedge(t+h)\vee t} - \langle M\rangle_{\tau_k\wedge(t+h)\vee t} \,|\, \mathscr{F}_{\tau_k\wedge(t+h)\vee t}] \,\Big|\, \mathscr{F}_t\Big] \\
&= \sum_{k\geqslant 0} \mathbb{E}\Big[\xi_k^2 (\langle M\rangle_{\tau_{k+1}\wedge(t+h)\vee t} - \langle M\rangle_{\tau_k\wedge(t+h)\vee t}) \,\Big|\, \mathscr{F}_t\Big] \\
&= \mathbb{E}\Big[\int_t^{t+h} X_s^2 d\langle M\rangle_s \,\Big|\, \mathscr{F}_t\Big].
\end{aligned}$$

We conclude

$$\mathbb{E}\Big[\Big(\int_0^{t+h} X_s dM_s\Big)^2 \,\Big|\, \mathscr{F}_t\Big] - \Big(\int_0^t X_s dM_s\Big)^2 = \mathbb{E}\Big[\int_0^{t+h} X_s^2 d\langle M\rangle_s \,\Big|\, \mathscr{F}_t\Big] - \int_0^t X_s^2 d\langle M\rangle_s.$$

Since $t \mapsto \int_0^t X_s^2 d\langle M\rangle_s$ starts in zero, is adapted, continuous, increasing, the last identity shows that it is indeed the quadratic variation process.

The Itô isometry follows by taking the expected value of the quadratic variation of $\int_0^\bullet X_s dM_s$, which is a centred martingale. □

2.11 Definition. A process $(X_t, t \geqslant 0)$ is called progressively measurable with respect to $(\mathscr{F}_t)$ if it is $(\mathscr{F}_t)$-adapted and the function $(\omega, s) \mapsto X_s(\omega)$ on $\Omega \times [0,t]$ is $\mathscr{F}_t \otimes \mathscr{B}_{[0,t]}$-measurable for all $t \geqslant 0$.

2.12 Lemma. *Every adapted left- or right-continuous process is progressively measurable. In particular, every simple process is progressively measurable.*

2.13 Definition. For $M \in \mathscr{M}_T^2$ introduce the space

$$\mathscr{L}_T(M) := \Big\{(X_t, t \in [0,T]) \,\Big|\, X \text{ progressively measurable, } \mathbb{E}\Big[\int_0^T X_t^2 d\langle M\rangle_t\Big] < \infty\Big\}$$

with norm $\|X\|_{M,T} := \mathbb{E}[\int_0^T X_t^2 d\langle M\rangle_t]^{1/2}$ and scalar product $\langle X, Y\rangle_{M,T} := \mathbb{E}[\int_0^T X_t Y_t d\langle M\rangle_t]$, identifying X and Y with $\int_0^T (X_t - Y_t)^2 d\langle M\rangle_t = 0$ a.s.

2.14 Lemma. *$\mathscr{L}_T(M)$ is a Hilbert space.*

2.15 Theorem. *The set $\mathscr{E}_T := \{(X_t, t \in [0,T]) \,|\, X$ simple and bounded$\}$ is dense in $\mathscr{L}_T(M)$.*

2.16 Definition. The linear map $I_T : \mathscr{E}_T \subseteq \mathscr{L}_T(M) \to \mathscr{M}_T^2$ with $I_T(X) = \int_0^\bullet X_s dM_s$ is isometric and thus has a unique isometric extension $\tilde{I}_T : \mathscr{L}_T(M) \to \mathscr{M}_T^2$. For $X \in \mathscr{L}_T(M)$ we define the stochastic integral

$$\int_0^\bullet X_s dM_s := \tilde{I}_T(X) \in \mathscr{M}_T^2.$$

Equivalently, for $X \in \mathscr{L}_T(M)$ we choose $X^{(n)} \in \mathscr{E}_T$ with $\|X^{(n)} - X\|_{M,T} \to 0$ and define

$$\int_0^t X_s dM_s := \lim_{n\to\infty} \int_0^t X_s^{(n)} dM_s,\ t \in [0,T], \text{ in } \mathscr{M}_T^2.$$

2.17 Proposition. *(Properties of the stochastic integral) For $M \in \mathscr{M}_T^2$ and $X \in \mathscr{L}_T(M)$ we have*

(a) $(\int_0^t X_s dM_s, t \in [0,T]) \in \mathscr{M}_T^2$ *such that* $\mathbb{E}[\int_u^t X_s dM_s \,|\, \mathscr{F}_u] = 0$ *for all* $0 \leqslant u < t \leqslant T$ *and* $t \mapsto \int_0^t X_s dM_s$ *is continuous a.s.*

(b) $\langle \int_0^\bullet X_s dM_s \rangle_t = \int_0^t X_t^2 d\langle M \rangle_t$ *and* $\mathbb{E}[(\int_0^t X_s dM_s)^2] = \mathbb{E}[\int_0^t X_s^2 d\langle M \rangle_s]$ *(Itô isometry),* $t \in [0,T]$.

(c) $\forall\, \alpha, \beta \in \mathbb{R},\ X, Y \in \mathscr{L}_T(M),\ t \in [0,T] : \int_0^t (\alpha X_s + \beta Y_s)\, dM_s = \alpha \int_0^t X_s dM_s + \beta \int_0^t Y_s dM_s$.

2.18 Lemma. *For $M \in \mathscr{M}_T^2$, $X \in \mathscr{L}_T(M)$ and a stopping time τ we have*

$$\int_0^{t \wedge \tau} X_s dM_s = \int_0^t X_s dM_s^\tau = \int_0^t X_s \mathbf{1}(s \leqslant \tau)\, dM_s \quad \text{a.s.}$$

2.19 Definition. For a continuous local martingale $(M_t, t \in [0,T])$ with $M_0 = 0$ define

$$\mathscr{L}_{loc,T}(M) := \Big\{ (X_t, t \in [0,T]) \,\Big|\, X \text{ progr. measurable}, \int_0^T X_t^2 d\langle M \rangle_t < \infty \text{ a.s.} \Big\}.$$

If (σ_n) are stopping times localising M (such that $M^{\sigma_n} \in \mathscr{M}_T^2$), introduce $\tau_n := \sigma_n \wedge \inf\{t > 0 \,|\, \int_0^t X_s^2 d\langle M \rangle_s \geqslant n\}$ and define the stochastic integral for $X \in \mathscr{L}_{loc,T}(M)$ as

$$\int_0^t X_s dM_s := \lim_{n \to \infty} \int_0^t X_s dM_s^{\tau_n}, \quad t \in [0,T],$$

with a.s.-convergence.

2.20 Proposition. *For a continuous local martingale $(M_t, t \in [0,T])$ with $M_0 = 0$ and $X \in \mathscr{L}_{loc,T}(M)$ the stochastic integral $\int_0^\bullet X_s dM_s$ is well defined and satisfies:*

(a) $(\int_0^t X_s dM_s, t \in [0,T])$ *is a continuous local martingale;*

(b) $\langle \int_0^\bullet X_s dM_s \rangle_t = \int_0^t X_s^2 d\langle M \rangle_s$, $t \in [0,T]$, *where for a continuous local martingale N the quadratic variation $t \mapsto \langle N \rangle_t$ is an adapted, continuous, increasing process such that $N_t^2 - \langle N \rangle_t$, $t \in [0,T]$ forms a local martingale.*

(c) $\int_0^{t \wedge \tau} X_s dM_s = \int_0^t X_s dM_s^\tau = \int_0^t X_s \mathbf{1}(s \leqslant \tau)\, dM_s$ *holds a.s. for any stopping time τ.*

2.21 Theorem. *If $(M_t, t \in [0,T])$ is a continuous local martingale with $M_0 = 0$ and $(X_t, t \in [0,T])$ is an adapted, continuous process, then $X \in \mathscr{L}_{loc,T}(M)$ and for partitions $0 = t_0^{(m)} < \cdots < t_{n_m}^{(m)} = t$ of $[0,t] \subseteq [0,T]$ with $\max_i (t_i^{(m)} - t_{i-1}^{(m)}) \to 0$ as $m \to \infty$ we have*

$$\sum_{i=1}^{n_m} X_{t_{i-1}^{(m)}} (M_{t_i^{(m)}} - M_{t_{i-1}^{(m)}}) \xrightarrow{\mathbb{P}} \int_0^t X_s dM_s.$$

2.22 Corollary. *For a continuous local martingale* $(M_t, t \in [0,T])$ *with* $M_0 = 0$ *and partitions* $0 = t_0^{(m)} < \cdots < t_{n_m}^{(m)} = t$ *of* $[0,t] \subseteq [0,T]$ *with* $\max_i(t_i^{(m)} - t_{i-1}^{(m)}) \to 0$ *as* $m \to \infty$ *we have*

$$\sum_{i=1}^{n_m}(M_{t_i^{(m)}} - M_{t_{i-1}^{(m)}})^2 \xrightarrow{\mathbb{P}} \langle M\rangle_t \quad \text{and} \quad M_t^2 = 2\int_0^t M_s dM_s + \langle M\rangle_t.$$

3 Main theorems of stochastic analysis

3.1 The Itô formula

3.1 Definition. A continuous semi-martingale $(X_t, t \in [0,T])$ with respect to some filtration $(\mathscr{F}_t)$ is a continuous, $(\mathscr{F}_t)$-adapted process which can be decomposed as $X_t = M_t + A_t$, $t \in [0,T]$, with a continuous local $(\mathscr{F}_t)$-martingale M and a finite variation process A (which is then necessarily continuous, adapted). We define

$$\int_0^t Y_s dX_s := \int_0^t Y_s dM_s + \int_0^t Y_s dA_s, \quad t \in [0,T],$$

when $Y \in \mathscr{L}_{loc,T}(M)$ and $\int_0^t Y_s(\omega)dA_s(\omega)$ is a.s. well defined, i.e. $Y(\omega) \in L^1(\mu_1(\omega) + \mu_2(\omega))$ a.s., when $A_t = A_t^{(1)} - A_t^{(2)}$ for some increasing process $A^{(1)}, A^{(2)}$ and μ_i denote the by $A^{(i)}$ induced Lebesgue-Stieltjes measures, $i = 1,2$. In particular, $\int_0^t Y_s dX_s$ is well defined for continuous, $(\mathscr{F}_t)$-adapted processes Y.

3.2 Definition. For a continuous semi-martingale X its quadratic variation is given by

$$\langle X\rangle_t = \lim_{m\to\infty}\sum_{i=1}^{n_m}(X_{t_i^{(m)}\wedge t} - X_{t_{i-1}^{(m)}\wedge t})^2, \quad t \in [0,T],$$

with convergence in probability for any sequence of partitions $0 = t_0^{(m)} < \cdots < t_{n_m}^{(m)} = T$ with $\max_{i=1,\dots,n_m}(t_i^{(m)} - t_{i-1}^{(m)}) \to 0$ as $m \to \infty$ (if the limit exists).

For two continuous semi-martingales the quadratic covariation is defined via polarisation as

$$\langle X, Y\rangle_t = \frac{1}{4}\Big(\langle X+Y\rangle_t - \langle X-Y\rangle_t\Big), \quad t \in [0,T].$$

3.3 Proposition. *For a continuous semi-martingale* $X = M + A$ *the quadratic variation exists and satisfies* $\langle X\rangle_t = \langle M\rangle_t$.

In general, for any continuous semi-martingale X *and any continuous finite variation process* A *we have* $\langle X, A\rangle_t = 0$.

3.4 Theorem (Partial integration)**.** *For continuous semi-martingales* X, Y *we have*

$$X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t, \quad t \in [0,T], \ a.s.$$

and in particular

$$X_t^2 = X_0^2 + 2\int_0^t X_s dX_s + \langle X\rangle_t, \quad t\in[0,T], \text{ a.s.}$$

3.5 Theorem (Associativity of stochastic integration)**.** *Let $M \in \mathscr{M}_T^2$, $X \in \mathscr{L}_T(M)$ and $Y \in \mathscr{L}_T(N)$ with $N_t = \int_0^t X_s dM_s$. Then $YX \in \mathscr{L}_T(M)$ and*

$$\int_0^t Y_s dN_s = \int_0^t Y_s X_s dM_s, \quad t\in[0,T], \text{ a.s.}$$

This holds more generally for continuous semi-martingales M and processes X, Y for which the integrals make sense.

3.6 Theorem. *If X is a continuous semi-martingale and $f \in C^2(\mathbb{R})$, then $(f(X_t), t\in[0,T])$ is a continuous semi-martingale and the Itô formula holds:*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X\rangle_s, \quad t\in[0,T], \text{ a.s.}$$

3.7 Definition. A d-dimensional continuous semi-martingale $X_t = (X_t^{(1)}, \dots, X_t^{(d)})^\top$ is a vector of d (one-dimensional) continuous semi-martingales $X_t^{(1)}, \dots, X_t^{(d)}$. A d-dimensional Brownian motion $B_t = (B_t^{(1)}, \dots, B_t^{(d)})^\top$ is a vector of d independent Brownian motions $B_t^{(1)}, \dots, B_t^{(d)}$.

3.8 Theorem (Multi-dimensional Itô formula)**.** *If X is a d-dimensional continuous semi-martingale and $f \in C^2(\mathbb{R}^d)$, then $(f(X_t), t\in[0,T])$ is a continuous (one-dimensional) semi-martingale and the following Itô formula holds:*

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s)\,dX_s^{(i)} + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\,d\langle X^{(i)}, X^{(j)}\rangle_s.$$

3.9 Corollary. *If X is a continuous semi-martingale and $f \in C^{2,1}(\mathbb{R}\times\mathbb{R}^+)$ (i.e., $f\in C^1(\mathbb{R}\times\mathbb{R}^+)$ and $x\mapsto f(x,t)\in C^2(\mathbb{R})$ for all $t \geqslant 0$), then $(f(X_t,t), t\in[0,T])$ is a continuous semi-martingale and*

$$f(X_t,t) = f(X_0,0) + \int_0^t \frac{\partial f}{\partial x}(X_s,s)\,dX_s + \int_0^t \frac{\partial f}{\partial t}(X_s,s)\,ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(X_s,s)d\langle X\rangle_s.$$

3.10 Corollary. *If $f\in C^2(\mathbb{R}^d)$, $x\in\mathbb{R}^d$ and B is d-dimensional Brownian motion, then for $X_t = x + B_t$*

$$f(X_t) = f(x) + \int_0^t \langle \nabla f(X_s), dX_s\rangle + \frac{1}{2}\int_0^t \Delta f(X_s)\,ds,$$

where ∇ denotes the gradient and Δ the Laplace operator. In particular, for a harmonic function f (i.e. $\Delta f = 0$) the process $(f(X_t), t\in[0,T])$ is a continuous local martingale.

3.11 Corollary. *Consider $X_t = x + B_t$, $t \geqslant 0$ for a d-dimensional Brownian motion B and $x \in \mathbb{R}^d\setminus\{0\}$. Then for $0 < r < |x|$*

$$\mathbb{P}(\exists t\geqslant 0 : |X_t| \leqslant r) = \begin{cases} 1, & \text{for } d\in\{1,2\}, \\ (r/|x|)^{d-2} < 1, & \text{for } d \geqslant 3. \end{cases}$$

3.2 Important consequences

3.12 Theorem (Lévy's characterisation of Brownian motion). *For an $(\mathscr{F}_t)$-adapted stochastic process $(B_t, t \geqslant 0)$ the following are equivalent:*

(a) B is an $(\mathscr{F}_t)$-Brownian motion;

(b) B is a continuous local $(\mathscr{F}_t)$-martingale with $B_0 = 0$ and $\langle B \rangle_t = t$. $t \geqslant 0$.

3.13 Theorem (Burkholder-Davis-Gundy inequality). *For every $p > 0$ there are constants $c_p, C_p > 0$ such that for any continuous local martingale M with $M_0 = 0$*

$$c_p \, \mathbb{E}[\langle M \rangle_t^{p/2}] \leqslant \mathbb{E}\big[\max_{0 \leqslant s \leqslant t} |M_s|^p\big] \leqslant C_p \, \mathbb{E}[\langle M \rangle_t^{p/2}].$$

3.3 Martingale representation theorems

3.14 Proposition. *A continuous local martingale M with $M_0 = 0$ satisfies $M \in \mathscr{M}_T^2 \iff \mathbb{E}[\langle M \rangle_T] < \infty$. In particular, for a continuous local martingale M and $X \in \mathscr{L}_{loc,T}(M)$ we have $(\int_0^t X_s dM_s, t \in [0,T]) \in \mathscr{M}_T^2$ if and only if $\mathbb{E}[\int_0^T X_t^2 d\langle M \rangle_t] < \infty$.*

3.15 Theorem (Doob's martingale representation). *Let M be a continuous local martingale with $M_0 = 0$ and an absolutely continuous quadratic variation (i.e., $\langle M \rangle_t = \int_0^t G_s ds$, $t \geqslant 0$, for some $G_\bullet(\omega) \in L^1([0,\infty))$). Then there is a Brownian motion (possibly on an enlarged probability space) and $X \in \mathscr{L}_{loc,T}(B)$ such that a.s.*

$$M_t = \int_0^t X_s dB_s, \quad t \geqslant 0.$$

3.16 Theorem (Brownian martingale representation). *For a Brownian motion B let $(\mathscr{F}_t)$ be the completed canonical filtration of B. Then:*

(a) For a random variable $Z \in L^2(\Omega, \mathscr{F}_T, \mathbb{P})$ there exists a unique $h \in \mathscr{L}_T(B)$ with

$$Z = \mathbb{E}[Z] + \int_0^T h_s dB_s.$$

(b) For every L^2-martingale $(M_t, t \in [0,T])$ with respect to $(\mathscr{F}_t)$ there is a unique $h \in \mathscr{L}_T(B)$ with

$$M_t = M_0 + \int_0^t h_s dB_s, \ \text{a.s.}, \quad t \in [0,T].$$

(c) Statement (b) remains true for continuous local martingales M and $h \in \mathscr{L}_{loc,T}(B)$.

3.17 Corollary. *Every L^2-martingale with respect to the completed canonical Brownian filtration has a continuous version.*

3.18 Corollary. *The completed canonical Brownian filtration is right-continuous.*

3.19 Theorem (Dambins, Dubbins, Schwarz; time-changed Brownian motion). *Let M be a continuous local $(\mathscr{F}_t)$-martingale with $M_0 = 0$ and $\lim_{t\to\infty}\langle M\rangle_t = \infty$ a.s., where the filtration $(\mathscr{F}_t)$ is completed by null sets. Then there is a Brownian motion B such that a.s.*

$$\forall t \geqslant 0: \; M_t = B_{\langle M\rangle_t}.$$

3.20 Lemma. *Let M be a continuous local martingale. For all $0 \leqslant a < b$ the events $\{\forall t \in [a,b]: \; M_t = M_a\}$ and $\{\langle M\rangle_b = \langle M\rangle_a\}$ are a.s. equal.*

3.4 Change of measure

3.21 Lemma. *Let $(L_t, t \in [0,T])$ be a non-negative martingale on $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ with $\mathbb{E}[L_T] = 1$. Then defining the probability measure $\mathbb{Q}_T$ on $\mathscr{F}_T$ via $\frac{d\,\mathbb{Q}_T}{d\,\mathbb{P}} = L_T$ we have for any $\mathscr{F}_t$-measurable $Y \in L^1(\mathbb{Q}_T)$, $t \in [0,T]$:*

$$\forall s \in [0,t]: \; \mathbb{E}_{\mathbb{Q}_T}[Y \,|\, \mathscr{F}_s] = \frac{\mathbb{E}_{\mathbb{P}}[Y L_t \,|\, \mathscr{F}_s]}{L_s} \quad \mathbb{Q}_T\text{ -a.s.}$$

3.22 Corollary. *In the setting of the lemma assume that $(\bar{M}_t L_t, t \in [0,T])$ is a $\mathbb{P}$-martingale for some adapted process $\bar{M}$. Then $(\bar{M}_t, t \in [0,T])$ is a $\mathbb{Q}_T$-martingale.*

3.23 Theorem (Girsanov 1960). *Suppose $X \in \mathscr{L}_{loc,T}(B)$ for a Brownian motion B and let*

$$L_t := \exp\Big(\int_0^t X_s dB_s - \frac{1}{2}\int_0^t X_s^2 ds\Big), \quad t \in [0,T].$$

If L is a martingale (for which $\mathbb{E}[L_T] = 1$ suffices), then

$$\bar{B}_t := B_t - \int_0^t X_s ds, \quad t \in [0,T],$$

defines a Brownian motion under the probability measure $\mathbb{Q}_T$ given by $d\,\mathbb{Q}_T / d\,\mathbb{P} = L_T$.

3.24 Definition. The <u>Cameron-Martin space</u> of Brownian motion on $[0,T]$ is given by

$$\mathscr{H} := \Big\{ f \in C([0,T]) \,\Big|\, \exists g \in L^2([0,T])\, \forall t \in [0,T]: \; f(t) = \int_0^t g(s)\, ds \Big\}.$$

3.25 Corollary. *For all functions $h \in \mathscr{H}$ the laws of a Brownian motion $(B_t, t \in [0,T])$ and of $(B_t + h(t), t \in [0,T])$ on $C([0,T])$ are equivalent (have the same null sets).*

3.26 Definition. The <u>support</u> of a probability measure $\mathbb{P}$ on the Borel σ-algebra of a metric space is given by

$$\operatorname{supp}(\mathbb{P}) := \bigcap \{A \text{ closed} \,|\, \mathbb{P}(A) = 1\}.$$

3.27 Corollary. *The support of the law $\mathbb{P}_T^B$ of Brownian motion on $C([0,T])$ (Wiener measure) is given by* $\operatorname{supp}(\mathbb{P}_T^B) = \overline{\mathscr{H}} = \{f \in C([0,T]) \,|\, f(0) = 0\}$.

3.28 Theorem. *Suppose $X \in \mathscr{L}_{loc,T}(B)$ for a Brownian motion B and let $Y_t = \int_0^t X_s dB_s$, $t \in [0,T]$. Then the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) hold with*

- *(a) (Novikov condition): $\mathbb{E}[\exp(\frac{1}{2}\langle Y\rangle_T)] < \infty$;*
- *(b) (Kazamaki condition): Y is a martingale and $\mathbb{E}[\exp(\frac{1}{2}Y_T)] < \infty$;*
- *(c) (Girsanov hypothesis): $L_t = \exp(Y_t - \frac{1}{2}\langle Y\rangle_t)$, $t \in [0,T]$, is a martingale.*

3.29 Corollary (piecewise Novikov condition)**.** *The Girsanov condition in part (c) of the theorem holds already if there are $0 = t_0 < t_1 < \cdots < t_m = T$ such that*

$$\forall i = 1, \ldots, m: \ \mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_{t_{i-1}}^{t_i} X_s^2 ds\Big)\Big] < \infty.$$

3.30 Lemma. *For $a, x_0 \in \mathbb{R}$ deterministic and a Brownian motion B the Ornstein-Uhlenbeck process $X_t := e^{at}x_0 + e^{at}\int_0^t e^{-as}dB_s$ solves the stochastic differential equation*

$$dX_t = aX_t dt + dB_t, \quad t \geqslant 0; \quad X_0 = x_0$$

in the sense that a.s.

$$X_t = x_0 + \int_0^t aX_s ds + B_t, \quad t \geqslant 0.$$

3.31 Lemma. *For $a \in \mathbb{R}$ let $L_t^{(a)}(X) = \exp(\int_0^t aX_s dX_s - \frac{1}{2}\int_0^t a^2 X_s^2 ds)$. Then for a Brownian motion B the process $(L_t^{(a)}(B), t \in [0,T])$ is a martingale. Under $\mathbb{Q}_T^{(a)}$ on $C([0,T])$, given by $\frac{d\,\mathbb{Q}_T^{(a)}}{d\,\mathbb{P}_T^B} = L_T^{(a)}(X)$ for the coordinate process $X \in C([0,T])$ and the law $\mathbb{P}_T^B$ of Brownian motion on $C([0,T])$, the process*

$$\bar{B}_t^{(a)} := X_t - \int_0^t aX_s ds, \quad t \in [0,T],$$

is a Brownian motion.

3.32 Corollary. *The law $\mathbb{Q}_T^{(a)}$ of the Ornstein-Uhlenbeck process with parameter $a \in \mathbb{R}$ and initial value $x_0 = 0$ on $C([0,T])$ has the likelihood function (density)*

$$L_T(a) = \exp\Big(\int_0^T aX_t dX_t - \frac{1}{2}\int_0^T a^2 X_t^2 dt\Big)$$

with respect to $\mathbb{P}_T^B = \mathbb{Q}_T^{(0)}$. The maximum-likelihood estimator $\hat{a}_T$ of the parameter $a \in \mathbb{R}$, given the continuous observation of an Ornstein-Uhlenbeck process $(X_t, t \in [0,T])$, is given by

$$\hat{a}_T = \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt}.$$

3.33 Lemma. *If $a_0 < 0$ holds, then we have $\frac{1}{T}\int_0^T X_t^2 dt \to |2a_0|^{-1}$ as $T \to \infty$ in $L^2(\mathbb{Q}_T^{(a_0)})$-convergence.*

3.34 Theorem (martingale CLT)**.** *For every $T > 0$ let $X \in \mathscr{L}_T(B)$ for a Brownian motion B and assume that there are deterministic numbers $\varphi_T > 0$ such that $\varphi_T^{-2}\int_0^T X_t^2 dt \xrightarrow{\mathbb{P}} 1$. Then*

$$\varphi_T^{-1}\int_0^T X_t dB_t \xrightarrow{d} N(0,1) \text{ as } T \to \infty.$$

3.35 Corollary. *Under $\mathbb{Q}_T^{(a_0)}$ we have*

$$\hat{a}_T = a_0 + \frac{\int_0^T X_t d\bar{B}_t^{(a_0)}}{\int_0^T X_t^2 dt}$$

and the estimation error satisfies for $a_0 < 0$

$$\sqrt{T}(\hat{a}_T - a_0) \xrightarrow{d} N(0, 2|a_0|).$$

3.36 Corollary. *For $\alpha \in (0,1)$ the interval*

$$\hat{I}_{1-\alpha}^T := \left[\hat{a}_T - \sqrt{\tfrac{2|\hat{a}_T|}{T}}q_{1-\alpha/2}, \hat{a}_T + \sqrt{\tfrac{2|\hat{a}_T|}{T}}q_{1-\alpha/2}\right]$$

is an asymptotic $(1-\alpha)$-confidence interval for $a_0 < 0$, where $q_{1-\alpha/2}$ denotes the $(1-\alpha/2)$-quantile of $N(0,1)$.

3.5 Kunita-Watanabe theory

3.37 Proposition. *Let M and N be two continuous local martingales.*

(a) $(\langle M, N\rangle_t, t \geqslant 0)$ is the unique (up to indistinguishability) continuous, adapted, finite-variation process starting in zero such that

$$(M_t N_t - \langle M, N\rangle_t, t \geqslant 0)$$

is a local martingale.

(b) The map $(M,N) \mapsto \langle M, N\rangle$ is bilinear and symmetric.

(c) $\langle M, N\rangle_t = \langle M - M_0, N - N_0\rangle_t$ for all $t \in [0,T]$.

3.38 Remark. Recall from Exercise 6.1 that every function $f\colon [0,T] \to \mathbb{R}$ of finite variation can be written as $f = f_1 - f_2$ for monotone functions f_1 and f_2. The variation of f is defined as

$$V(f)_t\colon = f_1(t) + f_2(t), \quad t \in [0,T].$$

3.39 Lemma (Kunita-Watanabe inequality)**.** *Let M and N be continuous local martingales and let H and K be measurable processes such that $\int_0^T |H_t K_t| dV(\langle M, N\rangle)_t < \infty$. Then*

$$\left|\int_0^T H_t K_t d\langle M, N\rangle_t\right| \leqslant \int_0^T |H_t K_t| dV(\langle M, N\rangle)_t \leqslant \left(\int_0^T H_t^2 d\langle M\rangle_t\right)^{1/2}\left(\int_0^T K_t^2 d\langle N\rangle_t\right)^{1/2}.$$

3.40 Theorem. *Let $M \in \mathscr{M}_T^2$ and $H \in \mathscr{L}_T(M)$. The stochastic integral $\int_0^\bullet H_s dM_s$ is the unique element in $\mathscr{M}_T^2$ that satisfies*

$$\left\langle \int_0^\bullet H_s dM_S, N \right\rangle = \int_0^\bullet H_s d\langle M, N\rangle_s \quad \forall N \in \mathscr{M}_T^2.$$

3.41 Corollary. *For $M, N \in \mathscr{M}_T^2$ and $H \in \mathscr{L}_T(M)$, $K \in \mathscr{L}_T(M)$ we have*

$$\left\langle \int_0^\bullet H_s dM_s, \int_0^\bullet K_s dN_s \right\rangle_t = \int_0^t H_s K_s d\langle M_s, N_s\rangle \quad \forall t \in [0, T].$$

4 Stochastic differential equations

4.1 Strong solutions

4.1 Definition. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ be a filtered probability space with a filtration $(\mathscr{F}_t)$ satisfying the usual conditions and carrying an m-dimensional $(\mathscr{F}_t)$-Brownian motion and an $\mathscr{F}_0$-measurable random variable ξ. A strong solution to the SDE

$$dX_t = b(X_t, t)\, dt + \sigma(X_t, t)\, dB_t, \quad t \geqslant 0; \qquad X_0 = \xi \tag{4.1}$$

is a continuous, $(\mathscr{F}_t)$-adapted d-dimensional process $(X_t, t \geqslant 0)$ satisfying a.s.

$$X_t = \xi + \int_0^t b(X_s, s)\, ds + \int_0^t \sigma(X_s, s)\, dB_s, \quad t \geqslant 0.$$

Here $b : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^{d\times m}$ are measurable, locally bounded (e.g. continuous) functions and we understand coordinatewise

$$X_{i,t} = \xi_i + \int_0^t b_i(X_s, s)\, ds + \sum_{j=1}^m \int_0^t \sigma_{i,j}(X_s, s)\, dB_{j,s}, \quad t \geqslant 0.$$

4.2 Lemma (Gronwall)**.** *Let $f : [0, T] \to \mathbb{R}^d$ be bounded, measurable. Assume for some $a, b \geqslant 0$ that*

$$f(t) \leqslant a + b \int_0^t f(s)\, ds, \quad t \in [0, T].$$

Then $f(t) \leqslant ae^{bt}$ holds for all $t \in [0, T]$.

4.3 Definition. The Frobenius or Hilbert-Schmidt norm of a matrix $M \in \mathbb{R}^{d\times m}$ is given by $\|M\|_F := (\sum_{i=1,\dots,d,j=1,\dots,m} M_{ij}^2)^{1/2}$.

4.4 Lemma. *Let b, σ be of linear growth, i.e.*

$$\exists K > 0\, \forall x \in \mathbb{R}^d,\, t \geqslant 0 : \; |b(x, t)| + \|\sigma(x, t)\|_F \leqslant K(1 + |x|),$$

and assume $\mathbb{E}[|\xi|^2] < \infty$. Then for any $T \geqslant 1$ every strong solution of the SDE (4.1) *satisfies*

$$\forall t \in [0, T] : \; \mathbb{E}\Big[\sup_{0\leqslant s\leqslant t} |X_s|^2\Big] \leqslant C(K^2T^2 + \mathbb{E}[|\xi|^2])e^{CTK^2t} < \infty$$

with some numerical constant $C > 0$.

4.5 Theorem. *Assume that there is a $K > 0$ such that*

(a) $\forall x, y \in \mathbb{R}^d$, $t \geqslant 0$: $|b(x,t) - b(y,t)| + \|\sigma(x,t) - \sigma(y,t)\|_F \leqslant K|x-y|$ *(global Lipschitz condition) and*

(b) $\forall x \in \mathbb{R}^d$, $t \geqslant 0$: $|b(x,t)| + \|\sigma(x,t))\|_F \leqslant K(1+|x|)$ *(linear growth).*

If $\mathbb{E}[|\xi|^2] < \infty$ *holds, then the SDE* (4.1) *has a strong solution which is unique in the sense that two solutions are indistinguishable.*

4.2 Weak solutions

4.6 Definition. A weak solution to the above SDE is an adapted, continuous process $(X_t, t \geqslant 0)$, defined on some(!) filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, satisfying the usual conditions, carrying an $(\mathscr{F}_t)$-Brownian motion B and an $\mathscr{F}_0$-measurable random variable ξ, such that a.s.

$$X_t = \xi + \int_0^t b(X_s, s)\, ds + \int_0^t \sigma(X_s, s)\, dB_s, \quad t \geqslant 0.$$

We say that weak uniqueness holds if any two weak solutions have the same finite-dimensional distributions, i.e. generate the same law on $C([0,\infty))$.

4.7 Theorem. *If $b : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ is measurable, bounded, then the SDE*

$$dX_t = b(X_t, t)\, dt + dB_t, \quad t \in [0,T]; \quad X_0 = \xi$$

has a weak solution and weak uniqueness holds.

4.3 Connections to PDEs

4.8 Lemma. *For a (weak or strong) solution X of the time-homogeneous SDE*

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dB_t, \quad t \geqslant 0; \qquad X_0 = \xi \tag{4.2}$$

and $f \in C^2(\mathbb{R}^d)$ we have a.s.

$$f(X_t) = f(\xi) + \int_0^t (\mathscr{L} f)(X_s)\, ds + \int_0^t \langle \nabla f(X_s), \sigma(X_s) dB_s \rangle, \quad t \geqslant 0,$$

in terms of the differential operator

$$\mathscr{L} f(x) := \sum_{i=1}^d b_i(x) \partial_{x_i} f(x) + \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \partial_{x_i} \partial_{x_j} f(x)$$

with

$$a_{ij}(x) = \sum_{k=1}^m \sigma_{ik}(x) \sigma_{jk}(x) = (\sigma \sigma^\top)_{ij}(x).$$

If $\mathscr{L} f(x) = 0$ for all $x \in \mathbb{R}^d$, then $f(X_t) - f(\xi)$ is a local martingale.

4.9 Definition. Let $D \subseteq \mathbb{R}^d$ be a bounded domain (i.e., open and connected) and $\varphi \in C(\partial D)$. Then $u \in C^2(D) \cap C(\bar{D})$ solves the Dirichlet problem $(\mathscr{L}, \varphi)$ if

$$\mathscr{L}u(x) = 0, \quad x \in D, \qquad \text{and} \qquad u(x) = \varphi(x), \quad x \in \partial D.$$

4.10 Theorem. *Assume that u solves the Dirichlet problem $(\mathscr{L}, \varphi)$ and that the SDE (4.2) admits a solution X^x with initial value $X_0^x = x$ for some $x \in D$. If the stopping time $\tau_{\partial D} := \inf\{t \geqslant 0 \,|\, X_t^x \in \partial D\}$ is a.s. finite, then we have*

$$u(x) = \mathbb{E}[\varphi(X^x_{\tau_{\partial D}})].$$

If the assumptions hold for all $x \in D$, then the solution for the Dirichlet problem is unique, and it satisfies $u(x) \in [\min_{y \in \partial D} \varphi(y), \max_{y \in \partial D} \varphi(y)]$.