Stochastic Analysis (Stochastic Processes II) course notes summer semester 2024

Markus Reiß Humboldt-Universität zu Berlin

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1 Construction and properties of Brownian motion

1.1 Motivation

1.1 Definition. A process $(B_t, t \ge 0)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ is called <u>Brownian motion</u> (Brownsche Bewegung) if

- (a) $B_0 = 0$ and $B_t \sim N(0, t), t > 0$, holds;
- (b) the increments are stationary and independent: for $0 \le t_0 < t_1 < \cdots < t_m$ we have

 $(B_{t_1} - B_{t_0}, \dots, B_{t_m} - B_{t_{m-1}}) \sim N(0, \operatorname{diag}(t_1 - t_0, \dots, t_m - t_{m-1})).$

(c) B has continuous sample paths, i.e. $t \mapsto B_t(\omega)$ is continuous (for \mathbb{P} -almost all $\omega \in \Omega$).

1.2 Definition. Brownian motion $(B_t, t \in [0, T])$ induces an image measure (law) $\mathbb{P}^W := \mathbb{P}^{(B_t, t \in [0,T])}$ on the path space $(C([0,T]), \mathfrak{B}_{C([0,T])})$, called Wiener measure.

1.3 Remark. Recall the construction of Brownian motion as a limit of rescaled, interpolated random walks via Donsker's invariance principle.

1.4 Lemma. Let $(B_t, t \ge 0)$ be a Brownian motion. Then the following processes are also Brownian motions:

- (a) $(-B_t, t \ge 0);$
- (b) $(a^{-1/2}B_{at}, t \ge 0)$ for any a > 0 ('time change');
- (c) $(X_t, t \ge 0)$ with $X_t = tB_{1/t}$ for t > 0 and $X_0 = 0$ ('time inversion').

1.2 Construction of Brownian motion

1.5 Lemma. Brownian motion $(B_t, t \ge 0)$ is a centred Gaussian process with covariance function $\operatorname{Cov}(B_t, B_s) = t \land s, t, s \ge 0$. Conversely, a continuous Gaussian process $(X_t, t \ge 0)$ with $\mathbb{E}[X_t] = 0$, $\operatorname{Cov}(X_t, X_s) = t \land s, t, s \ge 0$, is a Brownian motion.

1.6 Definition. Two processes $(X_t, t \in T), (Y_t, t \in T)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ are called

- (a) indistinguishable (ununterscheidbar) if $\mathbb{P}(\forall t \in T : X_t = Y_t) = 1;$
- (b) versions or modifications (Versionen, Modifikationen) of each other if we have $\forall t \in T : \mathbb{P}(X_t = Y_t) = 1$.

1.7 Theorem. (Kolmogorov, Centsov, 1956) Let $(X_t)_{t \in [0,T]}$ be a stochastic process on $(\Omega, \mathscr{F}, \mathbb{P})$. If there are constants C > 0, $\alpha, \beta > 0$ such that

$$\forall s, t \in [0, T] : \mathbb{E}[|X_t - X_s|^{\alpha}] \leqslant C|t - s|^{1+\beta},$$

then X has a continuous version \tilde{X} , which has even γ -Hölder continuous paths for any $\gamma \in (0, \beta/\alpha)$, i.e.

$$\forall \omega \in \Omega \, \exists L(\omega) > 0 \, \forall t, s \in [0, T] : \ |\tilde{X}_t(\omega) - \tilde{X}_s(\omega)| \leqslant L(\omega) |t - s|^{\gamma}.$$

1.8 Corollary. Brownian motion exists and has a.s. γ -Hölder-continuous sample paths for any $\gamma \in (0, 1/2)$.

1.3 Properties of Brownian sample paths

1.9 Theorem. (Quadratic variation, Lévy) Let $\tau_n = \{t_0^{(n)}, \ldots, t_{m_n}^{(n)}\}$ with $0 = t_0^{(n)} < \cdots < t_{m_n}^{(n)} = 1$, $n \ge 1$, be partitions of [0,1] with $\tau_n \subseteq \tau_{n+1}$ (refinement) and $\max_{i=1,\ldots,m_n} |t_i^{(n)} - t_{i-1}^{(n)}| \to 0$ as $n \to \infty$ (asymptotically dense). Then for a Brownian motion B

$$\lim_{n \to \infty} \sum_{i=1}^{m_n} (B_{t_i^{(n)} \wedge t} - B_{t_{i-1}^{(n)} \wedge t})^2 = t$$

holds in L^2 and almost surely.

1.10 Corollary. Brownian motion is a.s. not of bounded variation on any interval [0, t] and in particular not continuously differentiable.

1.11 Theorem. (Law of the iterated logarithm, Khinchine 1933) For Brownian motion B we have almost surely:

(a) $\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{2t \log(\log(t^{-1}))}} = 1;$

(b)
$$\liminf_{t\downarrow 0} \frac{B_t}{\sqrt{2t\log(\log(t^{-1}))}} = -1;$$

(c)
$$\limsup_{t\uparrow\infty} \frac{B_t}{\sqrt{2t\log(\log(t))}} = 1;$$

(d)
$$\liminf_{t\uparrow\infty} \frac{B_t}{\sqrt{2t\log(\log(t))}} = -1.$$

1.12 Lemma. For $Z \sim N(0,1)$ and a > 0 we have the bounds

$$\frac{1}{\sqrt{2\pi}}\frac{1}{a+1/a}e^{-a^2/2} \leqslant \mathbb{P}(Z \geqslant a) \leqslant \frac{1}{\sqrt{2\pi}}\frac{1}{a}e^{-a^2/2}.$$

1.4 Brownian motion as a martingale and Markov process

1.13 Definition. A process $(X_t, t \ge 0)$ is called

- (a) adapted to a filtration $(\mathscr{F}_t)_{t\geq 0}$ if X_t is \mathscr{F}_t -measurable for all $t\geq 0$;
- (b) (\mathscr{F}_t) -martingale (sub-/super-martingale) if it is adapted, $X_t \in L^1(\mathbb{P})$ and $\mathbb{E}[X_t | \mathscr{F}_s] = X_s \ (\mathbb{E}[X_t | \mathscr{F}_s] \ge X_s, \ \mathbb{E}[X_t | \mathscr{F}_s] \le X_s)$ for all $0 \le s \le t$;
- (c) (\mathscr{F}_t) -Brownian motion if it is adapted, continuous, $X_0 = 0$, the increments $X_t X_s$ are independent of \mathscr{F}_s and $X_t X_s \sim N(0, t-s)$ for all $0 \leq s < t$.

1.14 Proposition. If B is an (\mathscr{F}_t) -Brownian motion, then the following processes are (\mathscr{F}_t) -martingales:

$$B_t; \quad B_t^2 - t; \quad \exp(\lambda B_t - rac{1}{2}\lambda^2 t) \text{ for any } \lambda \in \mathbb{R}$$
 .

1.15 Proposition. If B is a Brownian motion with respect to a filtration $(\mathscr{F}^0_t)_{t\geq 0}$, then also with respect to its <u>right-continuous extension</u> $\mathscr{F}_t = \mathscr{F}^0_{t+} := \bigcap_{s>t} \mathscr{F}^0_s$.

1.16 Definition. A random variable τ with values in $[0, +\infty]$ is called (\mathscr{F}_t) -stopping time if $\{\tau \leq t\} \in \mathscr{F}_t$ holds for all $t \geq 0$. The $\underline{\sigma}$ -algebra of $\underline{\tau}$ -history is given by $\mathscr{F}_{\tau} := \{A \in \mathscr{F} \mid A \cap \{\tau \leq t\} \in \mathscr{F}_t \text{ for all } t \geq 0\}.$

From now on we always assume a right-continuous filtration $(\mathscr{F}_t)_{t \ge 0}$.

1.17 Lemma. For an adapted right-continuous process X and a finite stopping time τ , the map $\omega \mapsto X_{\tau(\omega)}(\omega)$ is \mathscr{F}_{τ} -measurable.

1.18 Theorem. Let $(X_t, t \ge 0)$ be an adapted right-continuous process with $X_t \in L^1(\mathbb{P})$ for all $t \ge 0$. Then the following are equivalent:

- (a) X is a martingale;
- (b) for any bounded stopping time τ we have $\mathbb{E}[X_{\tau}] = \mathbb{E}[X_0]$;
- (c) for all bounded stopping times $\sigma \leq \tau$ we have $\mathbb{E}[X_{\tau} | \mathscr{F}_{\sigma}] = X_{\sigma}$ (optional sampling);
- (d) for all stopping times τ the process $(X_{t\wedge\tau}, t \ge 0)$ is a martingale (optional stopping).

1.19 Corollary. For a right-continuous martingale $(M_t, t \ge 0)$ and a finite stopping time τ we have $\mathbb{E}[M_{\tau}] = \mathbb{E}[M_0]$ provided $(M_{t \land \tau}, t \ge 0)$ is uniformly integrable (e.g. dominated or bounded).

1.20 Proposition. For a Brownian motion $(B_t, t \ge 0)$ and the stopping time $\tau_{a,b} := \inf\{t \ge 0 \mid X_t \notin (a,b)\}$ of first hitting a < 0 or b > 0 we have

$$\mathbb{P}(B_{\tau_{a,b}} = b) = \frac{|a|}{|a| + b}, \quad \mathbb{P}(B_{\tau_{a,b}} = a) = \frac{b}{|a| + b}, \quad \mathbb{E}[\tau_{a,b}] = |a|b.$$

1.21 Proposition. For a Brownian motion $(B_t, t \ge 0)$ and the passage time $\tau_b := \inf\{t \ge 0 \mid X_t = b\}$ at b > 0 we have

$$\mathbb{E}[e^{-\lambda\tau_b}] = e^{-b\sqrt{2\lambda}}, \quad \lambda \ge 0,$$

which yields (using inverse Laplace transfrom) that τ_b has the density

$$f_b(t) = \frac{1}{\sqrt{2\pi t^3}} e^{-b^2/(2t)}, \quad t > 0.$$

1.22 Theorem. Brownian motion B is a strong Markov process in the sense that for any finite stopping time τ the process $\tilde{B}_t := B_{\tau+t} - B_{\tau}$, $t \ge 0$, is again a Brownian motion, independent of \mathscr{F}_{τ} .

1.23 Corollary. (Reflection principle) We have $\mathbb{P}(\max_{0 \le s \le t} B_s \ge b) = 2\mathbb{P}(B_t \ge b)$ for a Brownian motion B and $t, b \ge 0$.

1.24 Corollary. The random variables $M_t = \max_{0 \le s \le t} B_t$, $|B_t|$ and $M_t - B_t$ have the same distribution for a Brownian motion B and $t \ge 0$.

2 Continuous martingales and stochastic integration

2.1 Local martingales and simple stochastic integrals

2.1 Definition. An (\mathscr{F}_t) -adapted continuous process $(M_t, t \ge 0)$ is called <u>continuous local martingale</u> if there are (\mathscr{F}_t) -stopping times $\tau_1 \le \tau_2 \le \cdots$ with $\tau_n \uparrow +\infty$ a.s. such that $M_t^{\tau_n} := M_{t \land \tau_n}, t \ge 0$, are (\mathscr{F}_t) -martingales for all $n \ge 1$. The sequence (τ_n) is called localising sequence of stopping times for M.

2.2 Definition. A piecewise constant process $(X_t, t \ge 0)$ of the form

$$X_t(\omega) = \sum_{k=0}^{\infty} \xi_k(\omega) \mathbf{1}_{(\tau_k(\omega), \tau_{k+1}(\omega)]}(t), \quad t \ge 0, \, \omega \in \Omega,$$

is a simple process if $0 = \tau_0 \leq \tau_1 \leq \cdots$ are (\mathscr{F}_t) -stopping times with $\tau_k \uparrow \infty$ a.s. and each ξ_k is an (\mathscr{F}_{τ_k}) -measurable (real) random variable. For any other adapted process $(Y_t, t \geq 0)$ we call

$$\int_0^t X_s dY_s := \sum_{k=0}^\infty \xi_k (Y_{t \wedge \tau_{k+1}} - Y_{t \wedge \tau_k}), \quad t \ge 0,$$

the stochastic integral of X with respect to Y.

2.3 Proposition. If X is a simple bounded process and M a continuous L^2 -martingale, then $(\int_0^t X_s dM_s, t \ge 0)$ is a continuous L^2 -martingale as well.

If X is a simple process and M is a continuous martingale, then $(\int_0^t X_s dM_s, t \ge 0)$ is a continuous local martingale.

2.4 Proposition. Let $X_t = \sum_{k=0}^{\infty} \xi_k \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t)$ be a bounded simple process with $|X_t| \leq C$ for all $t \geq 0$ and a deterministic constant $C \geq 0$. If M is a continuous L^2 -martingale, then

$$\mathbb{E}\left[\left(\int_0^t X_s dM_s\right)^2\right] = \sum_{k=0}^\infty \mathbb{E}\left[\xi_k^2 \mathbb{E}[M_{t\wedge\tau_{k+1}}^2 - M_{t\wedge\tau_k}^2 \,|\,\mathscr{F}_{t\wedge\tau_k}]\right] \leqslant C^2 \mathbb{E}[M_t^2].$$

2.2 Quadratic variation

2.5 Definition. From now on we always suppose that the filtration (\mathscr{F}_t) contains all null sets (it is completed). Together with right-continuity it fulfills the 'usual conditions'.

 \mathscr{M}_T^2 denotes the space of all continuous (\mathscr{F}_t) -martingales $(M_t, 0 \leq t \leq T)$ with $M_0 = 0, M_T \in L^2(\mathbb{P})$. For $M, N \in \mathscr{M}_T^2$ we set $\|M\|_{\mathscr{M}_T^2} := \|M_T\|_{L^2(\mathbb{P})},$ $\langle M, N \rangle_{\mathscr{M}_T^2} := \langle M_T, N_T \rangle_{L^2(\mathbb{P})} = \mathbb{E}[M_T N_T].$

2.6 Proposition. $(\mathscr{M}_T^2, \langle \bullet, \bullet \rangle_{\mathscr{M}_T^2})$ is a Hilbert space, i.e. a complete space with scalar product, if indistinguishable $M, M' \in \mathscr{M}_T^2$ are identified.

2.7 Theorem. Suppose the martingale $M \in \mathcal{M}_T^2$ has finite variation, i.e. $V_T(M) = \sup_{n \ge 1} \sum_{i=1}^{m_n} |M_{t_i^{(n)}} - M_{t_{i-1}^{(n)}}| < \infty$ a.s. for partitions $= 0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{m_n}^{(n)} = T$ with $\max_i |t_i^{(n)} - t_{i-1}^{(n)}| \to 0$. Then M = 0 a.s.