Stochastic Analysis
(Stochastic Processes II)
course notes
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1 Construction and properties of Brownian motion

1.1 Motivation

1.1 Definition. A process $(B_t,\, t \geqslant 0)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ is called Brownian motion (Brownsche Bewegung) if

(a) $B_0 = 0$ and $B_t \sim N(0,t)$, $t > 0$, holds;

(b) the increments are stationary and independent: for $0 \leqslant t_0 < t_1 < \cdots < t_m$ we have
$$(B_{t_1} - B_{t_0}, \ldots, B_{t_m} - B_{t_{m-1}}) \sim N\big(0, \operatorname{diag}(t_1 - t_0, \ldots, t_m - t_{m-1})\big).$$

(c) B has continuous sample paths, i.e. $t \mapsto B_t(\omega)$ is continuous (for $\mathbb{P}$-almost all $\omega \in \Omega$).

1.2 Definition. Brownian motion $(B_t, t \in [0,T])$ induces an image measure (law) $\mathbb{P}^W := \mathbb{P}^{(B_t, t\in[0,T])}$ on the path space $(C([0,T]), \mathfrak{B}_{C([0,T])})$, called Wiener measure.

1.3 Remark. Recall the construction of Brownian motion as a limit of rescaled, interpolated random walks via Donsker's invariance principle.

1.4 Lemma. *Let $(B_t, t \geqslant 0)$ be a Brownian motion. Then the following processes are also Brownian motions:*

(a) $(-B_t, t \geqslant 0)$*;*

(b) $(a^{-1/2} B_{at}, t \geqslant 0)$ *for any* $a > 0$ *('time change');*

(c) $(X_t, t \geqslant 0)$ *with* $X_t = tB_{1/t}$ *for* $t > 0$ *and* $X_0 = 0$ *('time inversion').*

1.2 Construction of Brownian motion

1.5 Lemma. *Brownian motion* $(B_t,\, t \geqslant 0)$ *is a centred Gaussian process with covariance function* $\operatorname{Cov}(B_t, B_s) = t \wedge s$, $t, s \geqslant 0$. *Conversely, a continuous Gaussian process* $(X_t,\, t \geqslant 0)$ *with* $\mathbb{E}[X_t] = 0$, $\operatorname{Cov}(X_t, X_s) = t \wedge s$, $t, s \geqslant 0$, *is a Brownian motion.*

1.6 Definition. Two processes $(X_t,\, t \in T)$, $(Y_t,\, t \in T)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ are called

(a) indistinguishable (ununterscheidbar) if $\mathbb{P}(\forall\, t \in T :\ X_t = Y_t) = 1$;

(b) versions or modifications (Versionen, Modifikationen) of each other if we have $\forall\, t \in T :\ \mathbb{P}(X_t = Y_t) = 1$.

1.7 Theorem. *(Kolmogorov, Centsov, 1956) Let* $(X_t)_{t\in[0,T]}$ *be a stochastic process on* $(\Omega, \mathscr{F}, \mathbb{P})$. *If there are constants* $C > 0$, $\alpha, \beta > 0$ *such that*
$$\forall s, t \in [0,T] :\ \mathbb{E}[|X_t - X_s|^\alpha] \leqslant C|t-s|^{1+\beta},$$
then X *has a continuous version* $\tilde{X}$, *which has even* γ*-Hölder continuous paths for any* $\gamma \in (0, \beta/\alpha)$, *i.e.*
$$\forall \omega \in \Omega\, \exists L(\omega) > 0\, \forall t, s \in [0,T] :\ |\tilde{X}_t(\omega) - \tilde{X}_s(\omega)| \leqslant L(\omega)|t-s|^\gamma.$$

1.8 Corollary. *Brownian motion exists and has a.s.* γ*-Hölder-continuous sample paths for any* $\gamma \in (0, 1/2)$.

1.3 Properties of Brownian sample paths

1.9 Theorem. *(Quadratic variation, Lévy) Let* $\tau_n = \{t_0^{(n)}, \dots, t_{m_n}^{(n)}\}$ *with* $0 = t_0^{(n)} < \dots < t_{m_n}^{(n)} = 1$, $n \geqslant 1$, *be partitions of* $[0,1]$ *with* $\tau_n \subseteq \tau_{n+1}$ *(refinement) and* $\max_{i=1,\dots m_n} |t_i^{(n)} - t_{i-1}^{(n)}| \to 0$ *as* $n \to \infty$ *(asymptotically dense). Then for a Brownian motion* B

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} (B_{t_i^{(n)}\wedge t} - B_{t_{i-1}^{(n)}\wedge t})^2 = t$$

holds in L^2 *and almost surely.*

1.10 Corollary. *Brownian motion is a.s. not of bounded variation on any interval* $[0,t]$ *and in particular not continuously differentiable.*

1.11 Theorem. *(Law of the iterated logarithm, Khinchine 1933) For Brownian motion* B *we have almost surely:*

(a) $\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{2t\log(\log(t^{-1}))}} = 1;$

(b) $\liminf_{t\downarrow 0} \frac{B_t}{\sqrt{2t\log(\log(t^{-1}))}} = -1;$

(c) $\limsup_{t\uparrow\infty} \frac{B_t}{\sqrt{2t\log(\log(t))}} = 1;$

(d) $\liminf_{t\uparrow\infty} \frac{B_t}{\sqrt{2t\log(\log(t))}} = -1.$

1.12 Lemma. *For* $Z \sim N(0,1)$ *and* $a > 0$ *we have the bounds*

$$\frac{1}{\sqrt{2\pi}} \frac{1}{a + 1/a} e^{-a^2/2} \leqslant \mathbb{P}(Z \geqslant a) \leqslant \frac{1}{\sqrt{2\pi}} \frac{1}{a} e^{-a^2/2}.$$

1.4 Brownian motion as a martingale and Markov process

1.13 Definition. A process $(X_t, t \geqslant 0)$ is called

(a) adapted to a filtration $(\mathscr{F}_t)_{t\geqslant 0}$ if X_t is $\mathscr{F}_t$-measurable for all $t \geqslant 0$;

(b) $(\mathscr{F}_t)$-martingale (sub-/super-martingale) if it is adapted, $X_t \in L^1(\mathbb{P})$ and $\mathbb{E}[X_t \,|\, \mathscr{F}_s] = X_s$ ($\mathbb{E}[X_t \,|\, \mathscr{F}_s] \geqslant X_s$, $\mathbb{E}[X_t \,|\, \mathscr{F}_s] \leqslant X_s$) for all $0 \leqslant s \leqslant t$;

(c) $(\mathscr{F}_t)$-Brownian motion if it is adapted, continuous, $X_0 = 0$, the increments $X_t - X_s$ are independent of $\mathscr{F}_s$ and $X_t - X_s \sim N(0, t-s)$ for all $0 \leqslant s < t$.

1.14 Proposition. *If* B *is an* $(\mathscr{F}_t)$*-Brownian motion, then the following processes are* $(\mathscr{F}_t)$*-martingales:*

$$B_t; \quad B_t^2 - t; \quad \exp(\lambda B_t - \tfrac{1}{2}\lambda^2 t) \text{ for any } \lambda \in \mathbb{R}\,.$$

1.15 Proposition. *If* B *is a Brownian motion with respect to a filtration* $(\mathscr{F}_t^0)_{t\geqslant 0}$*, then also with respect to its right-continuous extension* $\mathscr{F}_t = \mathscr{F}_{t+}^0 := \bigcap_{s>t} \mathscr{F}_s^0$.

1.16 Definition. A random variable τ with values in $[0, +\infty]$ is called $(\mathscr{F}_t)$-stopping time if $\{\tau \leqslant t\} \in \mathscr{F}_t$ holds for all $t \geqslant 0$. The σ-algebra of τ-history is given by $\mathscr{F}_\tau := \{A \in \mathscr{F} \mid A \cap \{\tau \leqslant t\} \in \mathscr{F}_t \text{ for all } t \geqslant 0\}$.

From now on we always assume a right-continuous filtration $(\mathscr{F}_t)_{t \geqslant 0}$.

1.17 Lemma. *For an adapted right-continuous process X and a finite stopping time τ, the map $\omega \mapsto X_{\tau(\omega)}(\omega)$ is $\mathscr{F}_\tau$-measurable.*