Markus Reiß

Stochastic Analysis / Stochastic Processes II
Summer 2024
Humboldt-Universität zu Berlin

Exercises: sheet 1

1. For a Brownian motion $(B_t, t \geqslant 0)$ and $h > 0$ consider the process of difference quotients $X_t^{(h)} := (B_{t+h} - B_t)/h$, $t \geqslant 0$.

   (a) Show that $X^{(h)}$ is a centred Gaussian process and determine its covariance function.

   (b) For $h \downarrow 0$ show that the covariance function becomes a δ-function in the sense that

   $$\lim_{h\downarrow 0} \mathbb{E}\Big[\int_0^1 f(t)X_t^{(h)}dt \int_0^1 g(s)X_s^{(h)}ds\Big] = \int_0^1 f(t)g(t)\,dt$$

   for test functions $f, g : [0,1] \to \mathbb{R}$ (you may assume any regularity first, then try to find minimal assumptions).

2. A *Brownian bridge* $(B_t^0, t \in [0,1])$ is a Gaussian process with mean zero and covariance function $\operatorname{Cov}(B_t^0, B_s^0) = t \wedge s - ts$, $t, s \in [0,1]$. Prove that there is a γ-Hölder continuous version of a Brownian bridge for any $\gamma \in (0, 1/2)$.

3. Let $(\varphi_k)_{k\geqslant 1}$ be an orthonormal basis of $L^2([0,1])$, i.e. $\langle \varphi_k, \varphi_l\rangle_{L^2} = \delta_{k,l}$ and $\sum_{k=1}^n \langle f, \varphi_k\rangle_{L^2}\varphi_k$ converges in L^2 for $n \to \infty$ to f for $f \in L^2([0,1])$. For a sequence $(Y_k)_{k\geqslant 1}$ of independent $N(0,1)$-random variables put

   $$B_t := \sum_{k=1}^{\infty} Y_k \Phi_k(t), \quad t \in [0,1],$$

   with antiderivatives $\Phi_k(t) = \int_0^t \varphi_k(s)ds$. Prove:

   (a) For fixed $t \in [0,1]$ the process $M_n^{(t)} := \sum_{k=1}^n Y_k\Phi_k(t)$ converges almost surely and in $L^2(\mathbb{P})$ to some $M_\infty^{(t)} \in L^2(\mathbb{P})$ (use martingale convergence, $\Phi_k(t) = \langle \mathbf{1}_{[0,t]}, \varphi_k\rangle_{L^2}$ plus *Parseval identity*) and B_t is well defined as limit.

   (b) For $0 \leqslant t_0 < t_1 < \cdots < t_m$, $m \in \mathbb{N}$ show that the m-dimensional random vector $(M_n^{(t_1)} - M_n^{(t_0)}, \cdots, M_n^{(t_m)} - M_n^{(t_{m-1})})$ is centred Gaussian. Conclude from (a) convergence in distribution as $n \to \infty$. By calculating the covariance matrices deduce that $B_t \sim N(0,t)$ and $(B_t)_{t\in[0,1]}$ has stationary and independent Gaussian increments like Brownian motion.

Remark: One can show that B is indeed a Brownian motion on $[0,1]$ (has a.s. continuous sample paths) for any choice of $(\varphi_k)_{k\geqslant 1}$. The next problem gives a proof for the Haar basis.

4. Introduce the *Haar basis* $\varphi_0(t) = \mathbf{1}_{[0,1]}(t)$, $\psi_{0,0}(t) = \mathbf{1}_{[0,1/2)}(t) - \mathbf{1}_{[1/2,1)}(t)$ and generally $\psi_{j,k}(t) = 2^{j/2}\psi_{0,1}(2^j t - k)$ for $j \in \mathbb{N}_0$, $k = 0, \dots, 2^j - 1$, which forms an orthonormal basis in $L^2([0,1])$.

   (a) Define the *Schauder functions* $\Phi_0(t) = \int_0^t \varphi_0(s)\,ds$, $\Psi_{j,k}(t) = \int_0^t \psi_{j,k}(s)\,ds$ and draw them for the first (j,k). Sketch also realisations of
   $$B_t^{(J)} := Y_0\Phi_0(t) + \sum_{0\leqslant j\leqslant J, 0\leqslant k\leqslant 2^j-1} Y_{j,k}\Psi_{j,k}(t), \quad t \in [0,1],$$
   for independent $N(0,1)$-random variables $Y_0, (Y_{j,k})_{j,k}$ and some (small) values of $J \in \mathbb{N}_0$.

   (b) Verify for $j \geqslant 0$
   $$\Delta_j := \sup_{t\in[0,1]} \Big| \sum_{k=0}^{2^j-1} Y_{j,k}\Psi_{j,k}(t) \Big| = 2^{-(j+1)/2} \max_{0\leqslant k\leqslant 2^j-1} |Y_{j,k}|$$
   and deduce $\mathbb{P}(\Delta_j \geqslant \eta_j) \leqslant \sum_{k=0}^{2^j-1} \mathbb{P}(|Y_{j,k}| \geqslant 2^{(j+1)/2}\eta_j) \leqslant 2^j \exp(-2^j\eta_j^2)$ for $\eta_j > 0$.

   (c) Use (b) with a good choice of the η_j to prove for any $\varepsilon > 0$
   $$\lim_{J\to\infty} \mathbb{P}\Big(\sum_{j\geqslant J} \Delta_j > \varepsilon \Big) = 0.$$
   Deduce $\sup_{t\in[0,1]}|B_t^{(J)} - B_t| \xrightarrow{\mathbb{P}} 0$ for $J \to \infty$ and $B_t = B_t^{(\infty)}$ defined analogously to Problem 3. Conclude for a subsequence $(J_m)_{m\geqslant 1}$ that $B_t^{(J_m)} \to B_t$ uniformly on $[0,1]$ with probability one, whence B_t is a.s. continuous and thus a Brownian motion.

Submit the solutions *before* the lecture on Thursday, 25 April 2024.

Markus Reiß

Stochastic Analysis / Stochastic Processes II
Summer 2024
Humboldt-Universität zu Berlin

Exercises: sheet 2

1. Let B be an $(\mathscr{F}_t)$-Brownian motion. Verify that the following processes are $(\mathscr{F}_t)$-martingales:
   (a) B_t, $t \geqslant 0$;
   (b) $B_t^2 - t$, $t \geqslant 0$;
   (c) $\exp(\lambda B_t - \frac{1}{2}\lambda^2 t)$, $t \geqslant 0$, for any $\lambda \in \mathbb{R}$.

2. Let $(X_t, t \geqslant 0)$ be an $(\mathscr{F}_t)$-adapted process with right-continuous sample paths. Then for a finite $(\mathscr{F}_t)$-stopping time τ the map $\omega \mapsto X_{\tau(\omega)}(\omega)$ is $\mathscr{F}_\tau$-measurable. To prove this, assume a right-continuous filtration $(\mathscr{F}_t)$, consider the dyadic approximations $\tau_n := 2^{-n}\lceil 2^n \tau \rceil$ of τ from the right and establish that $X_\tau = \lim_{n\to\infty} X_{\tau_n}$, $\mathscr{F}_\tau = \bigcap_n \mathscr{F}_{\tau_n}$.

3. Extend the Doob inequalities from discrete to continuous time:
   (a) Maximal inequality: for any right-continuous submartingale $(M_t, t \geqslant 0)$ and $\alpha > 0$, $T > 0$ we have
   $$\mathbb{P}\Big(\sup_{0\leqslant t\leqslant T} M_t \geqslant \alpha\Big) \leqslant \frac{1}{\alpha}\,\mathbb{E}\Big[M_T \mathbf{1}\Big(\sup_{0\leqslant t\leqslant T} M_t \geqslant \alpha\Big)\Big].$$
   (b) L^p-inequality: for any right-continuous L^p-martingale $(M_t, t \geqslant 0)$ with $p > 1$ we have
   $$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} |M_t|^p\Big]^{1/p} \leqslant \frac{p}{p-1}\,\mathbb{E}\Big[|M_T|^p\Big]^{1/p}.$$

4. Let $(N_t, t \geqslant 0)$ be a Poisson process of intensity $\lambda > 0$. Check that $(N_t, t \geqslant 0)$ and $(N_t^2, t \geqslant 0)$ are right-continuous sub-martingales and that $(N_t - \lambda t, t \geqslant 0)$ forms a right-continuous martingale.

   Can you find a *continuous*, adapted and increasing process Q such that $(N_t^2 - Q_t, t \geqslant 0)$ forms a martingale? (*Hint*: determine $\lim_{h\downarrow 0} \frac{1}{h}\,\mathbb{E}[N_{t+h}^2 - N_t^2 \mid \mathscr{F}_t]$)

Submit the solutions *before* the lecture on Thursday, 2 May 2024.

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Stochastic Analysis / Stochastic Processes II
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Exercises: sheet 3

1. Let X be a complex Brownian motion starting in $X_0 = i$, i.e. $X_t = i + B_t^1 + iB_t^2$ for two independent standard Brownian motions B^1, B^2. Consider the first time X hits the real axis:
$$\tau = \inf\{t \geqslant 0 \,|\, \mathrm{Re}(X_t) = 0\}.$$
   (a) Sketch a typical path of X.
   (b) Verify that $(e^{i\lambda X_t}, t \geqslant 0)$ is a complex martingale for every $\lambda \in \mathbb{R}$ (i.e., real and imaginary parts are martingales).
   (c) Prove $\mathbb{E}[e^{i\lambda X_\tau}] = e^{-\lambda}$, $\lambda \geqslant 0$, and then $\mathbb{E}[e^{i\lambda X_\tau}] = e^{-|\lambda|}$, $\lambda \in \mathbb{R}$. Conclude that X_τ is Cauchy-distributed by using the characteristic function of a Cauchy distribution (from the literature).

2. Show the following properties of a continuous local martingale $(M_t, t \geqslant 0)$:
   (a) If M is dominated in the sense that there is a $Z \in L^1(\mathbb{P})$ with $|M_t| \leqslant Z$ for all $t \geqslant 0$, then M is a martingale.
   (b) If $M_0 = 0$, then the stopping times $\tau_n := \inf\{t \geqslant 0 \,|\, |M_t| \geqslant n\}$ localise M.
   Hint: use $|M_{t\wedge\tau_n}| \leqslant n$ and part (a).

3. Let $(X_t, t \geqslant 0)$ be a simple process and $(M_t, t \geqslant 0)$ be a continuous martingale. Prove:
   (a) If M is an L^2-martingale and X is bounded, then $(\int_0^t X_s dM_s, t \geqslant 0)$ is also a continuous L^2-martingale.
   (b) In any case $(\int_0^t X_s dM_s, t \geqslant 0)$ is a continuous local martingale.

   Does part (b) also hold if M is only a continuous local martingale?

4. Show that the continuous martingale $M_t = B_t^2 - t$ for a Brownian motion B has quadratic variation $\langle M \rangle_t = 4\int_0^t B_s^2 ds$.
   Remark: This will later follow directly from $M_t = 2\int_0^t B_s dB_s$.

Submit the solutions of Problems 1 & 2 via email to sascha.gaudlitz@hu-berlin.de by Friday 10 May, 9 a.m., and the solutions to Problems 3 & 4 *before* the lecture on Thursday, 16 May 2024.

Markus Reiß

Stochastic Analysis / Stochastic Processes II
Summer 2024
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Exercises: sheet 4

1. For a Brownian motion B and $n \in \mathbb{N}_0$, $t \geqslant 0$ consider

$$X_t^{(n)} := \sum_{k=0}^{\infty} B_{k2^{-n}} \mathbf{1}_{(k2^{-n},(k+1)2^{-n}]}(t).$$

Show for $t = k_0 2^{-n_0}$ with some $k_0, n_0 \in \mathbb{N}_0$

$$B_t^2 = \sum_{k=0}^{\infty} (B^2_{(k+1)2^{-n}\wedge t} - B^2_{k2^{-n}\wedge t}) = \sum_{k=0}^{\infty} (B_{(k+1)2^{-n}\wedge t} - B_{k2^{-n}\wedge t})^2 + 2\int_0^t X_s^{(n)} dB_s,$$

whenever $n \geqslant n_0$. Conclude for all $t \geqslant 0$ that

$$2\int_0^t X_s^{(n)} dB_s \xrightarrow{L^2(\mathbb{P})} B_t^2 - t$$

holds as $n \to \infty$.

2. (Stratonovich integral) For a simple process $X_t = \sum_{k=0}^{\infty} \xi_k \mathbf{1}_{(\tau_k,\tau_{k+1}]}(t)$, $t \geqslant 0$, with $(\mathscr{F}_t)$-stopping times $\tau_k \uparrow \infty$, ξ_k $\mathscr{F}_{\tau_k}$-measurable and an $(\mathscr{F}_t)$-adapted process Y set

$$\int_0^t X_s \circ dY_s := \sum_{k=0}^{\infty} \frac{\xi_k + \xi_{k+1}}{2} \big(Y_{\tau_{k+1}\wedge t} - Y_{\tau_k \wedge t}\big), \quad t \geqslant 0.$$

Consider B and $X^{(n)}$ from problem 1, show that $(\int_0^t X_s^{(n)} \circ dB_s, t \geqslant 0)$ is in general *not* a martingale and deduce for $n \to \infty$

$$2\int_0^t X_s^{(n)} \circ dB_s \xrightarrow{L^2(\mathbb{P})} B_t^2, \quad t \geqslant 0.$$

Submit the solutions *before* the lecture on Thursday, 16 May 2024.

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Stochastic Analysis / Stochastic Processes II
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Exercises: sheet 5

1. Let $f : [0,T] \to \mathbb{R}$ be a deterministic function which is simple, i.e. of the form $f(t) = \sum_{i=0}^{\infty} \xi_i \mathbf{1}_{(t_i,t_{i+1}]}(t)$, and satisfies $\int_0^T f(t)^2 dt < \infty$. Prove for a Brownian motion B:
   (a) $X_t := \int_0^t f(s)\, dB_s \sim N(0, \int_0^t f(s)^2 ds)$ for $t \in [0,T]$;
   (b) $(X_t, t \in [0,T])$ is a continuous Gaussian process with independent increments.
   (c) X has quadratic variation $\langle X \rangle_t = \int_0^t f(s)^2 ds$ (do not use the result from the lecture, but use independent increments).

2. Prove that any continuous local martingale $(M_t, t \geqslant 0)$, which has almost surely finite variation, is almost surely constant, i.e. $\mathbb{P}(\forall t \geqslant 0 : M_t = M_0) = 1$.
   Hint: reduce the problem for $(M_t - M_0, t \geqslant 0)$ by suitable stopping to continuous L^2-martingales starting in zero and extend the result from $t \in [0,T]$ to $t \geqslant 0$.

3. Show that an adapted process $(X_t, t \geqslant 0)$ is progressively measurable if it is left-continuous or right-continuous. Conclude that in particular every simple process is progressively measurable.

4. Recall that $\mathscr{L}_T(M)$ for $M \in \mathscr{M}_T^2$ is the set of all progressively measurable processes $(X_t, t \in [0,T])$ with $\mathbb{E}[\int_0^T X_t^2 d\langle M \rangle_t] < \infty$. Prove that $\mathscr{L}_T(M)$ is a Hilbert space with scalar product $\langle X, Y \rangle_{M,T} := \mathbb{E}[\int_0^T X_t Y_t d\langle M \rangle_t]$, $X, Y \in \mathscr{L}_T(M)$, if we identify X and Y with $\int_0^T (X_t - Y_t)^2 d\langle M \rangle_t = 0$ a.s.
   Hint: You may use that any L^2-space of (equivalence classes of) functions on a measure space is a Hilbert space.

Submit the solutions *before* the exercise class on Thursday, 23 May 2024.

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Exercises: sheet 6

1. A function $f : [0, T] \to \mathbb{R}$ is of *bounded variation* if

$$\|f\|_{BV} := \sup_{n \geqslant 1; 0 = t_0 < \cdots < t_n = T} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| < \infty$$

holds with the supremum over all partitions of $[0, T]$. Show:

   (a) Every function $f(t) = f(0) + \int_0^t f'(s)ds$ with some $f' \in L^1([0, T])$ is of bounded variation and $\|f\|_{BV} = \|f'\|_{L^1}$.

   (b) Every function $f = f_1 - f_2$ with $f_1, f_2 : [0, T] \to \mathbb{R}$ increasing is of bounded variation.

   (c) Define for a function f of bounded variation and $t \in [0, T]$

$$f_1(t) := f(0) + \sup_{n \geqslant 1; 0 = t_0 < \cdots < t_n = T} \sum_{i=1}^{n} (f(t_i \wedge t) - f(t_{i-1} \wedge t))_+,$$
$$f_2(t) := \sup_{n \geqslant 1; 0 = t_0 < \cdots < t_n = T} \sum_{i=1}^{n} (f(t_i \wedge t) - f(t_{i-1} \wedge t))_-.$$

   Then f_1, f_2 are increasing and $f = f_1 - f_2$ so that any bounded variation function is the difference of two monotone function. Determine f_1, f_2 in the case $f \in C^1([0, T])$.

2. Prove for a Brownian motion B and $t \geqslant 0$

$$\int_0^t B_s^2 dB_s = \tfrac{1}{3} B_t^3 - \int_0^t B_s ds,$$

using piecewise constant approximations $B^{(n)}$ of B as integrands.

3. Show for a continuous local martingale $(M_t, t \in [0, T])$ with $M_0 = 0$ and $X \in \mathscr{L}_{loc,T}(M)$ that the stochastic integral $\int_0^t X_s dM_s$, $t \in [0, T]$, is well defined as an almost sure limit and is itself a continuous local martingale. Prove further that $(\int_0^t X_s dM_s)^2 - \int_0^t X_s^2 d\langle M \rangle_s$, $t \in [0, T]$, is a continuous local martingale.

4. Use Fatou's Lemma to establish that a non-negative continuous local martingale $(M_t, t \geqslant 0)$ with $M_0 \in L^1(\mathbb{P})$ is always a super-martingale.

Submit the solutions *before* the lecture on Thursday, 30 May 2024.

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Stochastic Analysis / Stochastic Processes II
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Exercises: sheet 7

1. Let $M \in \mathscr{M}_T^2$, $X \in \mathscr{L}_T(M)$ and $Y \in \mathscr{L}_T(N)$ with $N_t = \int_0^t X_s dM_s$. Show:
   (a) $YX \in \mathscr{L}_T(M)$.
   (b) We have for simple, bounded X and Y the associativity
   $$\int_0^t Y_s dN_s = \int_0^t Y_s X_s dM_s, \quad t \in [0,T], \text{ a.s.}$$
   and then also for all $X \in \mathscr{L}_T(M)$ and finally for all $Y \in \mathscr{L}_T(N)$.
   (c) Formulate the corresponding associativity result for continuous local martingales M and then for semi-martingales M with adapted continuous integrands X, Y. Prove this or describe briefly the main arguments in the proof and cite a reference for full details.

2. For a Brownian motion B let $X \in \mathscr{L}_{loc,T}(B)$. Consider the *stochastic exponential*
   $$Z_t = \exp\Big(\int_0^t X_s dB_s - \frac{1}{2}\int_0^t X_s^2 ds\Big), \quad t \in [0,T].$$
   (a) Apply Itô's formula to $M_t = \int_0^t X_s dB_s$ and derive
   $$e^{M_t} = 1 + \int_0^t e^{M_s} X_s dB_s + \frac{1}{2}\int_0^t e^{M_s} X_s^2 ds, \quad t \in [0,T].$$
   (b) Argue that $Z_t e^{-M_t} = 1 - \frac{1}{2}\int_0^t Z_s e^{-M_s} X_s^2 ds$ and show with integration by parts
   $$Z_t = 1 + \int_0^t Z_s X_s dB_s, \quad t \in [0,T].$$
   (c) Conclude that Z is a continuous local martingale and by Problem 6.4 also a super-martingale. Is it a martingale for deterministic X?

3. For an open, bounded domain $D \subseteq \mathbb{R}^d$ assume that a solution $h \in C^2(D) \cap C(\bar{D})$ of the homogeneous *Poisson equation*

$$\Delta h = 0 \text{ on } D, \quad h = f \text{ on } \partial D$$

exists, where $f : \partial D \to \mathbb{R}$ is a continuous function on the boundary ∂D (you might think of electric charges on ∂D and the generated electro-static field inside D).

Let $X_t = x + B_t$, $t \geqslant 0$, for $x \in D$ and a d-dimensional Brownian motion B and define the stopping time $\tau = \inf\{t \geqslant 0 \,|\, X_t \notin D\}$ when X hits the boundary. Prove

$$h(x) = \mathbb{E}[f(X_\tau)].$$

Based on this formula explain how Monte-Carlo simulations of B can be used to determine the solution h of the Poisson equation.
Optional: Implement this in dimension two for the open unit disc D and some charge distributions f. Compare with the analytic result in case $x = 0$ (or for general $x \in D$).

4. The two-dimensional Brownian motion B_t, $t > 0$, does not hit a given point $x \in \mathbb{R}^2$ (even $x = 0$) a.s. We say that all singletons $\{x\}$ are *polar sets* for two-dimensional Brownian motion.

Prove $\mathbb{P}(\tau_x < \infty) = 0$ for $\tau_x = \inf\{t > 0 \,|\, B_t = x\}$ as follows:

(a) It is equivalent to the shifted problem whether $X_t = x_0 + B_t$, $t > 0$, hits zero for $x_0 \in \mathbb{R}^2$.

(b) For $x_0 \neq 0$ and $0 < A < |x_0| < B$ deduce with results from the lecture

$$\mathbb{P}(\tau^0 < \tau^B) \leqslant \inf_A \mathbb{P}(\tau^A < \tau^B) = 0 \text{ with } \tau^R := \inf\{t \geqslant 0 \,|\, |X_t| = R\}$$

and conclude $\mathbb{P}(\tau^0 < \infty) = 0$.

(c) For $x_0 = 0$ show

$$\mathbb{P}(\exists t > \varepsilon : B_t = 0) = \mathbb{E}[\mathbb{P}(\exists t > 0 : x_0 + B_t = 0)|_{x_0 = B_\varepsilon}] = 0$$

and let $\varepsilon \downarrow 0$.

Submit the solutions *before* the lecture on Thursday, 6 June 2024.

Markus Reiß

Stochastic Analysis / Stochastic Processes II
Summer 2024
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Exercises: sheet 8

1. Prove the lower bound of the Burkholder-Davis-Gundy-inequality (BDG): For any continuous local martingale $(M_t, t \geq 0)$ with $M_0 = 0$ and any $p \geq 4$ there exists a universal constant $c_p > 0$ (depending only on p) such that for all $t \geq 0$

$$c_p \, \mathbb{E}\left[\langle M \rangle_t^{p/2}\right] \leq \mathbb{E}\left[(M_t^*)^p\right],$$

where $M_t^* = \sup_{0 \leq s \leq t} |M|_s$. Use the following steps:

   (a) Assume first that M and $\langle M \rangle$ are bounded. Use the equality $M_t^2 = 2\int_0^t M_s \, dM_s + \langle M \rangle_t$ to show

$$\mathbb{E}\left[\langle M \rangle_t^{p/2}\right] \leq \tilde{c}_p \left(\mathbb{E}\left[(M_t^*)^p\right] + \mathbb{E}\left[\left|\int_0^t M_s \, dM_s\right|^{p/2}\right]\right)$$

   for some constant $\tilde{c}_p > 0$ and apply the upper bound of the BDG-inequality to the local martingale $\int_0^\bullet M_s \, dM_s$.

   (b) Conclude the general result by localisation.

2. Let B_t, $t > 0$, be a one-dimensional Brownian motion with $B_0 = 0$. Let f be a twice continuously differentiable function on $\mathbb{R}$, and let g be a continuous function on $\mathbb{R}$.

   (a) Verify that the process

$$X_t = f(B_t) \exp\left(-\int_0^t g(B_s) \, ds\right), \quad t \geqslant 0,$$

   is a semi-martingale, and give its decomposition as the sum of a continuous local martingale and a finite variation process.

   (b) Prove that X is a continuous local martingale if and only if the function f satisfies the differential equation

$$f'' = 2gf.$$

Submit the solutions *before* the lecture on Thursday, 13 June 2024.

Markus Reiß

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Exercises: sheet 9

1. Let M be a continuous local martingale. For all $0 \leqslant a < b$ we want to show that the events $A_{a,b} = \{\forall t \in [a,b] : M_t = M_a\}$ and $B_{a,b} = \{\langle M \rangle_b = \langle M \rangle_a\}$ are a.s. equal.
   (a) Prove $A_{a,b} \subseteq B_{a,b}$ a.s. by representing $\langle M \rangle_t$ as the limit of sums of squared increments over partitions.
   (b) For $B_{a,b} \subseteq A_{a,b}$ consider the continuous local martingale
   $$N_t = M_{t\wedge b} - M_{t\wedge a} = \int_0^t \mathbf{1}_{[a,b]}(s)\, dM_s$$
   and the stopping times $\tau_\varepsilon = \inf\{t \geqslant 0 \,|\, \langle N \rangle_t \geqslant \varepsilon\}$ for $\varepsilon > 0$. Verify that N^{τ_ε} is an L^2-martingale satisfying $\mathbb{E}[(N_t^{\tau_\varepsilon})^2] \leqslant \varepsilon$.
   Show that this implies $\mathbb{E}[N_t^2 \mathbf{1}(\langle M \rangle_b = \langle M \rangle_a)] \leqslant \varepsilon$ and conclude by letting $\varepsilon \downarrow 0$.

2. Consider the *Wiener space* $(\Omega, \mathscr{F}, \mathbb{P})$ with $\Omega = C([0,1])$, $\mathscr{F} = \mathscr{B}_{C([0,1])}$ and Wiener measure $\mathbb{P}$ (law of Brownian motion). Write $\Delta_{k,j} f = f(k2^{-j}) - f((k-1)2^{-j})$ for $j \geqslant 0$, $k = 1, \ldots, 2^j$ and $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$. Let
   $$L_j(B) = \exp\Big(\sum_{k=1}^{2^j} \frac{\Delta_{k,j} f}{2^{-j}} \Delta_{k,j} B - \frac{1}{2} \sum_{k=1}^{2^j} \Big(\frac{\Delta_{k,j} f}{2^{-j}}\Big)^2 2^{-j}\Big), \quad B \in \Omega.$$
   (a) Write $L_{j+1} = A_{j+1} L_j$ for some A_{j+1} in product form and prove that $(L_j)_{j\geqslant 1}$ is a non-negative martingale with $\mathbb{E}[L_j] = 1$ under $\mathbb{P}$ with respect to the filtration $\mathscr{F}_j = \sigma(\pi_{k2^{-j}} \,|\, k = 1, \ldots 2^j)$ with the coordinate projections $\pi_t(B) = B_t$ for $B \in \Omega$.
   (b) By Kakutani's dichtomy (Stochastik II) we have that $L_j \to L_\infty$ for some L_∞ in $L^1(\mathbb{P})$-convergence is equivalent to $\mathbb{E}[L_\infty] = 1$ and also to $\sum_{j=0}^\infty (1 - \mathbb{E}[A_j^{1/2}]) < \infty$ (put $A_0 := L_0$). Prove for smooth f (e.g. $f \in C^1([0,1])$) that
   $$L_\infty(B) = \exp\Big(\int_0^1 f'(s)\, dB_s - \frac{1}{2} \int_0^1 f'(s)^2 ds\Big), \quad \mathbb{P}\text{-a.s.}$$
   Optional: Extend this to all f of the form $f(t) = \int_0^t g(s) ds$ with $g \in L^2([0,1])$.
   (c) Conclude under one of the conditions of (c) that under Q_∞ given by $\frac{dQ_\infty}{d\mathbb{P}} = L_\infty$ the process $X_t = B_t - f(t)$ forms a Brownian motion.

Submit the solutions *before* the lecture on Thursday, 20 June 2024.

Markus Reiß

Stochastic Analysis / Stochastic Processes II
Summer 2024
Humboldt-Universität zu Berlin

Exercises: sheet 10

1. For a Brownian motion B let $B_t^* := \max_{0 \leqslant s \leqslant t} |B_s|$. For $\varepsilon > 0$ consider the stopping times $\tau = \inf\{t \geqslant 0 \,|\, |B_t| \geqslant \varepsilon/2\}$ and $\sigma = \inf\{t \geqslant \tau \,|\, B_t = 0\}$.

   (a) Argue that $\mathbb{P}(B_\sigma^* \leqslant \varepsilon) = 1/2$ holds.

   (b) Deduce that there are $\alpha, \gamma > 0$ such that $\mathbb{P}(\sigma \geqslant \alpha, B_\sigma^* \leqslant \varepsilon) = \gamma$.

   (c) Conclude that this implies $\mathbb{P}(B_{n\alpha}^* \leqslant \varepsilon) \geqslant \gamma^n$ for any $n \in \mathbb{N}$ and thus $\mathbb{P}(B_t^* \leqslant \varepsilon) > 0$ for all $t > 0$.

   Remark: One can even show $\lim_{\varepsilon \downarrow 0} \varepsilon^2 \log(\mathbb{P}(B_1^* \leqslant \varepsilon)) = -\pi^2/8$ (small ball probability of Brownian motion).

2. Show that $L_t = \exp(\int_0^t X_s dB_s - \frac{1}{2}\int_0^t X_s^2 ds)$, $t \in [0,T]$, for Brownian motion B and $X \in \mathscr{L}_{loc,T}(B)$ is a martingale if the following piecewise Novikov condition holds: There are $0 = t_0 < t_1 < \cdots < t_n = T$ deterministic such that $\mathbb{E}[\exp(\frac{1}{2}\int_{t_{i-1}}^{t_i} X_s^2 ds)] < \infty$ for $i = 1, \ldots, n$.

   For the proof write $\mathbb{E}[L_T] = \mathbb{E}[L_{t_{n-1}} \mathbb{E}[L_T/L_{t_{n-1}} \,|\, \mathscr{F}_{t_{n-1}}]]$ and show $\mathbb{E}[L_T/L_{t_{n-1}} \,|\, \mathscr{F}_{t_{n-1}}] = 1$. Then proceed inductively.

3. Establish the piecewise Novikov condition from Problem 2 for $X_t = aB_t$ and conclude that the coordinate process X on $C([0,T])$ under Q_T with $dQ_T/dP^B = L_T$ satisfies $X_0 = 0$ and $dX_t = aX_t + d\bar{B}_t$ for a Q_T-Brownian motion $\bar{B}$.

4. For $c \in \mathbb{R}$ and a Brownian motion B set $X_t = B_t + ct$. We want to determine the law of the stopping time

$$\tau = \tau(X) = \inf\{t \geqslant 0 \,|\, X_t = a\},$$

where $a > 0$. For $f \in C([0,T])$ and $t \in [0,T]$ let $\Phi_t(f) = \mathbf{1}(\max_{s\in[0,t]} f(s) \geqslant a)$.

   (a) Show for $t \in [0,T]$

   $$\mathbb{P}(\tau(X) \leqslant t) = \mathbb{E}[\Phi_t(X)] = \mathbb{E}[\Phi_t(B)\exp(cB_T - c^2T/2)].$$

   (b) Use optional stopping to obtain

   $$\mathbb{E}[\Phi_t(B)\exp(cB_t - c^2t/2)] = \mathbb{E}[\mathbf{1}(\tau(B) \leqslant t)\exp(ca - c^2\tau(B)/2)].$$

   (c) Derive, using the density of $\tau(B)$,

   $$\mathbb{P}(\tau(X) \leqslant t) = \int_0^t \frac{a}{\sqrt{2\pi s^3}} \exp\Big(-\frac{(a-cs)^2}{2s}\Big)\,ds.$$

   Determine $\mathbb{P}(\tau(X) < \infty)$ as a function of a and c (computer algebra permitted).

Submit the solutions *before* the lecture on Thursday, 27 June 2024.