Markus Reiß

Stochastic Analysis / Stochastic Processes II
Summer 2024
Humboldt-Universität zu Berlin

Exercises: sheet 1

1. For a Brownian motion $(B_t, t \geqslant 0)$ and $h > 0$ consider the process of difference quotients $X_t^{(h)} := (B_{t+h} - B_t)/h$, $t \geqslant 0$.

   (a) Show that $X^{(h)}$ is a centred Gaussian process and determine its covariance function.

   (b) For $h \downarrow 0$ show that the covariance function becomes a δ-function in the sense that

   $$\lim_{h\downarrow 0} \mathbb{E}\Big[\int_0^1 f(t)X_t^{(h)}dt \int_0^1 g(s)X_s^{(h)}ds\Big] = \int_0^1 f(t)g(t)\,dt$$

   for test functions $f, g : [0,1] \to \mathbb{R}$ (you may assume any regularity first, then try to find minimal assumptions).

2. A *Brownian bridge* $(B_t^0, t \in [0,1])$ is a Gaussian process with mean zero and covariance function $\operatorname{Cov}(B_t^0, B_s^0) = t \wedge s - ts$, $t, s \in [0,1]$. Prove that there is a γ-Hölder continuous version of a Brownian bridge for any $\gamma \in (0, 1/2)$.

3. Let $(\varphi_k)_{k\geqslant 1}$ be an orthonormal basis of $L^2([0,1])$, i.e. $\langle \varphi_k, \varphi_l\rangle_{L^2} = \delta_{k,l}$ and $\sum_{k=1}^n \langle f, \varphi_k\rangle_{L^2}\varphi_k$ converges in L^2 for $n \to \infty$ to f for $f \in L^2([0,1])$. For a sequence $(Y_k)_{k\geqslant 1}$ of independent $N(0,1)$-random variables put

   $$B_t := \sum_{k=1}^{\infty} Y_k \Phi_k(t), \quad t \in [0,1],$$

   with antiderivatives $\Phi_k(t) = \int_0^t \varphi_k(s)ds$. Prove:

   (a) For fixed $t \in [0,1]$ the process $M_n^{(t)} := \sum_{k=1}^n Y_k\Phi_k(t)$ converges almost surely and in $L^2(\mathbb{P})$ to some $M_\infty^{(t)} \in L^2(\mathbb{P})$ (use martingale convergence, $\Phi_k(t) = \langle \mathbf{1}_{[0,t]}, \varphi_k\rangle_{L^2}$ plus *Parseval identity*) and B_t is well defined as limit.

   (b) For $0 \leqslant t_0 < t_1 < \cdots < t_m$, $m \in \mathbb{N}$ show that the m-dimensional random vector $(M_n^{(t_1)} - M_n^{(t_0)}, \cdots, M_n^{(t_m)} - M_n^{(t_{m-1})})$ is centred Gaussian. Conclude from (a) convergence in distribution as $n \to \infty$. By calculating the covariance matrices deduce that $B_t \sim N(0,t)$ and $(B_t)_{t\in[0,1]}$ has stationary and independent Gaussian increments like Brownian motion.

Remark: One can show that B is indeed a Brownian motion on $[0,1]$ (has a.s. continuous sample paths) for any choice of $(\varphi_k)_{k\geqslant 1}$. The next problem gives a proof for the Haar basis.

4. Introduce the *Haar basis* $\varphi_0(t) = \mathbf{1}_{[0,1]}(t)$, $\psi_{0,0}(t) = \mathbf{1}_{[0,1/2)}(t) - \mathbf{1}_{[1/2,1)}(t)$ and generally $\psi_{j,k}(t) = 2^{j/2}\psi_{0,1}(2^j t - k)$ for $j \in \mathbb{N}_0$, $k = 0, \ldots, 2^j - 1$, which forms an orthonormal basis in $L^2([0,1])$.

   (a) Define the *Schauder functions* $\Phi_0(t) = \int_0^t \varphi_0(s)\,ds$, $\Psi_{j,k}(t) = \int_0^t \psi_{j,k}(s)\,ds$ and draw them for the first (j,k). Sketch also realisations of

   $$B_t^{(J)} := Y_0\Phi_0(t) + \sum_{0\leqslant j\leqslant J, 0\leqslant k\leqslant 2^j-1} Y_{j,k}\Psi_{j,k}(t), \quad t \in [0,1],$$

   for independent $N(0,1)$-random variables $Y_0, (Y_{j,k})_{j,k}$ and some (small) values of $J \in \mathbb{N}_0$.

   (b) Verify for $j \geqslant 0$

   $$\Delta_j := \sup_{t\in[0,1]} \Big|\sum_{k=0}^{2^j-1} Y_{j,k}\Psi_{j,k}(t)\Big| = 2^{-(j+1)/2} \max_{0\leqslant k\leqslant 2^j-1} |Y_{j,k}|$$

   and deduce $\mathbb{P}(\Delta_j \geqslant \eta_j) \leqslant \sum_{k=0}^{2^j-1} \mathbb{P}(|Y_{j,k}| \geqslant 2^{(j+1)/2}\eta_j) \leqslant 2^j \exp(-2^j\eta_j^2)$ for $\eta_j > 0$.

   (c) Use (b) with a good choice of the η_j to prove for any $\varepsilon > 0$

   $$\lim_{J\to\infty} \mathbb{P}\Big(\sum_{j\geqslant J} \Delta_j > \varepsilon\Big) = 0.$$

   Deduce $\sup_{t\in[0,1]}|B_t^{(J)} - B_t| \xrightarrow{\mathbb{P}} 0$ for $J \to \infty$ and $B_t = B_t^{(\infty)}$ defined analogously to Problem 3. Conclude for a subsequence $(J_m)_{m\geqslant 1}$ that $B_t^{(J_m)} \to B_t$ uniformly on $[0,1]$ with probability one, whence B_t is a.s. continuous and thus a Brownian motion.

Submit the solutions *before* the lecture on Thursday, 25 April 2024.

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Exercises: sheet 2

1. Let B be an $(\mathscr{F}_t)$-Brownian motion. Verify that the following processes are $(\mathscr{F}_t)$-martingales:
   (a) B_t, $t \geqslant 0$;
   (b) $B_t^2 - t$, $t \geqslant 0$;
   (c) $\exp(\lambda B_t - \frac{1}{2}\lambda^2 t)$, $t \geqslant 0$, for any $\lambda \in \mathbb{R}$.

2. Let $(X_t, t \geqslant 0)$ be an $(\mathscr{F}_t)$-adapted process with right-continuous sample paths. Then for a finite $(\mathscr{F}_t)$-stopping time τ the map $\omega \mapsto X_{\tau(\omega)}(\omega)$ is $\mathscr{F}_\tau$-measurable. To prove this, assume a right-continuous filtration $(\mathscr{F}_t)$, consider the dyadic approximations $\tau_n := 2^{-n}\lceil 2^n \tau \rceil$ of τ from the right and establish that $X_\tau = \lim_{n\to\infty} X_{\tau_n}$, $\mathscr{F}_\tau = \bigcap_n \mathscr{F}_{\tau_n}$.

3. Extend the Doob inequalities from discrete to continuous time:
   (a) Maximal inequality: for any right-continuous submartingale $(M_t, t \geqslant 0)$ and $\alpha > 0$, $T > 0$ we have
   $$\mathbb{P}\Big(\sup_{0\leqslant t\leqslant T} M_t \geqslant \alpha\Big) \leqslant \frac{1}{\alpha}\,\mathbb{E}\Big[M_T \mathbf{1}\Big(\sup_{0\leqslant t\leqslant T} M_t \geqslant \alpha\Big)\Big].$$
   (b) L^p-inequality: for any right-continuous L^p-martingale $(M_t, t \geqslant 0)$ with $p > 1$ we have
   $$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} |M_t|^p\Big]^{1/p} \leqslant \frac{p}{p-1}\,\mathbb{E}\Big[|M_T|^p\Big]^{1/p}.$$

4. Let $(N_t, t \geqslant 0)$ be a Poisson process of intensity $\lambda > 0$. Check that $(N_t, t \geqslant 0)$ and $(N_t^2, t \geqslant 0)$ are right-continuous sub-martingales and that $(N_t - \lambda t, t \geqslant 0)$ forms a right-continuous martingale.

   Can you find a *continuous*, adapted and increasing process Q such that $(N_t^2 - Q_t, t \geqslant 0)$ forms a martingale? (*Hint*: determine $\lim_{h\downarrow 0} \frac{1}{h}\,\mathbb{E}[N_{t+h}^2 - N_t^2 \mid \mathscr{F}_t]$)

Submit the solutions *before* the lecture on Thursday, 2 May 2024.

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Exercises: sheet 3

1. Let X be a complex Brownian motion starting in $X_0 = i$, i.e. $X_t = i + B_t^1 + iB_t^2$ for two independent standard Brownian motions B^1, B^2. Consider the first time X hits the real axis:

$$\tau = \inf\{t \geqslant 0 \,|\, X_t = 0\}.$$

   (a) Sketch a typical path of X.

   (b) Verify that $(e^{i\lambda X_t}, t \geqslant 0)$ is a complex martingale for every $\lambda \in \mathbb{R}$ (i.e., real and imaginary parts are martingales).

   (c) Prove $\mathbb{E}[e^{i\lambda X_\tau}] = e^{-\lambda}$, $\lambda \geqslant 0$, and then $\mathbb{E}[e^{i\lambda X_\tau}] = e^{-|\lambda|}$, $\lambda \in \mathbb{R}$. Conclude that X_τ is Cauchy-distributed by using the characteristic function of a Cauchy distribution (from the literature).

2. Show the following properties of a continuous local martingale $(M_t, t \geqslant 0)$:

   (a) If M is dominated in the sense that there is a $Z \in L^1(\mathbb{P})$ with $|M_t| \leqslant Z$ for all $t \geqslant 0$, then M is a martingale.

   (b) If $M_0 = 0$, then the stopping times $\tau_n := \inf\{t \geqslant 0 \,|\, |M_t| \geqslant n\}$ localise M.
   Hint: use $|M_{t\wedge\tau_n}| \leqslant n$ and part (a).

3. Let $(X_t, t \geqslant 0)$ be a simple process and $(M_t, t \geqslant 0)$ be a continuous martingale. Prove:

   (a) If M is an L^2-martingale and X is bounded, then $(\int_0^t X_s dM_s, t \geqslant 0)$ is also a continuous L^2-martingale.

   (b) In any case $(\int_0^t X_s dM_s, t \geqslant 0)$ is a continuous local martingale.

   Does part (b) also hold if M is only a continuous local martingale?

4. Show that the continuous martingale $M_t = B_t^2 - t$ for a Brownian motion B has quadratic variation $\langle M \rangle_t = 4\int_0^t B_s^2 ds$.
   Remark: This will later follow directly from $M_t = 2\int_0^t B_s dB_s$.

Submit the solutions of Problems 1 & 2 via email to sascha.gaudlitz@hu-berlin.de by Friday 10 May, 9 a.m., and the solutions to Problems 3 & 4 *before* the lecture on Thursday, 16 May 2024.

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Exercises: sheet 4

1. For a Brownian motion B and $n \in \mathbb{N}_0$, $t \geqslant 0$ consider

$$X_t^{(n)} := \sum_{k=0}^{\infty} B_{k2^{-n}} \mathbf{1}_{(k2^{-n},(k+1)2^{-n}]}(t).$$

Show for $t = k_0 2^{-n_0}$ with some $k_0, n_0 \in \mathbb{N}_0$

$$B_t^2 = \sum_{k=0}^{\infty} (B^2_{(k+1)2^{-n}\wedge t} - B^2_{k2^{-n}\wedge t}) = \sum_{k=0}^{\infty} (B_{(k+1)2^{-n}\wedge t} - B_{k2^{-n}\wedge t})^2 + 2\int_0^t X_s^{(n)} dB_s,$$

whenever $n \geqslant n_0$. Conclude for all $t \geqslant 0$ that

$$2\int_0^t X_s^{(n)} dB_s \xrightarrow{L^2(\mathbb{P})} B_t^2 - t$$

holds as $n \to \infty$.

2. (Stratonovich integral) For a simple process $X_t = \sum_{k=0}^{\infty} \xi_k \mathbf{1}_{(\tau_k,\tau_{k+1}]}(t)$, $t \geqslant 0$, with $(\mathscr{F}_t)$-stopping times $\tau_k \uparrow \infty$, ξ_k $\mathscr{F}_{\tau_k}$-measurable and an $(\mathscr{F}_t)$-adapted process Y set

$$\int_0^t X_s \circ dY_s := \sum_{k=0}^{\infty} \frac{\xi_k + \xi_{k+1}}{2} \big(Y_{\tau_{k+1}} - Y_{\tau_k}\big), \quad t \geqslant 0.$$

Consider B and $X^{(n)}$ from problem 1, show that $(\int_0^t X_s^{(n)} \circ dB_s, t \geqslant 0)$ is in general *not* a martingale and deduce for $n \to \infty$

$$2\int_0^t X_s^{(n)} \circ dB_s \xrightarrow{L^2(\mathbb{P})} B_t^2, \quad t \geqslant 0.$$

Submit the solutions *before* the lecture on Thursday, 16 May 2024.