Markus Reiß

Stochastik II / Stochastic Processes I
Winter 2023/24
Humboldt-Universität zu Berlin

Exercises: sheet 1

1. Flies and wasps land on your dinner plate in the manner of independent Poisson processes with respective intensities μ and λ. Show that the arrival of flying beasts forms a Poisson process of intensity $\lambda + \mu$ (*superposition*). The probability that an arriving fly is a blow-fly is p. Does the arrival of blow-flies also form a Poisson process? (*thinning*)

2. Let $(N_t,\, t \geqslant 0)$ be a Poisson process of intensity $\lambda > 0$ and let $(Y_k)_{k\geqslant 1}$ be a sequence of i.i.d. random variables, independent of N. Then $X_t := \sum_{k=1}^{N_t} Y_k$, $t \geqslant 0$, is called *compound Poisson process* ($X_t := 0$ if $N_t = 0$).

   (a) Show that $(X_t,\, t \geqslant 0)$ has independent and stationary increments.

   (b) Determine the expectation of X_t in the case $Y_k \in L^1$.

   (c) Introduce the *Lévy measure* $\nu(B) := \lambda P(Y_1 \in B)$, $B \in \mathfrak{B}_{\mathbb{R}}$. Show that X_t has characteristic function

   $$\varphi_t(u) = \mathbb{E}[e^{iuX_t}] = \exp\Big(t \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx)\Big), \quad u \in \mathbb{R}\,.$$

   (d) Find a sequence of compound Poisson processes $(X_t^{(n)}, t \geqslant 0)$ with Lévy measures ν_n such that $X_t^{(n)} \xrightarrow{d} N(0,1)$ as $n \to \infty$ for some fixed $t > 0$. Describe heuristically how the sample paths evolve.

   (e*) Characterize all sequences $(\nu_n)_{n\geqslant 1}$ with $X_1^{(n)} \xrightarrow{d} N(0,1)$ in (d).

3. The number of busses that arrive until time t at a bus stop follows a Poisson process with intensity $\lambda > 0$ (in our model). Adam and Berta arrive together at time $t_0 > 0$ at the bus stop and discuss how long they have to wait in the mean for the next bus.
   Adam: Since the waiting times are $\mathrm{Exp}(\lambda)$-distributed and the exponential distribution is memoryless, the mean is λ^{-1}.
   Berta: The time between the arrival of two busses is $\mathrm{Exp}(\lambda)$-distributed and has mean λ^{-1}. Since on average the same time elapses before our arrival and after our arrival, we obtain the mean waiting time $\frac{1}{2}\lambda^{-1}$ (at least assuming that at least one bus had arrived before time t_0).
   What is the correct answer to this *waiting time paradoxon*?

4. Let the processes $(X_t, t \geqslant 0)$ and $(Y_t, t \geqslant 0)$ be versions of each other and each have right-continuous sample paths. Prove that $(X_t, t \geqslant 0)$ and $(Y_t, t \geqslant 0)$ are indistinguishable.

Submit the solutions *before* the lecture on Thursday, 26 October 2023.

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Exercises: sheet 2

1. Show for a given Markov chain that the set M of invariant initial distributions μ is convex. Find examples where (a) M consists of one element only, (b) M has infinitely many elements and (c) M is empty.

2. Let $C([0,\infty))$ be equipped with the topology of uniform convergence on compacts using the metric $d(f,g) := \sum_{k\geqslant 1} 2^{-k}(\sup_{t\in[0,k]}|f(t)-g(t)| \wedge 1)$. Prove:

   (a) $(C([0,\infty)), d)$ is Polish.

   (b) The Borel σ-algebra is the smallest σ-algebra such that all coordinate projections $\pi_t : C([0,\infty)) \to \mathbb{R}$, $t \geqslant 0$, are measurable.

   (c) For any continuous stochastic process $(X_t,\, t \geqslant 0)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ the mapping $\bar{X} : \Omega \to C([0,\infty))$ with $\bar{X}(\omega)_t := X_t(\omega)$ is Borel-measurable.

   (d) The law of $\bar{X}$ is uniquely determined by the finite-dimensional distributions of X.

3. Prove the regularity lemma: Let $\mathbb{P}$ be a probability measure on the Borel σ-algebra $\mathfrak{B}$ of any metric space. Then

$$\mathcal{D} := \Big\{ B \in \mathfrak{B} \;\Big|\; \mathbb{P}(B) = \sup_{K\subseteq B \text{ compact}} \mathbb{P}(K) = \inf_{O\supseteq B \text{ open}} \mathbb{P}(O) \Big\}$$

   is closed under set differences and countable unions ($\mathcal{D}$ is a σ-*ring*).
   Conclude for a Polish space, using the lecture results, that $\mathcal{D}$ is a σ-algebra and $\mathcal{D} = \mathfrak{B}$.

4. Abstract construction of discrete-time Markov chains: Let $(S, \mathcal{P}(S))$ be a countable state space and let an initial counting density $\mu^{(0)}$ (i.e. $\mu_i^{(0)} \geqslant 0$, $\sum_{i \in S} \mu_i^{(0)} = 1$) as well as transition probabilities p_{ij} (i.e. $p_{ij} \geqslant 0$ and $\sum_{j \in S} p_{ij} = 1$) be given.

   (a) Show that $(S, \mathcal{P}(S))$ becomes a Polish space when equipped with the discrete metric $d(i,j) = \mathbf{1}(i \neq j)$, $i, j \in S$.

   (b) For $A \subseteq S^{n+1}$ define

   $$\mu_n(A) := \sum_{i_0 \in S} \cdots \sum_{i_n \in S} \mathbf{1}_A(i_0, \ldots, i_n) \mu_{i_0}^{(0)} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}.$$

   Show the one-step consistency condition

   $$\mu_{n+1}\left(\pi^{-1}_{\{0,\ldots,n+1\} \to \{0,\ldots,n\}}(A)\right) = \mu_n(A), \quad A \subseteq S^{n+1}.$$

   (c) Conclude that $\mu_{\{t_1,\ldots t_n\}}(B) := \mu_{t_n}(\pi^{-1}_{\{0,\ldots,t_n\} \to \{t_1,\ldots,t_n\}}(B))$ for $n \geqslant 1$, $0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ and $B \subseteq S^n$ defines a projective family and that a Markov chain $(X_n, n \geqslant 0)$ with $\mathbb{P}(X_0 = j) = \mu_j^{(0)}$, $\mathbb{P}(X_{n+1} = j \,|\, X_n = i) = p_{ij}$ for all $n \geqslant 0$ and $i, j \in S$ with $\mathbb{P}(X_n = i) > 0$ exists.

Submit the solutions *before* the lecture on Thursday, 2 November 2023.

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Exercises: sheet 3

1. A process $(B_t,\, t \geqslant 0)$ is called *Brownian motion* if

   (a) $B_t \sim N(0,t)$, $t \geqslant 0$, holds;

   (b) the increments are stationary and independent: for $0 \leqslant t_0 < t_1 < \dots < t_m$ we have
   $$(B_{t_1} - B_{t_0}, \dots, B_{t_m} - B_{t_{m-1}}) \sim N(0, \operatorname{diag}(t_1 - t_0, \dots, t_m - t_{m-1})).$$

   (c) B has continuous sample paths.

   Prove that a process $(B_t,\, t \geqslant 0)$ with properties (a) and (b) exists by showing that these properties are satisfied by a Gaussian process with mean zero and covariance function $c(t,s) = t \wedge s$, $t, s \geqslant 0$. For the existence of such a Gaussian process the representation $c(t,s) = \int_0^\infty \mathbf{1}_{[0,t]}(u)\mathbf{1}_{[0,s]}(u)\,du$ might be useful to derive positive-semidefiniteness.

2. (Proof of $C([0,1]) \notin \mathfrak{B}_{\mathbb{R}}^{\otimes [0,1]}$) We say that $A \subseteq \mathbb{R}^{[0,1]} := \{f : [0,1] \to \mathbb{R}\}$ only depends on countably many coordinates if there is a countable index set $T(A) \subseteq [0,1]$ with
   $$\forall f \in A,\, g \in \mathbb{R}^{[0,1]} :\ f|_{T(A)} = g|_{T(A)} \Rightarrow g \in A.$$

   Let $\mathcal{A} := \{A \subseteq \mathbb{R}^{[0,1]} \mid A \text{ only depends on countably many coordinates}\}$.

   (a) Show that $\{f \in \mathbb{R}^{[0,1]} \mid f(t_0) \in B\}$ for any $t_0 \in [0,1], B \in \mathfrak{B}_{\mathbb{R}}$ lies in $\mathcal{A}$.

   (b) Verify that $\mathcal{A}$ is a σ-algebra and deduce that $\mathfrak{B}_{\mathbb{R}}^{\otimes [0,1]} \subseteq \mathcal{A}$.

   (c) Prove $C([0,1]) = \{f \in \mathbb{R}^{[0,1]} \mid f \text{ is continuous}\} \notin \mathfrak{B}_{\mathbb{R}}^{\otimes [0,1]}$.

3. Let (X, Y) be a two-dimensional random vector with Lebesgue density $f^{X,Y}$.

   (a) For $x \in \mathbb{R}$ with $f^X(x) > 0$ (recall $f^X(x) = \int f^{X,Y}(x, \eta)\, d\eta$) consider the *conditional density*

   $$f^{Y|X=x}(y) := \frac{f^{X,Y}(x, y)}{f^X(x)}.$$

   Which regularity condition on $f^{X,Y}$ ensures for any Borel set B

   $$\lim_{h \downarrow 0} \mathbb{P}(Y \in B \,|\, X \in [x, x+h]) = \int_B f^{Y|X=x}(y)\, dy \quad ?$$

   (b) Show that for $Y \in L^2$ (without any condition on $f^{X,Y}$) the function

   $$\varphi_Y(x) := \begin{cases} \int y f^{Y|X=x}(y)\, dy, & \text{if } f^X(x) > 0 \\ 0, & \text{otherwise} \end{cases}$$

   minimizes the L^2-distance $\mathbb{E}[(Y - \varphi(X))^2]$ over all measurable functions φ. We write $\mathbb{E}[Y \,|\, X = x] := \varphi_Y(x)$ and $\mathbb{E}[Y \,|\, X] := \varphi_Y(X)$.

4. In the situation of problem 3 prove the following properties directly from the definition:

   (a) $\mathbb{E}[\mathbb{E}[Y \,|\, X]] = \mathbb{E}[Y]$;

   (b) if X and Y are independent, then $\mathbb{E}[Y \,|\, X] = \mathbb{E}[Y]$ holds a.s.;

   (c) if $Y \geqslant 0$ a.s., then $\mathbb{E}[Y \,|\, X] \geqslant 0$ a.s.;

   (d) for all $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$ we have $\mathbb{E}[\alpha Y + \beta \,|\, X] = \alpha\, \mathbb{E}[Y \,|\, X] + \beta$ a.s.;

   (e) if $g : \mathbb{R} \to \mathbb{R}$ is such that $(x, y) \mapsto (x, y g(x))$ is a diffeomorphism and $Y g(X) \in L^1$, then $\mathbb{E}[Y g(X) \,|\, X] = \mathbb{E}[Y \,|\, X] g(X)$ a.s.

Submit the solutions *before* the lecture on Thursday, 9 November 2023.

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Exercises: sheet 4

1. Let $\Omega = \bigcup_{n\in\mathbb{N}} B_n$, $B_m \cap B_n = \varnothing$ for $m \neq n$, be a measurable, countable partition for given $(\Omega, \mathscr{F}, \mathbb{P})$ and put $\mathcal{B} := \sigma(B_n,\, n \in \mathbb{N})$. Show:

   (a) Any $\mathcal{B}$-measurable random variable X can be written as $X = \sum_{n\in\mathbb{N}} \alpha_n \mathbf{1}_{B_n}$ with suitable $\alpha_n \in \mathbb{R}$. For $Y \in L^1$ we have
   $$\mathbb{E}[Y \,|\, \mathcal{B}] = \sum_{n:\,\mathbb{P}(B_n)>0} \Big(\frac{1}{\mathbb{P}(B_n)} \int_{B_n} Y \, d\,\mathbb{P}\Big) \mathbf{1}_{B_n} \quad \mathbb{P}\text{-a.s.}$$

   (b) Specify $\Omega = [0,1)$ with Borel σ-algebra and $\mathbb{P} = U([0,1))$, the uniform distribution. For $Y(\omega) := \omega$, $\omega \in [0,1)$, determine
   $$\mathbb{E}[Y \,|\, \sigma([(k-1)/n, k/n),\, k = 1, \dots, n)].$$
   For $n = 1, 3, 5, 10$ plot the conditional expectations and Y itself as functions on Ω.

2. Let (X, Y) be a two-dimensional $N(\mu, \Sigma)$-random vector.

   (a) For which $\alpha \in \mathbb{R}$ are X and $Y - \alpha X$ uncorrelated?

   (b) Conclude that X and $Y - (\alpha X + \beta)$ are independent for these values α and for arbitrary $\beta \in \mathbb{R}$ such that $\mathbb{E}[Y|X] = \alpha X + \beta$ with suitable $\beta \in \mathbb{R}$.

   Remark: In the Gaussian case the conditional expectation is linear!

3. Let $\mathscr{G}$ be a sub-σ-algebra of $\mathscr{F}$. Prove:

   (a) $Y_n \in \mathcal{M}^+(\Omega, \mathscr{F}) \Rightarrow \mathbb{E}[\liminf_{n\to\infty} Y_n \,|\, \mathscr{G}] \leqslant \liminf_{n\to\infty} \mathbb{E}[Y_n \,|\, \mathscr{G}]$ a.s. (Fatou's Lemma);

   (b) $Y_n \in \mathcal{M}(\Omega, \mathscr{F})$, $Y_n \to Y$, $|Y_n| \leqslant Z$ with $Z \in L^1(\Omega, \mathscr{F}, \mathbb{P})$ implies $\mathbb{E}[Y_n \,|\, \mathscr{G}] \to \mathbb{E}[Y \,|\, \mathscr{G}]$ a.s. as $n \to \infty$ (dominated convergence).

   Hint: Use the monotone convergence theorem for conditional expectations, recalling the arguments for the Lebesgue integral / expectation.

4. For $Y \in L^2$ define the *conditional variance* of Y given X by

$$\operatorname{Var}(Y|X) := \mathbb{E}[(Y - \mathbb{E}[Y \mid X])^2 \mid X].$$

(a) Why is $\operatorname{Var}(Y|X)$ well defined?

(b) Show $\operatorname{Var}(Y) = \operatorname{Var}(\mathbb{E}[Y \mid X]) + \mathbb{E}[\operatorname{Var}(Y|X)]$.

(c) Use (b) to prove for independent random variables $(Z_k)_{k \geqslant 1}$ and N in L^2 with (Z_k) identically distributed and N $\mathbb{N}$-valued:

$$\operatorname{Var}\left(\sum_{k=1}^{N} Z_k\right) = \mathbb{E}[Z_1]^2 \operatorname{Var}(N) + \mathbb{E}[N] \operatorname{Var}(Z_1).$$

What is the variance of the compound Poisson process $(X_t, t \geqslant 0)$ from Exercise 1.2 at time t (assuming $Y_k \in L^2$)?

Submit the solutions *before* the lecture on Thursday, 16 November 2023.

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Stochastik II / Stochastic Processes I
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Exercises: sheet 5

1. Let $(X_n)_{n\geqslant 0}$ be an $(\mathscr{F}_n)$-adapted family of random variables in L^1. Show that $(X_n)_{n\geqslant 0}$ is a martingale if and only if for all bounded $(\mathscr{F}_n)$-stopping times τ the identity $\mathbb{E}[X_\tau]=\mathbb{E}[X_0]$ holds.
   Hint for 'if': Deduce first $\mathbb{E}[X_n]=\mathbb{E}[X_0]$ and consider then $\tau=n\mathbf{1}_{G^C}+(n+1)\mathbf{1}_G$ for suitable events G.

2. Let $(\mathcal{F}_n^X)_{n\geqslant 0}$ be the natural filtration of a process $(X_n)_{n\geqslant 0}$ and consider a finite stopping time τ with respect to $(\mathcal{F}_n^X)$.

   (a) Prove $\mathcal{F}_\tau=\sigma(\tau,\, X_{\tau\wedge n},\, n\geqslant 0)$.
   Hint: for '$\subseteq$' write $A\in\mathcal{F}_\tau$ as $A=\bigcup_n A\cap\{\tau=n\}$.

   (b*) Show that even $\mathcal{F}_\tau=\sigma(X_{\tau\wedge n},\, n\geqslant 0)$ holds.

3. Let $(S_n)_{n\geqslant 0}$ be the symmetric simple random walk, that is $S_0=0$, $S_n=\sum_{i=1}^n X_i$, $n\geqslant 1$, with independent X_i and $\mathbb{P}(X_i=+1)=\mathbb{P}(X_i=-1)=1/2$.

   (a) Argue that $(|S_n|)_{n\geqslant 0}$ is a submartingale with respect to the natural filtration $(\mathscr{F}_n^S)_{n\geqslant 0}$ of (S_n) (and then also to the natural filtration $(\mathscr{F}_n^{|S|})_{n\geqslant 0}$ of $(|S_n|)$).

   (b) Verify that $A_n=\sum_{i=0}^{n-1}\mathbf{1}(|S_i|=0)$, $n\geqslant 1$, yields the compensator of $(|S_n|)_{n\geqslant 0}$. A_n is called *local time* of the random walk at zero.

   (c) Show $\mathbb{P}(S_{2j}=0)=\binom{2j}{j}2^{-2j}$ and conclude

$$\mathbb{E}[|S_n|]=\sum_{i=0}^{n-1}\mathbb{P}(S_i=0)=\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\binom{2j}{j}2^{-2j}.$$

4. Generating function of a random walk's first passage time:
Let $(S_n)_{n\geqslant 0}$ be a simple random walk with $S_0 = 0$, $S_n = \sum_{i=1}^n X_i$, $n \geqslant 1$, where the X_i are independent and $\mathbb{P}(X_i = +1) = p$, $\mathbb{P}(X_i = -1) = q = 1 - p$, $p \in (0, 1)$. Prove:

   (a) With $M(\lambda) = pe^{\lambda} + qe^{-\lambda}$, $\lambda \in \mathbb{R}$, the process
   $$Y_n^{(\lambda)} := e^{\lambda S_n} M(\lambda)^{-n}, \quad n \geqslant 0,$$
   is a martingale with respect to $(\mathscr{F}_n^S)$.

   (b) For $M(\lambda) \geqslant 1$, $a, b \in \mathbb{Z}$ with $a < 0 < b$ and the stopping time $\tau := \inf\{n \geqslant 0 \,|\, S_n \in \{a, b\}\}$ we have
   $$e^{a\lambda}\,\mathbb{E}[M(\lambda)^{-\tau}\mathbf{1}_{\{S_\tau = a\}}] + e^{b\lambda}\,\mathbb{E}[M(\lambda)^{-\tau}\mathbf{1}_{\{S_\tau = b\}}] = 1.$$

   (c) This implies for all $s \in (0, 1]$ (solve $s = M(\lambda)^{-1}$)
   $$\mathbb{E}[s^\tau \mathbf{1}_{\{S_\tau = a\}}] = \frac{\nu_+(s)^b - \nu_-(s)^b}{\nu_+(s)^b\nu_-(s)^a - \nu_+(s)^a\nu_-(s)^b},$$
   $$\mathbb{E}[s^\tau \mathbf{1}_{\{S_\tau = b\}}] = \frac{\nu_-(s)^a - \nu_+(s)^a}{\nu_+(s)^b\nu_-(s)^a - \nu_+(s)^a\nu_-(s)^b}$$
   with $\nu_\pm(s) = (1 \pm \sqrt{1 - 4pqs^2})/(2ps)$ and continuous extension of the quotient in the case $\nu_+(s) = \nu_-(s)$.

   (d) Now let $a \downarrow -\infty$ and infer that the generating function of the *first passage time* $\tau_b := \inf\{n \geqslant 0 \,|\, S_n = b\}$ is given by
   $$\varphi_{\tau_b}(s) := \mathbb{E}[s^{\tau_b}\mathbf{1}_{\{\tau_b < \infty\}}] = \nu_+(s)^{-b} = \Big(\frac{1 - \sqrt{1 - 4pqs^2}}{2qs}\Big)^b, \quad s \in (0, 1].$$
   In particular, we have $\mathbb{P}(\tau_b < \infty) = \varphi_{\tau_b}(1) = \min(1, p/q)^b$.

Submit the solutions *before* the lecture on Thursday, 23 November 2023.

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Exercises: sheet 6

1. Prove that a family $(X_i)_{i\in I}$ of real-valued random variables is uniformly integrable if and only if $\sup_{i\in I}\|X_i\|_{L^1} < \infty$ holds as well as

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall A \in \mathscr{F} :\ \mathbb{P}(A) < \delta \Rightarrow \sup_{i\in I} \mathbb{E}[|X_i|\mathbf{1}_A] < \varepsilon.$$

2. Show for an L^p-bounded martingale (M_n) (i.e. $\sup_n \mathbb{E}[|M_n|^p] < \infty$) with $p \in (1,\infty)$:
   (a) (M_n) converges a.s. and in L^1 to some $M_\infty \in L^1$.
   (b) Use $|M_\infty| \leqslant \sup_{n\geqslant 0}|M_n|$ and Doob's inequality to infer $M_\infty \in L^p$.
   (c) Prove with dominated convergence that (M_n) converges to M_∞ in L^p.

3. Give a martingale proof of Kolmogorov's 0-1 law:
   (a) Let $(\mathscr{F}_n)$ be a filtration and $\mathscr{F}_\infty = \sigma(\mathscr{F}_n, n \geqslant 0)$. Then for $A \in \mathscr{F}_\infty$ we have $\lim_{n\to\infty} \mathbb{E}[\mathbf{1}_A \,|\, \mathscr{F}_n] = \mathbf{1}_A$ a.s.
   (b) For a sequence $(X_n)_{n\geqslant 1}$ of independent random variables consider the natural filtration $(\mathscr{F}_n^X)$ and the terminal σ-algebra $\mathcal{T} := \bigcap_{n\geqslant 1} \sigma(X_k,\, k \geqslant n)$. Then for $A \in \mathcal{T}$ deduce $\mathbb{P}(A) = \mathbb{E}[\mathbf{1}_A \,|\, \mathscr{F}_n^X] \to \mathbf{1}_A$ a.s. for $n \to \infty$, implying $\mathbb{P}(A) \in \{0, 1\}$.

4. A monkey types at random the 26 capital letters of the Latin alphabet. Let τ be the first time by which the monkey has completed the sequence ABRACADABRA. Prove that τ is almost surely finite and satisfies

$$\mathbb{E}[\tau] = 26^{11} + 26^4 + 26.$$

   Give an example of an 11-letter word where $\mathbb{E}[\tau] = 26^{11}$.
   Hint: You may look at a fair game with gamblers G_n arriving before times $n = 1, 2, \ldots$ Then G_n bets 1 Euro on 'A' for letter n; if she wins, she puts 26 Euro on 'B' for letter $n+1$, otherwise she stops. If she wins again, she puts 26^2 Euro on 'R', otherwise she stops etc. What is the balance of the game maker at time τ?

Submit the solutions *before* the lecture on Thursday, 30 November 2023.

Markus Reiß

Stochastik II / Stochastic Processes I
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Exercises: sheet 7

1. Let $Z_n(x) = (3/2)^n \sum_{k\in\{0,2\}^n} \mathbf{1}_{I(k,n)}(x)$, $x \in [0,1)$, with intervals $I(k,n) := [\sum_{i=1}^n k_i 3^{-i}, \sum_{i=1}^n k_i 3^{-i} + 3^{-n})$. Show:

   (a) $(Z_n)_{n\geqslant 0}$ with $Z_0 = 1$ forms a martingale on $([0,1), \mathfrak{B}_{[0,1]}, \lambda, (\mathscr{F}_n))$ with Lebesgue measure λ on $[0,1)$ and $\mathscr{F}_n := \sigma(I(k,n),\, k \in \{0,1,2\}^n)$.

   (b) (Z_n) converges λ-a.s., but not in $L^1([0,1], \mathfrak{B}_{[0,1]}, \lambda)$.

   (c) Interpret Z_n as the density of a probability measure $\mathbb{P}_n$ with respect to λ. Then $(\mathbb{P}_n)$ converges weakly to some probability measure $\mathbb{P}_\infty$ ($\mathbb{P}_\infty$ is called *Cantor measure*). Identify a Borel set $C \subseteq [0,1]$ with $\mathbb{P}_\infty(C) = 1$, $\lambda(C) = 0$ so that $\mathbb{P}_\infty \perp \lambda$.
   Hint: Show that the distribution functions converge.

2. Let $(X_k)_{k\geqslant 1}$ be a sequence of i.i.d. $\{-1,+1\}$-valued random variables. Under the probability measure $\mathbb{P}_0$ (the null hypothesis H_0) we have $\mathbb{P}_0(X_k = +1) = p_0$ with $p_0 \in (0,1)$, while under $\mathbb{P}_1$ (the alternative H_1) we have $\mathbb{P}_1(X_k = +1) = p_1$ with $p_1 \in (0,1)$, $p_1 \neq p_0$.

   (a) Explain why the *likelihood quotient* $L_n = \frac{d(\otimes_{i=1}^n \mathbb{P}_1^{X_i})}{d(\otimes_{i=1}^n \mathbb{P}_0^{X_i})}$ after n observations $X_1, \dots, X_n$ is given by

   $$L_n = \frac{p_1^{(n+S_n)/2}(1-p_1)^{(n-S_n)/2}}{p_0^{(n+S_n)/2}(1-p_0)^{(n-S_n)/2}} \text{ with } S_n = \sum_{k=1}^n X_k.$$

   (b) Show that the *likelihood process* $(L_n)_{n\geqslant 0}$ (put $L_0 := 1$) forms a non-negative martingale under the hypothesis H_0 (i.e. under $\mathbb{P}_0$) with respect to its natural filtration.

   (c) A *sequential likelihood-quotient test*, based on $0 < A < B$ and the stopping time

   $$\tau_{A,B} := \inf\{n \geqslant 1 \,|\, L_n \geqslant B \text{ or } L_n \leqslant A\},$$

   rejects H_0 if $L_{\tau_{A,B}} \geqslant B$, and accepts H_0 if $L_{\tau_{A,B}} \leqslant A$. Determine the probability for errors of the first and second kind (i.e., $\mathbb{P}_0(L_{\tau_{A,B}} \geqslant B)$ and $\mathbb{P}_1(L_{\tau_{A,B}} \leqslant A)$) in the case $p_0 = 0.4$, $p_1 = 0.6$, $A = (2/3)^5$, $B = (3/2)^5$. Calculate $\mathbb{E}[\tau_{A,B}]$.

   (d*) Compare the error probabilities of this sequential test with those of the test which after $n = \lfloor \mathbb{E}[\tau_{A,B}] \rfloor$ observations rejects H_0 if $L_n \geqslant 1$ and accepts H_0 if $L_n < 1$.

3. Prove in detail for probability measures $\mathbb{Q} \ll \mathbb{P}$, $Z = \frac{d\,\mathbb{Q}}{d\,\mathbb{P}}$ and $Y \in L^1(\mathbb{Q})$ that YZ is in $L^1(\mathbb{P})$ and that the identity

$$\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[YZ], \text{ i.e. } \int Y d\,\mathbb{Q} = \int Y \frac{d\,\mathbb{Q}}{d\,\mathbb{P}} d\,\mathbb{P}$$

holds. Give an example where Y is in $L^1(\mathbb{Q})$, but not in $L^1(\mathbb{P})$.

4. Suppose μ_0, μ_1, μ_2 are measures on $(\Omega, \mathscr{F})$ so that μ_2 has a μ_1-density $f_{2,1}$ and μ_1 has a μ_0-density $f_{1,0}$ (i.e., $\mu_1(A) = \int_A f_{1,0} d\mu_0$ etc.). Show:

   (a) μ_0 and μ_1 are *equivalent* if and only if $f_{1,0} > 0$ holds μ_0-a.e. In that case $f_{0,1} := f_{1,0}^{-1}$ is μ_0-a.e. and μ_1-a.e. the μ_1-density of μ_0.
   Short-hand notation: $\frac{d\mu_0}{d\mu_1} = (\frac{d\mu_1}{d\mu_0})^{-1}$.

   (b) We have $\mu_2 \ll \mu_0$ and $f_{2,0} := f_{2,1} f_{1,0}$ is μ_0-a.e. the μ_0-density of μ_2.
   Short-hand notation: $\frac{d\mu_2}{d\mu_0} = \frac{d\mu_2}{d\mu_1} \frac{d\mu_1}{d\mu_0}$.

Submit the solutions *before* the lecture on Thursday, 7 December 2023.

Markus Reiß

Stochastik II / Stochastic Processes I
Winter 2023/24
Humboldt-Universität zu Berlin

Exercises: sheet 8

1. The recursion $X_n = aX_{n-1} + \varepsilon_n$ for $n \geqslant 1$ with $a \in \mathbb{R}$ and independent random variables $\varepsilon_n \sim N(0, \sigma^2)$, $X_0 \sim N(\mu_0, \sigma_0^2)$ defines a so called *autoregressive process of order one*.
   (a) Show that $(X_n, n \geqslant 0)$ forms a Gaussian process.
   (b) For which values of a do $\mu_0 \in \mathbb{R}$, $\sigma_0 > 0$ exist such that $(X_n, n \geqslant 0)$ is stationary?
   (c*) (*optional*) Simulate several trajectories for $a \in \{-1, -0.5, 0, 1, 2\}$ and different μ_0, σ_0. Explain what you see.

2. Let $(X_n)_{n\geqslant 0}$ be a time-homogeneous Markov chain with initial distribution μ. Show that the following are equivalent:
   (a) (X_n) is a stationary process;
   (b) μ is an invariant initial distribution, i.e. $\mathbb{P}_\mu(X_1 \in B) = \mu(B)$ for all $B \subseteq S$.

   Consider the one-step transition matrix of a Markov chain on $S = \{1, 2, 3\}$

$$P(1) = \begin{pmatrix} p_{11} & p_{12} & 0 \\ 0 & p_{22} & p_{23} \\ 0 & p_{32} & p_{33} \end{pmatrix}$$

   with each $p_{ij} > 0$. Visualise this by a graph with directed edges along positive transition probabilities. Then determine an invariant initial distribution μ.

3. Let $\mathscr{I}_T$ be the σ-algebra of invariant events for the measure-preserving map T on $(\Omega, \mathscr{F}, \mathbb{P})$. Show:
   (a) A random variable Y is $\mathscr{I}_T$-measurable if and only if $Y \circ T = Y$ holds $\mathbb{P}$-a.s.
   (b) T is ergodic if and only if all bounded random variables Y with $Y \circ T = Y$ $\mathbb{P}$-a.s. are constant $\mathbb{P}$-a.s.
   (c) For all invariant events A there is a *strictly invariant* event B (i.e., $T^{-1}(B) = B$ holds) such that $\mathbb{P}(A \Delta B) := \mathbb{P}(A \setminus B \cup B \setminus A) = 0$.

4. Read Ryan Tibshirani's slides on Google's PageRank algorithm (lecture 3 under `http://www.stat.cmu.edu/~ryantibs/datamining`) and explain briefly the main ideas.

Submit the solutions *before* the lecture on Thursday, 14 December 2023.

Markus Reiß

Stochastik II / Stochastic Processes I
Winter 2023/24
Humboldt-Universität zu Berlin

Exercises: sheet 9

1. Extend Birkhoff's ergodic theorem to an L^p-ergodic theorem:
For measure-preserving T and $X \in L^p$, $p \geqslant 1$, consider $A_n := \frac{1}{n}\sum_{i=0}^{n-1} X \circ T^i$.
Then $(|A_n|^p)_{n\geqslant 1}$ is uniformly integrable and $A_n \to \mathbb{E}[X \mid \mathscr{I}_T]$ holds in L^p.

2. Show that a measure-preserving map T on $(\Omega, \mathscr{F}, \mathbb{P})$ is ergodic if and only if for all $A, B \in \mathscr{F}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{P}(A \cap T^{-k}B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

Hint: For one direction apply an ergodic theorem to $\mathbf{1}_B$.
(*optional) Extension: If even $\lim_{n\to\infty} \mathbb{P}(A \cap T^{-n}B) = \mathbb{P}(A)\,\mathbb{P}(B)$ holds, then T is called *mixing*. Show that T mixing implies T ergodic, but not conversely (e.g., consider rotation by an irrational angle).

3. *Gelfand's Problem:* Does the decimal representation of 2^n ever start with the initial digit 7? Study this as follows:

   (a) Determine the relative frequencies of the initial digits of $(2^n)_{1\leqslant n\leqslant 30}$.

   (b) Let $A \sim U([0,1])$. Prove that the relative frequency of the initial digit k in $(10^A 2^n)_{1\leqslant n\leqslant m}$ converges as $m \to \infty$ a.s. to $\log_{10}(k+1) - \log_{10}(k)$.
   Hint: consider $X_n = A + n\log_{10}(2) \mod 1$ and argue via ergodicity.

   (c) Prove that the convergence in (b) even holds everywhere. In particular, the relative frequency of the initial digit 7 in the powers of 2 converges to $\log_{10}(8/7) \approx 0{,}058$.
   Hint: Show for trigonometric polynomials $p(a) = \sum_{|m|\leqslant M} c_m e^{2\pi i m a}$ that $\frac{1}{n}\sum_{k=0}^{n-1} p(a+k\eta) \to \int_0^1 p(x)dx$ holds for all $\eta \in \mathbb{R}\setminus\mathbb{Q}$, $a \in [0,1]$ (calculate explicitly for monomials!) and approximate (you may use Weierstraß's Theorem: trigonometric polynomials are dense in $(C([0,1]), \|\bullet\|_\infty)$).

Suggested reading: Benford's law and fraud detection for election results, tax declarations and corona statistics, e.g. `https://en.wikipedia.org/wiki/Benford%27s_law`.

4. Consider the set $\mathscr{I}$ of all invariant initial distributions of a recurrent Markov chain on a state space S. Prove:

   (a) $\mathscr{I}$ is convex.

   (b) If $\pi \in \mathscr{I}$ is even ergodic (that is $\mathbb{P}_\pi$ is ergodic), then there is a connected component $[x]$ with $\pi([x]) = 1$ and $\pi(\{y\}) > 0$ for all $y \in [x]$.

   (c) If $\pi, \pi' \in \mathscr{I}$ are both ergodic, then $\pi = \pi'$ or $\pi \perp \pi'$ follows.

   (d) Suppose $\mathscr{I} \neq \varnothing$. By decomposing $S = \bigcup_n [x_n]$ into pairwise disjoint components $[x_n]$, there are ergodic $\pi_n \in \mathscr{I}$ with $\pi_n([x_n]) = 1$ so that we can write any $\pi \in \mathscr{I}$ as convex combination $\pi = \sum_n \alpha_n \pi_n$ with $\alpha_n \geqslant 0$, $\sum_n \alpha_n = 1$. In particular, for an irreducible chain $\mathscr{I}$ contains at most one element, which is then ergodic.
   Here, the union and the sum extend over finitely or countably many n.

Submit the solutions *before* the lecture on Thursday, 18 January 2024.

Markus Reiß

Stochastik II / Stochastic Processes I
Winter 2023/24
Humboldt-Universität zu Berlin

Exercises: sheet 10

1. Let the initial distribution π of a Markov chain with one-step transition probabilities $p_{xy}(1)$ satisfy $\pi(\{x\})p_{xy}(1) = \pi(\{y\})p_{yx}(1)$ for all states x, y (then π is called *reversible*). Show:

   (a) π is an invariant initial distribution.

   (b) $\mathbb{P}_\pi(X_0 = x_0, \ldots, X_n = x_n) = \mathbb{P}_\pi(X_n = x_0, \ldots, X_0 = x_n)$ holds for all $x_0, \ldots, x_n \in S$ (use induction over $n \geqslant 1$). Explain in your words what this reversibility of the Markov chain means.

   (c) The transition operator P is $L^2(\pi)$-self-adjoint in the sense $\langle Pf, g\rangle_\pi = \langle f, Pg\rangle_\pi$ for all $f, g \in L^2(\pi)$.

2. For random variables X, Y on $(\Omega, \mathscr{F}, \mathbb{P})$ with values in a Polish space (S, d) with Borel σ-algebra define $d_0(X, Y) := \mathbb{E}[d(X, Y) \wedge 1]$. Show:

   (a) $\omega \mapsto d(X(\omega), Y(\omega))$ is measurable and d_0 defines a metric on the space $L^0(\Omega; S)$ of all S-valued random variables on $(\Omega, \mathscr{F}, \mathbb{P})$, when $\mathbb{P}$-a.s. equal random variables are identified.

   (b) $d_0(X_n, X) \to 0 \iff X_n \xrightarrow{\mathbb{P}} X$ (stochastic convergence).

   (c) $X_n \xrightarrow{\mathbb{P}} X$ implies $X_n \xrightarrow{d} X$ (convergence in distribution).

   (d) $X_n \xrightarrow{d} c$ for some constant $c \in S$ implies $X_n \xrightarrow{\mathbb{P}} c$.

3. Let $(X_k)_{k\geqslant 1}$ be an i.i.d. sequence of random variables in L^2 with $\mu = \mathbb{E}[X_k]$. Introduce the sample mean $\bar{X}_n := \frac{1}{n}\sum_{k=1}^n X_k$ and the sample variance $\bar{\sigma}_n^2 := \frac{1}{n-1}\sum_{k=1}^n (X_k - \bar{X}_n)^2$. Use a CLT and Slutsky's Lemma to prove for $n \to \infty$

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\bar{\sigma}_n} \xrightarrow{d} N(0,1).$$

   Determine approximately a real number $c > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\mu \in \left[\bar{X} - c\tfrac{\bar{\sigma}_n}{\sqrt{n}}, \bar{X} + c\tfrac{\bar{\sigma}_n}{\sqrt{n}}\right]\right) = 0.95.$$

4. Let $\alpha \in (0,1)$. Choose $X_0 \in [0,1]$ and perform the following independent iterations for $n \in \mathbb{N}$: given $X_{n-1} \in [0,1]$, go with probability 1/2 left, setting $X_n = \alpha X_{n-1}$, and with probability 1/2 right, setting $X_n = (1-\alpha) + \alpha X_{n-1}$.

   (a) Write $X_n = \alpha X_{n-1} + (1-\alpha) Z_n$, $n \in \mathbb{N}$, with suitable i.i.d. random variables (Z_n). Interpret $(X_n, n \geqslant 0)$ as a Markov process on $([0,1], \mathfrak{B}_{[0,1]})$.

   (b) For $\alpha = 1/2$ and $\alpha = 1/3$ determine an invariant initial distribution μ such that $(X_n, n \geqslant 0)$ becomes stationary with $X_0 \sim \mu$.
   Hint: Represent $x \in [0,1]$ in a dyadic or triadic expansion.

   (c) Show that, whatever the initial distribution of X_0 is, we have $X_n \xrightarrow{d} \mu$ in (b). Conclude that with $X_0 \sim \mu$ the process $(X_n, n \geqslant 0)$ is ergodic.

   (d*) *(Optional, but beautiful!)* Consider the triangle Δ spanned by the corner points $(0,0)$, (1,0), $(0,1)$ in $\mathbb{R}^2$. Perform iterations, where for given $X_{n-1} \in \Delta$ with probability 1/3 one of the corners is selected and X_n is obtained as the middle point between that corner and X_{n-1}. Expand $x \in \Delta$ as $x = \sum_i b_i 2^{-i}$ with certain $b_i \in \{0,1\}^2$ and describe the unique invariant initial distribution μ. Plot the support set of μ approximately by simulating (X_n). Try to understand and explore further!
   Application: a treasure is hidden in the triangle spanned by three pyramids. A treasure hunter starts digging somewhere in the triangle and then moves half way to one of the pyramids at random to dig next etc. Does he asymptotically dig in a dense subset of the triangle and thus find the treasure eventually?

Markus Reiß

Stochastik II / Stochastic Processes I
Winter 2023/24
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Exercises: sheet 11

1. Consider a distribution π on S with $\pi(\{x\}) > 0$ for all $x \in S$ and an irreducible Markov chain on S with transition probabilities $q_{xy}(1)$, $x, y \in S$, satisfying $q_{xy}(1) > 0$ if and only if $q_{yx}(1) > 0$ for $x, y \in S$. Prove that the Markov chain with transition probabilities

$$p_{xy}(1) := \begin{cases} \min\left(q_{xy}(1), \frac{\pi(\{y\})}{\pi(\{x\})} q_{yx}(1)\right), & \text{if } x \neq y, \\ 1 - \sum_{z \neq x} p_{xz}(1), & \text{if } x = y \end{cases}$$

is reversible with respect to π and irreducible. If the transition matrix $Q(1)$ is aperiodic or if π is not reversible with respect to $Q(1)$, deduce that the transition matrix $P(1)$ is aperiodic.

2. Read Example 18.16 (Ising model) in the book by Klenke. Write down the Boltzmann distribution π on $S = \{-1, +1\}^\Lambda$ and explain briefly the quantities appearing. Prove in detail that the proposal Markov chain there satisfies the properties in Problem 1. Then derive rigorously that the Markov chain $X_n = F_n(X_{n-1})$ has invariant distribution π. Is (X_n) aperiodic?

3. For probability measures $\mathbb{P}$ and $\mathbb{Q}$ on a measurable space $(\Omega, \mathcal{F})$ their total variation distance is given by $\|\mathbb{P} - \mathbb{Q}\|_{TV} = \sup_{A \in \mathcal{F}} |\mathbb{P}(A) - \mathbb{Q}(A)|$. Prove that convergence in total variation implies weak convergence on metric spaces.
Decide whether for $n \to \infty$ the probabilities $\mathbb{P}_n$ with the following Lebesgue densities f_n on $\mathbb{R}$ converge in total variation, weakly or not at all:

$$f_n(x) = ne^{-nx}\mathbf{1}_{[0,\infty)}(x), \quad f_n(x) = \tfrac{n+1}{n} x^{1/n} \mathbf{1}_{[0,1]}(x), \quad f_n(x) = \tfrac{1}{n} \mathbf{1}_{[0,n]}(x).$$

4. Prove: Every relatively (weakly) compact family $(\mathbb{P}_i)_{i \in I}$ of probability measures on a Polish space $(S, \mathfrak{B}_S)$ is uniformly tight. Proceed as follows (compare the proof of Ulam's Theorem):
   (a) For $k \geqslant 1$ consider open balls $(A_{k,m})_{m \geqslant 1}$ of radius $1/k$ that cover S. If $\lim_{M \to \infty} \inf_i \mathbb{P}_i(\bigcup_{m=1}^M A_{k,m}) < 1$ were true, then by assumption and by the Portmanteau Theorem we would have $\lim_{M \to \infty} \mathbb{Q}(\bigcup_{m=1}^M A_{k,m}) < 1$ for some limiting probability measure $\mathbb{Q}$, which is contradictory.
   (b) Conclude that for any $\varepsilon > 0$, $k \geqslant 1$ there are indices $M_{k,\varepsilon} \geqslant 1$ such that $\inf_i \mathbb{P}_i(K) > 1 - \varepsilon$ holds with $K := \bigcap_{k \geqslant 1} \bigcup_{m=1}^{M_{k,\varepsilon}} A_{k,m}$. Moreover, K is relatively compact in S, which suffices.

Submit the solutions *before* the lecture on Thursday, 1 February 2024.

Markus Reiß

Stochastik II / Stochastic Processes I
Winter 2023/24
Humboldt-Universität zu Berlin

Exercises: sheet 12

1. We say that a family of real-valued random variables $(X_i)_{i\in I}$ is *stochastically bounded*, notation $X_i = O_{\mathbb{P}}(1)$, if $\lim_{R\to\infty}\sup_{i\in I}\mathbb{P}(|X_i|>R)=0$.
   (a) Show $X_i = O_{\mathbb{P}}(1)$ if and only if the laws $(\mathbb{P}^{X_i})_{i\in I}$ are uniformly tight.
   (b) Prove that any L^p-bounded family of random variables is stochastically bounded, hence has uniformly tight laws.
   (c) If $X_n \xrightarrow{\mathbb{P}} 0$ holds, then we write $X_n = o_{\mathbb{P}}(1)$. Check the symbolic rules $O_{\mathbb{P}}(1)+O_{\mathbb{P}}(1)=O_{\mathbb{P}}(1)$ and $O_{\mathbb{P}}(1)o_{\mathbb{P}}(1)=o_{\mathbb{P}}(1)$.

2. For probability measures $\mathbb{P},\mathbb{Q}$ on a metric space (S,d) with Borel σ-algebra define the *Bounded-Lipschitz metric*

$$d_{BL}(\mathbb{P},\mathbb{Q}) := \sup\Big\{\Big|\int_S f\,d\,\mathbb{P} - \int_S f\,d\,\mathbb{Q}\Big|\,\Big|\, f\in BL_1(S)\Big\}$$

   with $BL_1(S)=\{f:S\to\mathbb{R}\mid \|f\|_\infty\leqslant 1, \forall\, x,y\in S:\ |f(x)-f(y)|\leqslant d(x,y)\}$. Prove that d_{BL} is indeed a metric and that $d_{BL}(\mathbb{P}_n,\mathbb{P})\to 0 \Rightarrow \mathbb{P}_n \xrightarrow{w} \mathbb{P}$.

   For $S=[0,T]$ use the Arzelà-Ascoli Theorem to prove

$$d_{BL}(\mathbb{P}_n,\mathbb{P})\to 0 \iff \mathbb{P}_n \xrightarrow{w} \mathbb{P}\,.$$

   Remark: This holds in fact on any Polish space (S,d).

3. Let $(B_t, t\geqslant 0)$ be a Brownian motion. Verify that the following processes are also Brownian motions:
   (a) $(-B_t, t\geqslant 0)$;
   (b) $(a^{-1/2}B_{at}, t\geqslant 0)$ for any $a>0$ ('time change');
   (c) $(X_t, t\geqslant 0)$ with $X_t = tB_{1/t}$ for $t>0$ and $X_0=0$ ('time inversion').

*4. (Optional) We want to show that a Brownian motion B is a.s. not 1/2-Hölder continuous at zero and a.s. hits zero again immediately after start in zero.

(a) Let $A_s := \{\exists\, t \in (0, s] : B_t \geqslant K\sqrt{t}\}$, $s > 0$, for some $K > 0$. Use invariance of B under time changes to prove $\mathbb{P}(A_s) = \mathbb{P}(A_1)$ for all $s > 0$.

(b) By letting $s \downarrow 0$ deduce

$$\mathbb{P}(\inf\{t > 0 \,|\, B_t \geqslant K\sqrt{t}\} = 0) \geqslant \mathbb{P}(B_1 \geqslant K) > 0.$$

(c) Apply Blumenthal's 0-1 law (follows from Kolmogorov's 0-1 law, e.g. Thm. 21.15 in Klenke) to deduce that $\inf\{t > 0 \,|\, B_t \geqslant K\sqrt{t}\} = 0$ almost surely.

(d) This implies that with probability one there is for any $\varepsilon > 0$ a sequence $(t_K)_{K \geqslant 1} \subseteq (0, \varepsilon)$ with $B_{t_K} \geqslant K\sqrt{t_K}$ for all $K \in \mathbb{N}$, that is $\limsup_{t \to 0} t^{-1/2} B_t = \infty$ a.s.

(e) By Problem 3(a) we obtain further $\liminf_{t \to 0} t^{-1/2} B_t = -\infty$ a.s. so that $\inf\{t > 0 \,|\, B_t = 0\} = 0$ a.s. By Problem 3(c) we get with probability one

$$\limsup_{t \to \infty} t^{-1/2} B_t = \infty,\ \liminf_{t \to \infty} t^{-1/2} B_t = -\infty,\ \sup\{t > 0 \,|\, B_t = 0\} = \infty.$$

Submit the solutions *before* the lecture on Thursday, 8 February 2024.