Splitting Methods for Linear Circuit DAEs of Index 1

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Abstract Operator Splitting is a powerful method for numerical investigation of complex models. This method was successfully used for ordinary and partial differential equations (ODEs and PDEs) [10]. In constrained dynamical problems as electric circuits or energy transport networks, differential-algebraic equations (DAEs) arise [12]. The constraints prevent a simple transfer of operator splitting from ODEs to DAEs. Here, we present an approach for splitting linear circuit DAEs of index 1 [6] based on a topological decoupling of the circuit that is modeled by loop and cutset equations. Finally, we present convergence results for the proposed DAE operator splitting.

1 Introduction

The idea of operator splitting methods is based on the splitting of a complex problem into a sequence of simpler subproblems. Usually, one exploits some structural properties of the separated operators belonging to the subproblems, for example, the linear behavior, the symmetric behavior or the stiff behavior that allows the application of efficient integration methods to the subproblems, see for instance [8, 9, 11]. For dynamical problems like ODEs or parabolic PDEs, additive operator splitting are well established and appropriate. However, for constrained problems an additive operator splitting method would usually fail. This becomes obvious when comparing the simple problems

$$u' = Au = A_1u + A_2u \qquad \text{and} \qquad Ax = A_1x + A_2x = b.$$

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Solving $u' = A_1 u$ and afterwards using its solution as an initial condition to solve $u' = A_2 u$ yields an approximate solution of $u' = Au$, while solving $Ax = b$ in the same manner, does not make sense. Here, a multiplicative splitting $Ax = A_1 A_2 x$ would be appropriate to solve $Ax = b$ by $A_1 y = b$ and afterwards $A_2 x = y$. It shows us that there is no simple extension of operator splitting for ODEs to DAEs. One has to adapt the operator splitting for DAEs to the different nature of inherent DAE parts.

The next section describes the branch oriented circuit modeling. It provides a natural decoupling of the circuit DAEs that can be exploited for a suitable operator splitting. Section 3 describes our operator splitting approach for linear circuit DAEs of index 1. It includes a convergence analysis and a discussion of some structural properties of the subsystems (for instance Hamiltonian structure of the first subsystem). Finally we demonstrate numerical results for a benchmark circuit in Section 4.

2 Circuit Modeling

In contrast to standard circuit modeling using the modified nodal analysis [4] we consider the branch oriented loop-cutset modeling [3, 5]. It allows us to split the operators in a natural way exploiting physical properties.

For a given circuit graph $\mathscr{G}$ with n node and b branches, select any tree and remove all its links. Then replace each link once at a time, it will form a loop that is called as fundamental loop. We select an orientation of the loop to coincide with that of the link completing it. On the other hand, a fundamental cutset with reference to a tree is a cutset formed with one tree branch and remaining links. The orientation of a cutset is the same of that of the tree branch.

Definition 1. The fundamental loop matrix $B \in \mathrm{I\!R}^{\mathrm{b}-(\mathrm{n}-1)\times \mathrm{b}}$ is defined by its entries

$$b_{ij} = \begin{cases} 1, & \text{if the branch } j \text{ has the same orientation of fundamental loop } i \\ -1, & \text{if the branch } j \text{ has the opposite orientation of fundamental loop } i \\ 0, & \text{else.} \end{cases}$$

Lemma 1 (Loop Equations, KVL [3]). *Let v be the vector of branch voltages in an electric network, then we have*

$$Bv = 0 \tag{1}$$

In general, matrix B is arranged such that the first columns corresponding to entries of links and then the columns corresponding to entries of tree branches, therefore

$$B = \begin{pmatrix} B_l & B_t \end{pmatrix} = \begin{pmatrix} I & B_t \end{pmatrix}$$

Definition 2. The fundamental cutset matrix $Q \in \mathrm{I\!R}^{(\mathrm{n}-1)\times \mathrm{b}}$ is defined by its entries

$$q_{ij} = \begin{cases} 1, & \text{if the branch } j \text{ has the same orientation of cutset } i \\ -1, & \text{if the branch } j \text{ has the opposite orientation of cutset } i \\ 0, & \text{else.} \end{cases}$$

Lemma 2 (Cut-set Equations, KCL [3]). *Let i be the vector of branch currents in an electric network, then we have*

$$Qi = 0 \tag{2}$$

and similar to the columns re-arrangement of B, we get $Q = \begin{pmatrix} Q_l & Q_t \end{pmatrix} = \begin{pmatrix} Q_l & I \end{pmatrix}$.

Theorem 1 (Orthogonality Relation [3]). *For a given connected graph $\mathcal{G}$, the orthogonality relation between the fundamental loop matrix B and the fundamental cutset matrix Q is given by $BQ^\top = 0$.*

The circuit equations consist of the loop equations (1) and cutset equations (2) reflecting the Kirchhoff's laws together with elements constitutive equations

$$i_C = Cv'_C, \quad v_L = Li'_L, \quad i_G = Gv_G, \quad v_R = Ri_R, \quad i_I = i_s(t), \quad v_V = v_s(t). \tag{3}$$

For simplicity, we consider only RLC circuits since our focus is to demonstrate the new splitting approach. We assume that all resistances, conductances, capacitances and inductances show a globally passive behavior, i.e. their corresponding matrices R, G, C and L are positive definite. In addition, the independent functions v_s and i_s for voltage and current sources are assumed to be continuously differentiable. Notice that, we used in our approach the conductive description for all resistances that belong to the tree and the resistive description for all resistances that does not belong to the tree, see below.

An index-1 circuit DAE models a circuit network that does neither have an LI-cutset nor a CV-loop, see [6]. Then we can construct a tree as follows [13]:

1. All capacitive elements and voltage sources belong to the tree.
2. All inductive elements and current sources do not belong to the tree.
3. Split resistors in such a way that all G-resistances belong to the tree and all R-resistances do not belong to the tree.

Then, the loop and cutset equations have the form

$$\begin{pmatrix} v_L \\ v_R \\ v_I \end{pmatrix} + B_t \begin{pmatrix} v_C \\ v_G \\ v_V \end{pmatrix} = 0, \qquad Q_l \begin{pmatrix} i_L \\ i_R \\ i_I \end{pmatrix} + \begin{pmatrix} i_C \\ i_G \\ i_V \end{pmatrix} = 0.$$

Inserting the element constitutive equations we get the DAE system

$$\begin{pmatrix} Li'_L \\ Ri_R \\ v_I \end{pmatrix} + B_t \begin{pmatrix} v_C \\ v_G \\ v_s(t) \end{pmatrix} = 0, \qquad Q_l \begin{pmatrix} i_L \\ i_R \\ i_s(t) \end{pmatrix} + \begin{pmatrix} Cv'_C \\ Gv_G \\ i_V \end{pmatrix} = 0.$$

Notice that $Q_l = -B_t^\top$ due to Lemmas 1, 2 and Theorem 1. Introducing

$$B_t =: \begin{pmatrix} B_{LC} & B_{LG} & B_{LV} \\ B_{RC} & B_{RG} & B_{RV} \\ B_{IC} & B_{IG} & B_{IV} \end{pmatrix}, \quad Q_l =: \begin{pmatrix} Q_{CL} & Q_{CR} & Q_{CI} \\ Q_{GL} & Q_{GR} & Q_{GI} \\ Q_{VL} & Q_{VR} & Q_{VI} \end{pmatrix}$$

and reordering the equations, we obtain a system of the form

$$Dx'(t) + Jx(t) + My(t) = r_x(t) \tag{4a}$$

$$-M^T x(t) + Sy(t) = r_y(t) \tag{4b}$$

$$z(t) + K_x x(t) + K_y y(t) = r_z(t) \tag{4c}$$

with $x = \binom{i_L}{v_C}$, $y = \binom{i_R}{v_G}$, $z = \binom{i_V}{v_I}$,

$$D = \begin{pmatrix} L & 0 \\ 0 & C \end{pmatrix}, \quad J = \begin{pmatrix} 0 & B_{LC} \\ Q_{CL} & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & B_{LG} \\ Q_{CR} & 0 \end{pmatrix}, \quad S = \begin{pmatrix} R & B_{RG} \\ Q_{GR} & G \end{pmatrix}$$

and

$$K_x = \begin{pmatrix} 0 & B_{IC} \\ Q_{VL} & 0 \end{pmatrix}, K_y = \begin{pmatrix} 0 & B_{IG} \\ Q_{VR} & 0 \end{pmatrix}, r_x = -\begin{pmatrix} B_{LV} v_s \\ Q_{CI} i_s \end{pmatrix}, r_y = -\begin{pmatrix} B_{RV} v_s \\ Q_{GI} i_s \end{pmatrix}, r_z = -\begin{pmatrix} B_{IV} v_s \\ Q_{VI} i_s \end{pmatrix}.$$

Notice that J is skew-symmetric since $B_{LC} = -Q_{CL}^\top$. Furthermore,

$$S = S_1 + S_2 := \begin{pmatrix} R & 0 \\ 0 & G \end{pmatrix} + \begin{pmatrix} 0 & B_{RG} \\ Q_{GR} & 0 \end{pmatrix}$$

with the positive definite diagonal matrix S_1 and the skew-symmetric matrix S_2 since $B_{RG} = -Q_{GR}^\top$. Consequently, S is not symmetric (unless $B_{RG} = 0$) but positive definite and hence non-singular. Since (4c) represents a simple evalution procedure for calculating z, we consider only the reduced DAE system (4a)-(4b) in the following.

3 Operator Splitting for Index-1 Circuit DAEs

Regarding the fact that additive splitting makes no sense for solving the constraints (4b), we propose a splitting approach based on the inherent ODE. Therefore, we rewrite the DAE system (4a)-(4b) equivalently as

$$Dx' + Jx + MS^{-1}M^\top x = r_x(t) - MS^{-1}r_y(t) \tag{5a}$$

$$y = S^{-1}\left(r_y(t) + M^\top x\right). \tag{5b}$$

Now, we split (5a), using Lie-Trotter splitting method, into the subsystems

$$Dx' + Jx = 0$$

and

$$Dx' + MS^{-1}M^T x = r_x(t) - MS^{-1} r_y(t). \tag{6}$$

Next, we reformulate (6) with (5b) back as DAE and obtain the following splitting approach (SADAE) for circuit index-1 DAEs.

1. Initialize $x_2(t_0) := x_0$ and $n = 0$.
2. Solve on $[t_n, t_{n+1}]$ the first subsystem

$$Dx_1' + Jx_1 = 0, \quad x_1(t_n) = x_2(t_n) \qquad \text{(splitDAE 1)}$$

3. Solve on $[t_n, t_{n+1}]$ the second subsystem

$$Dx_2'(t) + My(t) = r_x(t), \quad x_2(t_n) = x_1(t_{n+1}) \qquad \text{(splitDAE 2a)}$$
$$-M^T x_2(t) + Sy(t) = r_y(t). \qquad \text{(splitDAE 2b)}$$

4. Set $n = n+1$ and go to 2. unless t_n is the final time point.

3.1 Subsystem Properties

The first subsystem (splitDAE 1) is in fact a Hamiltonian ODE system with the Hamiltonian

$$H(x) = \frac{1}{2} x^\top D x = \frac{1}{2} v_C^\top C v_C + \frac{1}{2} i_L^\top L i_L =: H(v_C, i_L) \tag{7}$$

describing the total energy stored in the capacitors and inductors. In fact, we have

$$\frac{d}{dt} H(x) = x^\top D x' = -x^\top J x = 0$$

since J is skew-symmetric. Obviously, H is a quadratic form. Consequently, we can apply symplectic numerical methods to (splitDAE 1). They have the advantage to preserve the total energy H stored in the capacitors and inductors [7].

The second subsystem (splitDAE 2a)-(splitDAE 2b) leads to non-symmetric but positive definite linear systems after time discretization that allows the exploitation of suitable iterative methods [2].

3.2 Convergence Analysis

In order to verify the convergence of DAE operator splitting method, one has to rely on the convergence of the ODE operator splitting method. For this reason, we define the non-homogeneous Cauchy problem

$$u'(t) = A_1 u(t) + A_2 u(t) + r(t), \quad u(t_0) = u_0 \tag{8}$$

where the initial condition u_0 and the source function r are bounded. Let Δt denotes the time step such that the following stability condition is satisfied

$$||e^{\Delta t(A_1+A_2)}|| \leq 1, \quad ||e^{\Delta t A_1}|| \leq 1, \quad \text{and} \quad ||e^{\Delta t A_2}|| \leq 1$$

After time discretization, apply the following operator splitting algorithm (OSA)

$$\begin{cases} u_1'(t) = A_1 u_1(t), & t \in [t_n, t_{n+1}] \quad \text{and} \quad u_1(t_n) = u_{sp}^n \\ u_2'(t) = A_2 u_2(t) + r(t), & t \in [t_n, t_{n+1}] \quad \text{and} \quad u_2(t_n) = u_1(t_{n+1}) \end{cases}$$

where $u_{sp}^0 = u_0$, and the approximated splitting solution at $t = t_{n+1}$ is $u_{sp}^{n+1} = u_2(t_{n+1})$.

Theorem 2. *(see [1]) Under the boundedness and stability conditions formulated above, the approximated splitting solution obtained from the operator splitting algorithm (OSA) converges to the exact solution of the ODE* (8).

In other words, if we denote by $T(t_n)$ the solution operator of (8) at the n-th time step, and by $T_s(t_n)$ the splitting solution operator, then we have:

$$||T(t_n)u_0 - T_s(t_n)u_0|| \longrightarrow 0 \quad \text{as} \quad \Delta t \longrightarrow 0.$$

Regarding the equivalence of the DAE system (splitDAE 2a)-(splitDAE 2b) to the system

$$Dx_2' + MS^{-1}M^\top x_2 = r_x(t) - MS^{-1}r_y(t) \tag{9a}$$

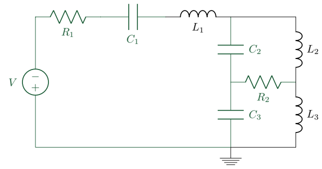
$$y = S^{-1}\big(r_y(t) + M^\top x_2\big). \tag{9b}$$

we may directly conclude from Theorem 2 the following theorem (choosing $A_1 = -D^{-1}J$ and $A_2 = -D^{-1}MS^{-1}M^\top$ and $r = D^{-1}(r_x - MS^{-1}r_y)$).

Theorem 3. *Let the time stepsize Δt be sufficiently small, the initial currents and voltages as well as the source functions of current and voltage sources be bounded. Then, the approximated solution of the circuit DAE operator splitting approach (SADAE) on page 5 converges to the exact solution of the DAE* (4a)-(4b).

4 Numerical Simulation

We use a small RLC circuit example in order to demonstrate the operator splitting approach for DAEs. It operates in a GHz regime as often used in chip design.
Using the tree in Figure 1, we get for the circuit DAE system (4a)-(4c) the matrices



$$C_1 = 10^{-12}\text{ F}, \quad C_2 = 5\cdot 10^{-13}\text{ F}, \quad C_3 = 10^{-12}\text{ F}$$
$$L_1 = 5\cdot 10^{-7}\text{ H}, \quad L_2 = 5\cdot 10^{-7}\text{ H}, \quad L_3 = 5\cdot 10^{-7}\text{ H}$$
$$G_1 = 50\ \Omega, \quad G_2 = 50\ \Omega, \quad v_s(t) = \sin(10^9 t)$$

Fig. 1 Benchmark RLC-circuit. The green branches form the tree considered for the model equations.

$$D = \begin{pmatrix} L_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_3 \end{pmatrix}, \; J = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \; M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \; r_x = \begin{pmatrix} -v_s \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$S = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}, \quad K_x = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_y = 0, \quad r_y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad r_z = 0.$$

For comparison, we consider the following three variants of numerical simulation of the circuit:

1. Solve (4a)-(4c) by implicit Euler method.
2. Solve (splitDAE 1) and (splitDAE 2a)-(splitDAE 2b) by implicit Euler method
3. Solve (splitDAE 1) by symplectic Euler and (splitDAE 2a)-(splitDAE 2b) by implicit Euler method

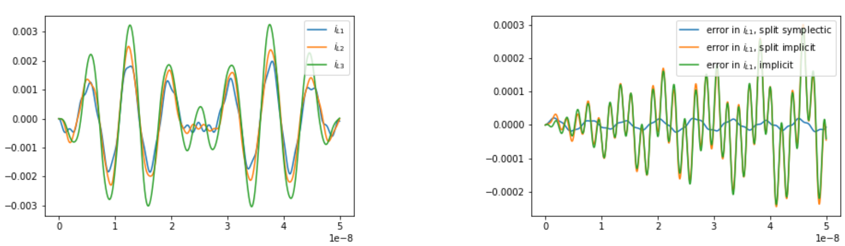


Fig. 2 Reference solution for inductive currents for circuit in Figure 1(left). Error for numerical solution of the three simulation variants with time stepsize $h = 1e-11$ (right).

In figure 2 we see the reference solution computed by time stepsize $h = 1e-13$ and the error between the numerical solution for the three simulation variants with time stepsize $h = 1e-11$ and the reference solution. The results show that the solution of the DAE splitting approach (variant 2) is almost the same as for the non-splitted solution (variant 1) whereas the use of the the DAE splitting approach with the symplectic Euler method (variant 3) gives better results and even faster since the

symplectic Euler method for the first subsystem is in fact an explicit method.
Remark: The numerical approximations used are convergent, and therefore the overall algorithm is convergent.

5 Conclusions and Outlook

In this paper, we extended the operator splitting method from ODEs to circuit linear DAEs. Followed by the topological decoupling of circuit DAEs of index 1 in loop-cutset formulation, we were able to construct a suitable decomposition of the matrices so that a convergent splitting approach was achieved. Since most of the circuit solvers are based on modfied nodal analysis (MNA), we are interested in developing an adapted splitting approach for MNA formulations.

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