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The deformed Inozemtsev spin chain

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We present two new quantum-integrable models with long-range spin interactions. First, a partially isotropic (XXZ-type) spin chain that unifies the Inozemtsev and partially isotropic Haldane–Shastry chains. Its short-range limit is a variant of the twisted Heisenberg XXZ chain. Second, a quantum many-body system that generalises the elliptic Ruijsenaars model by including spins with interactions mediated by dynamical R-matrices. It unifies the elliptic Calogero–Sutherland and trigonometric Ruijsenaars–Macdonald models with spins, and gives our spin chain by 'freezing'.

I. INTRODUCTION

Long-range spin interactions are inherent in atomic, molecular and optical physics, and may be relevant in quantum computers. Their influence on physical properties can be studied analytically in *(quantum) integrable* models, but such models are rare. In this Letter we present new long-range models — a (quantum) spin chain and a quantum many-body system (QMBS) — that are integrable in that they have a hierarchy of commuting hamiltonians ('*abelian symmetries*').

The nearest-neighbour Heisenberg chain [1, 2] has implications ranging from mathematics to e.g. $KCuF_3$ in the lab [3]. It has three levels (Fig. 1): *isotropic* (fully $SU(2)$ symmetric), *partially isotropic* ($U(1)$, i.e. spin-z, symmetric) and *anisotropic*. Until recently, the study of integrable long-range spin chains focused on the isotropic level (Fig. 1). A famous example is the Haldane–Shastry (HS) chain [4, 5], which exhibits rich mathematics [6] and serves as a lattice toy model for the fractional quantum Hall effect [7, 8] and Wess–Zumino–Witten CFT [9–12]. Another example is the Inozemtsev chain [13], an exactly solvable interpolation between Heisenberg and HS [14, 15], enabling the analytical study of spin interactions with increasing range. It famously played a guest role in AdS/CFT integrability [16], where long-range interactions appear beyond first order in perturbation theory. The Inozemtsev chain is believed to be integrable, with a conjecture for its abelian symmetries [17], but no underlying algebraic structure is known. Controlled symmetry breaking may reveal such structure.

The HS chain has a partially isotropic extension with more complex interactions [18, 19] precisely so that its key properties persist [6, 18, 20]: the *deformed* HS (DHS) chain [18–20]. *Does such a generalisation exist for the Inozemtsev chain?* (See Fig. 1.) Recently, Matushko–Zotov (MZ) introduced a (completely) anisotropic spin chain with abelian symmetries [21]. It resembles the DHS chain, but features Baxter's eight-vertex R-matrix, and does not fit in Fig. 1. We remedy this using Felder's dynamical R-matrix to obtain a partially isotropic spin chain with abelian symmetries that unifies the Inozemtsev and DHS chains as in Fig. 1. The short-range limit is a twisted variant of the Heisenberg XXZ chain that appears to be new and is related to the affine Temperley–Lieb algebra in the spirit of [22].

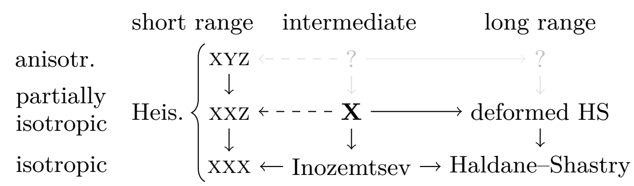


Figure 1. Landscape of integrable long-range spin chains, including the Heisenberg and Haldane–Shastry chains and their partially isotropic extensions. We find the spot marked '**X**'.

Integrability for long-range spin chains hinges on connections to QMBS. This is best understood for HS: (i) its explicit wave functions come from eigenfunctions of a *spinless* trigonometric Calogero–Sutherland (CS) model [6, 7]; (ii) its abelian symmetries stem from a trigonometric CS model *with spins* via 'freezing' [6, 23, 24]. The latter also underpins the enhanced (Yangian) spin symmetry (or 'nonabelian symmetries') of HS [6, 25]. This generalises to the partially isotropic level, with CS replaced by its 'relativistic' version, the trigonometric Ruijsenaars–Macdonald model [6, 18, 20]. For the Inozemtsev chain only (i) is understood, via the *elliptic* CS model [14, 15]. The MZ chain originates as in (ii) [21] from an elliptic Ruijsenaars model with spins based on Baxter's eight-vertex R-matrix [26]. Similarly, our chain arises by freezing an *elliptic dynamical spin-Ruijsenaars model*. Despite its supporting role in this Letter, this new QMBS is clearly of independent interest. We will return to it, and the commutativity of its hamiltonians, soon.

This Letter represents major progress for the *isotropic* Inozemtsev chain too. First, we put it in the framework of freezing as in (ii) at last. Second, we give strong evidence, if not a proof, that it does indeed have abelian symmetries. Third, our work hints at an underlying algebraic structure via the appearance of R-matrices.

Without further ado we present our models. We take spin 1/2 for simplicity, but everything extends to multi-component versions with r particle 'species' ('colours').[1]

II. THE SPIN CHAIN

A. Chiral hamiltonians

Consider N spin-1/2 sites equally spaced on a circle. The deformed Inozemtsev chain has two 'chiral' hamiltonians of the pairwise form (cf. [19, 20])

$$H^{\mathrm{L}} = \sum_{i<j}^{N} V(i-j)\, S^{\mathrm{L}}_{[i,j]}\,, \quad H^{\mathrm{R}} = \sum_{i<j}^{N} V(i-j)\, S^{\mathrm{R}}_{[i,j]}\,. \tag{1}$$

Let $\theta(x)$ be the odd Jacobi theta function with quasiperiods $\mathrm{i}\pi/\kappa$ and N, normalised by $\theta'(0)=1$; see the Supplemental Material. Here $\kappa>0$ sets the interaction range. Define the 'prepotential' $\rho(x)=\theta'(x)/\theta(x)$ and fix an anisotropy parameter η. As anticipated in [15], the potential in (1) is an (even, N-periodic) 'point splitting' of the Weierstrass $\wp$ pair potential of Inozemtsev

$$V(x) = -\frac{\rho(x+\eta)-\rho(x-\eta)}{\theta(2\eta)}\,. \tag{2}$$

It is essentially $1/\mathrm{sn}(x+\eta)\,\mathrm{sn}(x+\eta)$, see again the Supplemental Material.

It remains to give the $S_{[i,j]}$. The isotropic long-range spin exchange $E_{ij} = 1-P_{ij} = (1-\vec{\sigma}_i\cdot\vec{\sigma}_j)/2$ admits two 'chiral' decompositions into nearest-neighbour steps:

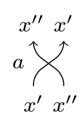
$$\begin{aligned} E_{ij} &= P_{j-1,j}\cdots P_{i+1,i+2}\, E_{i,i+1}\, P_{i+1,i+2}\cdots P_{j-1,j} \\ &= P_{i,i+1}\cdots P_{j-2,j-1}\, E_{j-1,j}\, P_{j-2,j-1}\cdots P_{i,i+1}\,. \end{aligned} \tag{3}$$

The structure on the right-hand sides persist to the partially isotropic level. Define the combinations

$$f(x,a) = \frac{\theta(\eta+a)\,\theta(x)}{\theta(a)\,\theta(x+\eta)}\,, \quad g(x,a) = \frac{\theta(x+a)\,\theta(\eta)}{\theta(a)\,\theta(x+\eta)}\,, \tag{4}$$

and the dynamical R-matrix [29]

$$\check{R}(x,a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g(x,\eta\, a) & f(x,\eta\, a) & 0 \\ 0 & f(x,-\eta\, a) & g(x,-\eta\, a) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \tag{5}$$



where $x = x'-x''$, and a is a 'dynamical' parameter. In diagrams, 'inhomogeneity' parameters $x', x'', \dots$ follow the lines, but (labels of) spaces do not. The role of the permutations P in (3) is taken over by

$$P_{i,i+1}(x) = \check{R}_{i,i+1}\big(x, a-(\sigma^z_1+\cdots+\sigma^z_{i-1})\big)\,, \tag{6a}$$

with a shifted by the spin-z to the left of the R-matrix. On the usual spin (or computational) basis this means

$$\begin{aligned} P_{i,i+1}(x)\,|s_1,\dots,s_N\rangle &= |s_1,\dots,s_{i-1}\rangle \\ &\quad\otimes \check{R}\big(x, a-\textstyle\sum_{k=1}^{i-1} s_k\big)\,|s_i,s_{i+1}\rangle \\ &\quad\otimes |s_{i+1},\dots,s_N\rangle\,, \end{aligned} \tag{6b}$$

so e.g. $P_{23}(x) = |{\uparrow}\rangle\langle{\uparrow}|\otimes\check{R}(x,a-1) + |{\downarrow}\rangle\langle{\downarrow}|\otimes\check{R}(x,a+1)$. These obey the (braid-like) Yang–Baxter equation

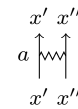
$$\begin{aligned} &P_{i,i+1}(x-y)\,P_{i+1,i+2}(x)\,P_{i,i+1}(y) \\ &\quad = P_{i+1,i+2}(y)\,P_{i,i+1}(x)\,P_{i+1,i+2}(x-y)\,, \end{aligned} \tag{7}$$

commutativity $[P_{i,i+1}(x), P_{j,j+1}(y)]=0$ for $|i-j|>1$, and unitarity (inversion) relation

$$P_{i,i+1}(-x)\,P_{i,i+1}(x) = 1\,. \tag{8}$$

Recall that the local hamiltonian for Heisenberg chains is $\partial\log\check{R} = \check{R}^{-1}\check{R}'$. Likewise we need (again $x=x'-x''$)

$$E(x,a) = \frac{1}{\theta(\eta)\,V(x)}\,\check{R}(-x,a)\,\partial_x\check{R}(x,a) = \tag{9}$$



normalised so that V, E have suitable limits, see Sec. II B. The nearest-neighbour exchange in (3) is replaced by

$$E_{i,i+1}(x) = \frac{1}{\theta(\eta)\,V(x)}\,P_{i,i+1}(-x)\,P'_{i,i+1}(x)\,. \tag{10}$$

Its dependence on x (and, of course, a) is new compared to the Inozemtsev and DHS chains; but cf. [21]. An explicit expression is given in the Supplemental Material.

To put everything together we give each site an 'inhomogeneity' parameter $x^\star_k = k$, i.e. its (equispaced) position on the chain. Then the 'left' spin interactions from (1) have the same structure as in [19]:

$$\begin{aligned} S^{\mathrm{L}}_{[i,j]} &= P_{j-1,j}(1)\cdots P_{i+1,i+2}(j-i-1)\;E_{i,i+1}(i-j) \\ &\quad\times P_{i+1,i+2}(i-j+1)\cdots P_{j-1,j}(-1)\,, \end{aligned} \tag{11}$$

cf. the first line in (3). For example,

$$\begin{aligned} &S^{\mathrm{L}}_{[1,2]} = E_{12}(-1)\,, \qquad S^{\mathrm{L}}_{[2,3]} = E_{23}(-1)\,, \\ &S^{\mathrm{L}}_{[1,3]} = P_{23}(1)\,E_{12}(-2)\,P_{23}(-1)\,, \\ &S^{\mathrm{L}}_{[1,4]} = P_{34}(1)\,P_{23}(2)\,E_{12}(-3)\,P_{23}(-2)\,P_{34}(-1)\,. \end{aligned} \tag{12}$$

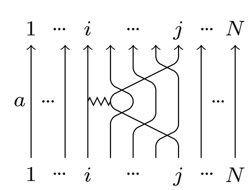
Its 'right' counterpart looks like in [20]:

$$\begin{aligned} S^{\mathrm{R}}_{[i,j]} &= P_{i,i+1}(1)\cdots P_{j-2,j-1}(j-i-1)\;E_{j-1,j}(i-j) \\ &\quad\times P_{j-2,j-1}(i-j+1)\cdots P_{i,i+1}(-1)\,. \end{aligned} \tag{13}$$

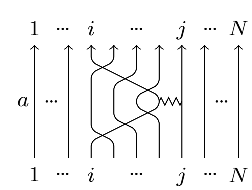
as illustrated by

$$S^{\mathrm{R}}_{[1,4]} = P_{12}(1)\,P_{23}(2)\,E_{34}(-3)\,P_{23}(-2)\,P_{12}(-1)\,. \tag{14}$$

Diagrammatically, (11) and (13) respectively equal



and



. (15)

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[1] Simply replace (5) by the dynamical $\mathfrak{gl}_r$ R-matrix [27], see [28].

B. Properties and limits

Our chain has four free parameters: the length $N \geq 2$, $\kappa > 0$ to tune the interaction range, the anisotropy η, and the dynamical parameter a.[2] The spectrum is real when η is imaginary (i.e. $|\Delta| > 1$ in terms of the usual parameter of Heisenberg XXZ), cf. [19], and a is real.

a. Defining properties. Our spin chain unifies the Inozemtsev and DHS chains and is integrable.

In the isotropic limit $\eta \to 0$ we get the Inozemtsev chain in the form of [14, 15]. Indeed, (2) becomes $-\rho'(x) = \wp(x) + \text{cst}$, and, since $P(x) \to P$ and $E(x) \to E$ if $a \to -\mathrm{i}\infty$, both (11) and (13) yield (3) up to a similarity transformation that can be removed by sending $a \to -\mathrm{i}\infty$. When $\eta \neq 0$ the spin symmetry breaks down to $U(1) \subset SU(2)$ generated by $S^z = \sum_j \sigma^z_j/2$.

In the long-range limit $\kappa \to 0$ we recover the DHS chain, again up to a conjugation that becomes trivial as $a \to -\mathrm{i}\,\infty$. Indeed, (2) gives the long-range potential $(\frac{\pi}{N})^2/\sin[\pi(\frac{x}{N}+\gamma)]\sin[\pi(\frac{x}{N}-\gamma)]$ of [19] with $\eta = N\gamma$. As $a \to -\mathrm{i}\infty$, (9) reduces to the Temperley–Lieb generator

$$e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-\pi\mathrm{i}\gamma} & -\mathrm{e}^{\pi\mathrm{i}\gamma} & 0 \\ 0 & -\mathrm{e}^{-\pi\mathrm{i}\gamma} & \mathrm{e}^{\pi\mathrm{i}\gamma} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{16}$$

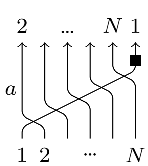
and $P(x) \to 1 - \sin(\pi\frac{x}{N})\, e/\sin[\pi(\frac{x}{N}+\gamma)]$ to Jimbo's R-matrix of quantum $\mathfrak{sl}_2$. This gives the DHS chain.

Finally, the chiral hamiltonians (1) commute,

$$[H^{\mathrm{L}}, H^{\mathrm{R}}] = 0\,. \tag{17}$$

They belong to a hierarchy of commuting operators whose expressions parallel those in [20, 21], see [28].

b. Further properties. Our chain shares some features with the DHS chain. In particular, (11) and (13) involve multispin interactions that affect all intermediate spins, whence the notation '$[i,j]$'. In addition, $\eta \neq 0$ breaks periodicity, but our chain has some sort of quasiperiodic (twisted) boundary conditions. Indeed, the commuting family (17) contains the modified (left) translation operator (cf. [19])

$G =$ 

$= K_N^{-1} P_{N-1,N}(1-N) \cdots P_{12}(-1)\,,$ (18)

with $K_N = k_N\big(a - (\sigma^z_1 + \cdots + \sigma^z_{N-1})\big)$ for $k(a) = \mathrm{e}^{\kappa\eta\, a\,\sigma^z} = \mathrm{diag}(\mathrm{e}^{\kappa\eta a}, \mathrm{e}^{-\kappa\eta a})$ a diagonal twist. The quasiperiodicity

$$S^{\mathrm{L}}_{[1,N]} = G\, S^{\mathrm{L}}_{[1,2]}\, G^{-1}\,, \quad S^{\mathrm{R}}_{[1,N]} = G^{-1} S^{\mathrm{R}}_{[N-1,N]}\, G\,, \tag{19}$$

underlines the chirality of the hamiltonians (1). Upon normalisation, (18) provides a notion of momentum, as well as all N eigenvectors at $S^z = N/2 - 1$ (cf. §1.2.6 in [20]), i.e. the magnons of our chain.

c. New limits Our chain has various new limits, including an a-dependent extension of the Inozemtsev chain that is not quite isotropic [28]. When $N \to \infty$ we formally get a hyperbolic counterpart of the DHS model, with $N \rightsquigarrow \mathrm{i}\pi/\kappa$ and sum in (1) over all $i < j$ in $\{1, 2, \ldots\}$. Numerics suggests that its matrix entries converge.

We focus on the short-range limit. Set $\eta = -\mathrm{i}\pi\,\bar{\gamma}/\kappa$ and renormalise (1) by $n_H(\kappa) \sim \mathrm{e}^{2\kappa}/\kappa^2$ as in [13, 15]. For $\kappa \to \infty$ we get a chain of the nearest-neighbour form

$$H^{\mathrm{XXZ}} = \sum_{i=1}^{N-1} S^{\mathrm{H}}_{[i,i+1]} + S^{\mathrm{H}}_{[1,N]}\,. \tag{20}$$

Here $S^{\mathrm{H}}_{[i,i+1]} = e_{i,i+1}\big(a - (\sigma^z_1 + \cdots + \sigma^z_{i-1})\big)$ is defined as in (6) in terms of a dynamical generalisation of (16):

$$e^{\mathrm{H}}(a) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\sin[\pi\bar{\gamma}(a-1)]}{\sin[\pi\bar{\gamma}a]} & -\frac{\sin[\pi\bar{\gamma}(a+1)]}{\sin[\pi\bar{\gamma}a]} & 0 \\ 0 & -\frac{\sin[\pi\bar{\gamma}(a-1)]}{\sin[\pi\bar{\gamma}a]} & \frac{\sin[\pi\bar{\gamma}(a+1)]}{\sin[\pi\bar{\gamma}a]} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

The last term in (20) admits two forms coming from (19),

$$S^{\mathrm{H}}_{[1,N]} = G^{\mathrm{H}}\, S^{\mathrm{H}}_{[1,2]}\, G^{\mathrm{H}\,-1} = G^{\mathrm{H}\,-1} S^{\mathrm{H}}_{[N-1,N]}\, G^{\mathrm{H}}\,, \tag{22}$$

now in terms of the limit of (18),

$$G^{\mathrm{H}} = K^{\mathrm{H}}_N\, P^{\mathrm{H}}_{N,N-1} \cdots P^{\mathrm{H}}_{12}\,. \tag{23}$$

Here the twist involves $k^{\mathrm{H}}(a) = \mathrm{e}^{\mathrm{i}\pi\bar{\gamma}\,a\,\sigma^z}$, and $P^{\mathrm{H}}_{i,i+1}$ is again defined as in (6) via the trigonometric dynamical R-matrix without spectral parameter

$$\check{R}^{\mathrm{H}}(a) = 1 - \mathrm{e}^{-\mathrm{i}\pi\bar{\gamma}}\, e^{\mathrm{H}}(a)\,. \tag{24}$$

The short-range limit (20) thus is a dynamical variant of the Heisenberg XXZ chain. It is no longer chiral, but remains quasiperiodic, because the twist in the boundary term prevents removing a. In the isotropic limit $\bar{\gamma} \to 0$ we obtain, once more up to a conjugation that vanishes as $a \to -\mathrm{i}\,\infty$, the usual periodic Heisenberg XXX chain. This completes our description of the limits in Fig. 1.

C. Discussion

Our spin chain arises from 'freezing', see Sec. III. In fact, this produces a modular family of integrable spin chains, one for each classical equilibrium position of the scalar elliptic Ruijsenaars model. Only two amongst them have a real spectrum for a suitable parameter range, of which only the chain given here has a short-range limit. Concretely, this amounts to thinking of $\theta(x)$ as a periodisation of $\sinh(\kappa x)/\kappa$ (rather than a sine, cf. the Supplemental Material). At the isotropic level this corresponds

[2] The potential (2) has poles at $2\eta = Nk + \mathrm{i}\pi\, l/\kappa$ for $k, l \in \mathbb{Z}$, and the functions (4) further require avoiding $\eta\, a = N\,k + \mathrm{i}\pi\, l/\kappa$.

to shifting $\wp(x)$ to $-\rho'(x)$ by a constant that regularises the short-range limit $\kappa \to \infty$ [13, 14], simplifies the dispersion relation and Bethe-ansatz equations, and allows the latter to be recast in rational form [15]. Note that, nevertheless, $\theta(x) \to N \sin(\pi x/N)/\pi$ for $\kappa \to 0$.

The spin chains in Fig. 1 each have two 'types'. In *positions* (cf. the potential), the HS chain is trigonometric (and rational as $N \to \infty$), Inozemtsev is elliptic (hyperbolic as $N \to \infty$), and Heisenberg is of 'contact' type; cf. Sec. 2.2 in [15]. In *spins*, the isotropic chains are rational (related to the Yangian), and the Heisenberg XXZ and DHS chains are trigonometric (associated to quantum affine algebras). What about our chain? On the one hand, it has an elliptic R-matrix (5). On the other hand it belongs at the partially isotropic level (cf. Fig. 1). We argue that it is spin-trigonometric. The hidden trigonometric side of our chain will become more clear in Sec. III C. We suspect that it might also manifest itself in the spectrum, with energies and Bethe equations that are trigonometric in suitable rapidities.

The long-range spin interactions (11)–(15) are fine-tuned generalisations of the isotropic case (3), in contrast with the simple and 'robust' form of the traditional Heisenberg chains. The need for fine tuning is more clear for the DHS chain, which is a *good* generalisation of the HS chain, maintaining (deforming, rather than breaking) the integrability ('abelian symmetries'), enhanced spin symmetry ('nonabelian symmetries'), and remarkably simple and explicit spectrum. Generalising the DHS chain, our model must have complicated spin interactions too. They ensure that our chain keeps its abelian symmetries and, we expect, exact solvability for any κ.

Let us examine one fine-tuned ingredient in more detail: the choice of R-matrix. For the DHS chain, the enhanced spin symmetry requires [6, 25] its R-matrix to be related (by 'Baxterisation') to the Hecke algebra and, for spin 1/2, the Temperley–Lieb algebra.[3] This leads to some asymmetry ($P \check{R} P \neq \check{R}$): in the terminology of [30], the R-matrix should be in the homogeneous, rather than principal, grading. At the partially isotropic level, an elliptic potential asks for an elliptic R-matrix, cf. (9). There are two standard choices. The first choice is Baxter's eight-vertex R-matrix, which was used in [26], is symmetric, and generalises the symmetric (principal) trigonometric R-matrix. Whilst at the trigonometric limit the two gradings are related by a conjugation by a (spectral-parameter dependent) matrix, there does not seem to be a conjugation of the eight-vertex R-matrix that limits to the homogeneous trigonometric R-matrix.[4] The other choice is the elliptic dynamical R-matrix (5), which does limit to the homogeneous R-matrix, and (unlike the eight-vertex R-matrix) commutes with S^z. This is why our model fits in Fig. 1, while the (fully) anisotropic MZ chain [26] belongs to a separate landscape of integrable spin chains [26, 28].

Algebraic structures show up in the short-range limit too. As for the DHS chain, the (dynamical) operators $e_i = S^{\text{H}}_{[i,i+1]}$ obey the Temperley–Lieb relations $e_i^2 = 2\cos(\pi\bar{\gamma})\, e_i$ and $e_i\, e_{i\pm1}\, e_i = e_i$. The boundary term (22) is a 'braid translation' [32], and $e_0 = S^{\text{H}}_{[1,N]}$ obeys the *periodic* Temperley–Lieb relations, i.e. the preceding extended to subscripts modulo N. The normalised translation $u \propto G^{\text{H}}$ enhances this to the *affine* Temperley–Lieb algebra, $u\, e_i\, u^{-1} = e_{i-1 \bmod N}$ and $e_{N-1} = u^2\, e_1 \cdots e_{N-1}$. More specifically, (20) looks like an unrestricted version of the RSOS model [33], and forms an S^z-symmetric alternative to the Temperley–Lieb representation from the conclusion of [22]; our e_i are minimal dynamical versions of the pair interaction in Eq. (35) in [22].

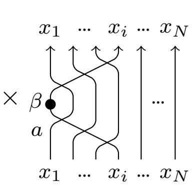
More broadly, our chain presents important progress for the development of a general theory of long-range integrability. It implies the existence of commuting hamiltonians for the Inozemtsev chain, although extracting explicit higher hamiltonians requires work, cf. Remark ii in §1.3.4 of [20]. Moreover, with explicit R-matrices, our work is a step towards an algebraic approach to the Inozemtsev chain. We believe that this will even offer valuable new insights for the Heisenberg XXX chain.

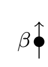
III. THE QUANTUM MANY-BODY SYSTEM

A. Hamiltonians

Now consider N spin-$\frac{1}{2}$ particles with coordinates x_j moving on a circle. Let $\Gamma_j = \exp(\mathrm{i}\frac{\kappa}{\pi}\beta\,\partial_{x_j})$ be the shift $x_i \mapsto x_i - \beta\,\delta_{ij}$, where β is a new ('reduced coupling') parameter. Set $x_{ij} = x_i - x_j$. Our QMBS is given by a family of matrix-valued difference operators based on (6), with structure like in [20, 26, 34]. The first one is

$$\widetilde{D}_1 = \sum_{i=1}^{N} A_i(\boldsymbol{x}) \times$$





$= \Gamma_i$

$$\begin{aligned} &= \sum_{j=1}^{N} A_i(\boldsymbol{x})\, P_{i-1,i}(x_{i,i-1}) \cdots P_{12}(x_{i,1}) \\ &\qquad \times \Gamma_i\, P_{12}(x_{1,i}) \cdots P_{i-1,j}(x_{i-1,i}) \\ &= \sum_{i=1}^{N} A_i(\boldsymbol{x})\, P_{i-1,i}(x_{i,i-1}) \cdots P_{12}(x_{i,1}) \\ &\qquad \times P_{12}(x_{1,i}+\beta) \cdots P_{i-1,i}(x_{i-1,i}+\beta)\, \Gamma_i\,, \end{aligned} \tag{25}$$

where $A_i(\boldsymbol{x}) = \prod_{k(\neq i)}^{N} \theta(x_{ik}+\eta)/\theta(x_{ik})$. The higher difference operators will be given in [28]. Here we only need the total shift operator $\widetilde{D}_N = \Gamma_1 \cdots \Gamma_N$ and an

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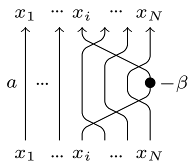
[3] This is also the algebraic reason why the HS and DHS chains can be extended to higher rank, but not to higher spin.

[4] This is supported by the fact that the principal grading operator is essential in the construction of the universal elliptic R-matrix of vertex type [31]. We thank H. Konno for pointing this out.

'antichiral' version of (25):

$$\widetilde{D}_{-1} = \sum_{i=1}^{N} A_{-i}(\boldsymbol{x}) \times$$



$$= \sum_{i=1}^{N} A_{-i}(\boldsymbol{x})\, P_{i,i+1}(x_{i+1,i}) \cdots P_{N-1,N}(x_{N,i}) \times \Gamma_i^{-1}\, P_{N-1,N}(x_{i,N}) \cdots P_{i,i+1}(x_{i,i+1})\,, \tag{26}$$

where $A_{-i}(\boldsymbol{x}) = \prod_{k(\neq i)}^{N} \theta(x_{ik}-\eta)/\theta(x_{ik})$. These operators define a version of the elliptic Ruijsenaars model [35] with spins that is integrable in the sense that the difference operators all commute. In particular,

$$[\widetilde{D}_1, \widetilde{D}_{-1}] = 0\,, \quad [\widetilde{D}_{\pm 1}, \widetilde{D}_N] = 0\,. \tag{27}$$

The second equality is clear as $\widetilde{D}_{\pm 1}$ only depends on differences x_{ij}. The first equality can be checked explicitly for low N. The proof will be published elsewhere.

B. Properties and limits

Our QMBS has the four free parameters of our spin chain plus the reduced coupling β. It admits various limits.

In the trigonometric limit $\kappa \to 0$, again with $a \to -\mathrm{i}\,\infty$, we readily obtain the trigonometric spin-Ruijsenaars–Macdonald model [20, 34] underlying the DHS chain [6, 18, 20]. In the isotropic limit $\eta \to 0$ plus $a \to -\mathrm{i}\,\infty$ one recovers the elliptic spin-Calogero–Sutherland model [36, 37]. Like our spin chain, our QMBS admits an intermediate a-dependent generalisation of this CS model [28].

Freezing amounts to the (semi)classical limit $\beta \to 0$, i.e. to an expansion in β of the difference operators evaluated at classical equilibria, see [6, 23, 24] and [21] for the elliptic case. Denote linearisation in β by $\delta = \frac{\pi}{\kappa}\, \partial_\beta|_{\beta=0}$. Due to (27), $\delta\widetilde{D}_{\pm 1}$ and the total momentum operator $-\delta\widetilde{D}_N = \sum_j(-\mathrm{i}\,\partial_{x_j})$ still commute at the equispaced classical equilibrium positions $x_k^\star = k$ [21, 28]. In particular, our spin-chain hamiltonians (1) are

$$H^{\mathrm{L,R}} = \frac{1}{\theta(\eta)} \left[\frac{1}{A^\star}\, \delta\widetilde{D}_{\pm 1} \mp \delta\widetilde{D}_N \right]_{x_k = x_k^\star}\,, \tag{28}$$

where $A^\star = A_{\pm i}(\boldsymbol{x}^\star) = \theta(\eta)|_{N=1}/[N\,\theta(\eta)]$. Hence the commutativity (17) for our spin chain follows from (27).

A new feature compared to the trigonometric case is that there is a whole modular family of classical equilibria associated to the quasiperiods $N, \mathrm{i}\pi/\kappa$. These equilibria are related by extending the action of $SL(2,\mathbb{Z})$ on elliptic functions to our setting, and can be identified at the QMBS level by suitable reparametrisations of the parameters and $\boldsymbol{x}$. Upon freezing, however, each equilibrium yields a *different* integrable spin chain.

C. Discussion

Like long-range spin chains, QMBSs have two 'types'. One is again in *positions* (cf. the potential energy). The other is in *momenta* (kinetic enery): rational here corresponds to differential operators, and trigonometric to difference operators (exponentiated differentials). Our QMBS is elliptic in positions and trigonometric in momenta.

The operators (25)–(26) only differ from MZ [26] in the choice of R-matrix. Since the dynamical R-matrix (5) is related to Baxter's eight-vertex R-matrix via the face-vertex transformation [38]

$$\check{R}^{8\mathrm{v}}(x_{ij})\, S(x_i, x_j, a) = S(x_j, x_i, a)\, \check{R}(x_{ij}, a)\,, \tag{29}$$

one might expect the two QMBS to be equivalent. However, the transformation S in (29) depends on coordinates x_k, and does not commute with the shift operators Γ_k. Thus our difference operators are *not* face-vertex transforms of those of MZ, and define *another* QMBS. This difference persists to all limiting spin chains.

It seems difficult to adapt the proof of integrability from [26], which heavily relies on the simple periodicity properties of $\check{R}^{8\mathrm{v}}$ in order to simplify expressions and set up a proof by induction. Unfortunately, the dynamical R-matrix does not have such simple properties. Our proof for the commutativity (27) is independent, and, in view of its technical nature, will appear elsewhere.

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IV. SUPPLEMENTAL MATERIAL

A. Elliptic functions

Here we summarise the definitions of the elliptic functions that we need. See [15] (where the functions θ and ρ defined below were decorated with a subscript '2') and [28] for more details and some references.

We use the (odd) Jacobi theta function with nome $p = \mathrm{e}^{-N\kappa}$,

$$\theta(x) = \frac{\sinh(\kappa\, x)}{\kappa} \prod_{n=1}^{\infty} \frac{\sinh[\kappa\,(N\,n + x)]\,\sinh[\kappa\,(N\,n - x)]}{\sinh^2(N\kappa\, n)} = \frac{\sinh(\kappa\, x)}{\kappa} + O(p^2)\,. \tag{30}$$

It is the unique odd entire function with double quasiperiodicity $\theta(x + \mathrm{i}\pi/\kappa) = -\theta(x)$, $\theta(x + N) = -\mathrm{e}^{\kappa(N+2x)}\,\theta(x)$ and normalisation $\theta'(0) = 1$. In terms of the Weierstraß sigma function with quasiperiods N and $\mathrm{i}\pi/\kappa$ it reads

$$\theta(x) = \mathrm{e}^{\mathrm{i}\,\kappa\,\eta_2\, x^2/2\pi}\,\sigma(x)\,, \qquad \eta_2 = 2\,\zeta(\mathrm{i}\pi/2\kappa)\,. \tag{31}$$

The prepotential is the logarithmic derivative

$$\rho(x) = \frac{\theta'(x)}{\theta(x)} = \zeta(x) + \frac{\mathrm{i}\,\kappa\,\eta_2}{\pi}\,x = \kappa\coth(\kappa\, x) + O(p^2)\,, \tag{32}$$

with $\zeta(x) = \sigma'(x)/\sigma(x)$ the Weierstraß zeta function. It is odd and obeys $\rho(x+\mathrm{i}\pi/\kappa) = \rho(x)$, $\rho(x+N) = \rho(x)-2\pi\mathrm{i}/N$.

Finally, the potential is defined as the symmetric difference quotient

$$V(x) = -\frac{\rho(x+\eta) - \rho(x-\eta)}{\theta(2\eta)} = \frac{b}{\mathrm{sn}[c\,(x+\eta),k]\;\mathrm{sn}[c\,(x-\eta),k]} + d\,, \qquad k = \frac{\sqrt{\wp(\mathrm{i}\pi/2\kappa) - \wp[(N+\mathrm{i}\pi/\kappa)/2]}}{\sqrt{\wp(N/2) - \wp[(N+\mathrm{i}\pi/\kappa)/2]}}\,, \tag{33}$$

where the equality with Jacobi's elliptic sine $\mathrm{sn}(x,k)$ with elliptic modulus k involves constants b,d and $c = \sqrt{\wp(N/2) - \wp(\mathrm{i}\pi/2\kappa)}$. The potential is even and doubly periodic, $V(x+\mathrm{i}\pi/\kappa) = V(x+N) = V(x)$. The sign in (33) is chosen such that $V(x) \to -\rho'(x) = \wp(x) - \mathrm{i}\kappa\eta_2/\pi$ becomes the Weierstraß elliptic function as $\eta \to 0$.

B. Nearest-neighbour exchange

The deformed long-range spin interactions contain the logarithmic derivative of the dynamical R-matrix, which determines the nearest-neighbour exchange $E(x,a)$ via (9). Explicitly we have

$$\check{R}(-x,a)\,\check{R}'(x,a) = \theta(\eta)\,V(x)\,E(x,a) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha(x,\eta\,a) & \beta(x,\eta\,a) & 0 \\ 0 & \beta(x,-\eta\,a) & \alpha(x,-\eta\,a) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{34}$$

where the coefficients are related to the functions f,g defined in (4) through

$$\begin{aligned} \alpha(x,a) &= g(x,a)\,g(-x,a)\,\big(\rho(x+a) - \rho(x)\big) - \big(\rho(x+\eta) - \rho(x)\big)\,, \\ \beta(x,a) &= f(x,a)\,g(-x,a)\,\big(\rho(x) - \rho(x-a)\big)\,. \end{aligned} \tag{35}$$

Here we simplified β using the addition formula for $\theta(x)$.

C. Limits

Most of the limits in Sec. II B and Sec. III B can be evaluated using the limits given in [15]. For the dynamical R-matrix (5) note that the functions (4) can be expressed via the Kronecker elliptic function (denoted by χ_2 in [15])

$$\phi(x,y) = \phi(y,x) = \frac{\theta(x+y)}{\theta(x)\,\theta(y)}\,, \qquad f(x,a) = \frac{\phi(\eta,a)}{\phi(\eta,x)}\,, \quad g(x,a) = \frac{\phi(a,x)}{\phi(\eta,x)}\,. \tag{36}$$

The short-range limit of the potential can be found using the convergent sum

$$\rho(x+\eta)-\rho(x-\eta)=4\,\kappa\sinh(2\kappa\eta)\sum_{n\in\mathbb{Z}}\frac{1}{\cosh(2\kappa\eta)-\cosh[2\kappa(N\,n+x)]}\,. \tag{37}$$

For a convergent but non-zero limit as $\kappa\to\infty$ we must also send $\eta\to 0$ with $\kappa\eta$ fixed so that $\cosh(2\kappa\eta)$ becomes constant. Thus we set $\eta=-\mathrm{i}\pi\,\bar{\gamma}/\kappa$ and rescale (37) by a prefactor behaving as $n_\eta(\kappa)\sim \mathrm{e}^{2\kappa}/(\kappa\sinh 2\eta)$ to obtain

$$n_{-\mathrm{i}\pi\bar{\gamma}/\kappa}(\kappa)\,\big(\rho(x-\mathrm{i}\pi\bar{\gamma}/\kappa)-\rho(x+\mathrm{i}\pi\bar{\gamma}/\kappa)\big)\;\to\;\delta_{x,1}+\delta_{x,N-1}\,,\quad \kappa\to\infty\,,\qquad x\in\{1,\dots,N-1\}\,. \tag{38}$$

A choice that fits with all other limits is $n_\eta(\kappa)=\sinh^2(\kappa)/[\kappa^2\,\theta(2\eta)]$. This is why we choose denominator $\theta(2\eta)$ in the potential (33) rather than the 2η from [15]; when $\eta\to 0$ the two behave the same.