A SHORT NOTE ON BKP FOR THE KONTSEVICH MATRIX MODEL WITH ARBITRARY POTENTIAL

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Abstract. We exhibit the Kontsevich matrix model with arbitrary potential as a BKP tau function with respect to polynomial deformations of the potential.

Contents

1. The formula

Let $\mathcal{H}_N$ be the space of hermitian $N \times N$ matrices equipped with the Lebesgue measure

$$\mathrm{d}H = \prod_{i=1}^{N} \mathrm{d}H_{ii} \prod_{1 \leq i < j \leq N} \mathrm{dRe}(H_{ij})\, \mathrm{dIm}(H_{ij}).$$

Given a positive matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, we introduce the Gaussian probability measure on $\mathcal{H}_N$

$$\mathrm{d}\mathbb{P}(H) = \frac{\sqrt{\Delta(\boldsymbol{\lambda}, \boldsymbol{\lambda})}}{2^{\frac{N}{2}} (2\pi)^{\frac{N^2}{2}}}\, \mathrm{d}H\, e^{-\frac{1}{2}\mathrm{Tr}(\Lambda H^2)}. \tag{1.1}$$

We use the notations $\Delta(\boldsymbol{\lambda}) = \prod_{1 \leq i < j \leq N}(\lambda_j - \lambda_i)$ and $\Delta(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \prod_{1 \leq i,j \leq N}(\lambda_i + \mu_j)$. We denote $[n] = \{1, \ldots, n\}$ and $\lambda_{\min} = \min\{\lambda_i \mid i \in [N]\}$.

Let V_0 be a continuous function on $\mathbb{R}$ such that the measure $e^{-\frac{1}{2}\lambda_{\min} x^2 + V_0(x)}$ has finite moments on $\mathbb{R}$ (take for instance V_0 to be a polynomial of even degree with negative top coefficient). Then the measure $\mathrm{d}\mathbb{P}(H)\, e^{\mathrm{Tr}\, V_0(H)}$ on $\mathcal{H}_N$ is finite . Let

$$V_{\mathbf{t}}(x) = V_0(x) + \sum_{k \geq 0} t_{2k+1}\, x^{2k+1},$$

where $\mathbf{t} = (t_{2k+1})_{k \geq 1}$ are formal parameters. The partition function of the Kontsevich model with arbitrary potential is defined by

$$Z(\mathbf{t}) = \int_{\mathcal{H}_N} \mathrm{d}\mathbb{P}(H)\, e^{\mathrm{Tr}\, V_{\mathbf{t}}(H)}. \tag{1.2}$$

This short note presents a derivation of the following result.

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Theorem 1.1. *For even N, we have*

$$Z(\mathbf{t}) = \frac{\sqrt{\Delta(\boldsymbol{\lambda},\boldsymbol{\lambda})}}{2^{\frac{N^2}{2}}\,(2\pi)^{\frac{N}{2}}\,\prod_{n=1}^{N-1} n!}\,\underset{0\leq m,n\leq N-1}{\mathrm{Pf}}\big(K_{m,n}(\mathbf{t})\big), \tag{1.3}$$

where $⨍ = \lim_{\epsilon\to 0}\int_{|x+y|\geq\epsilon}$ *is the Cauchy principal value integral and*

$$\begin{aligned} K_{m,n}(\mathbf{t}) &= ⨍_{\mathbb{R}^2} \frac{x-y}{x+y}\,F_m(x)F_n(y)\,e^{V_{\mathbf{t}}(x)+V_{\mathbf{t}}(y)}\mathrm{d}x\mathrm{d}y, \\ F_n(x) &= x^{2n} + \frac{(-2)^n\,n!}{\Delta(\boldsymbol{\lambda})}\,\det\Big(\lambda_i^0\,\big|\,\lambda_i^1\,\big|\,\ldots,\lambda_i^{n-1}\,\big|\,R_N\big(-\tfrac{1}{2}\lambda_i x^2\big)\,\big|\,\lambda_i^{n+1}\,\big|\,\cdots\,\big|\,\lambda_i^{N-1}\Big), \\ R_N(\xi) &= \frac{\xi^N}{(N-1)!}\int_0^1 \mathrm{d}u\,(1-u)^{N-1}\,e^{\xi u} = \frac{e^\xi}{\xi}\bigg(N - \frac{\Gamma(N+1;\xi)}{(N-1)!}\bigg). \end{aligned}$$

The proof proposed in Section 2 consists in classical algebraic manipulations with matrix integrals and an analysis argument — that one may find of independent interest, see Lemma 2.1 and Remark 2.2 — to extend de Bruijn's Pfaffian formula [8] to singular kernels like the one appearing in (1.3).

We recognize in (1.3) the expression of a BKP τ-function according to [17], see also [3, §7.1.2.3] where the functions y of equation 7.1.49 should be taken to

$$y_n(\mathbf{t}) = \int_{\mathbb{R}} F_n(x)\,e^{V_{\mathbf{t}}(x)}\,\mathrm{d}x$$

and depend on Λ. By the general theory, $Z(\mathbf{t})$ satisfies the Hirota bilinear relation of the BKP hierarchy [7].

Corollary 1.2. *For fixed Λ, $Z(\mathbf{t})$ is a BKP τ-function with respect to the times $\mathbf{t}$, i.e. it satisfies the Hirota bilinear equation of type B*

$$Z(\mathbf{t})Z(\mathbf{s}) = \operatorname*{Res}_{z=0}\frac{\mathrm{d}z}{z}\,e^{\sum_{k\geq 0} z^{2k+1}(t_{2k+1}-s_{2k+1})}Z\big(\mathbf{t}-2[z^{-1}]\big)\,Z\big(\mathbf{s}+2[z^{-1}]\big). \tag{1.4}$$

where $[z^{-1}] = \big(\frac{1}{z},\frac{1}{3z^3},\frac{1}{5z^5},\ldots\big)$.

Expanding the bilinear equation (1.4) near $\mathbf{t}=\mathbf{s}=0$ yields the BKP hierarchy of equations. In the present case, they give algebraic relations between the odd moments (or the cumulants) of the Kontsevich model with arbitrary potential, defined as

$$\begin{aligned} M_{2\ell_1+1,\ldots,2\ell_n+1} &= \frac{1}{Z(\mathbf{t})}\frac{\partial}{\partial t_{2\ell_1+1}}\cdots\frac{\partial}{\partial t_{2\ell_n+1}}Z(\mathbf{t})\Big|_{\mathbf{t}=0} \\ &= \frac{\int_{\mathcal{H}_N}\mathrm{d}\mathbb{P}(H)\,e^{\mathrm{Tr}(-\frac{1}{2}\Lambda H^2+V_0(H))}\,\mathrm{Tr}\,H^{2\ell_1+1}\cdots\mathrm{Tr}\,H^{2\ell_n+1}}{\int_{\mathcal{H}_N}\mathrm{d}\mathbb{P}(H)\,e^{\mathrm{Tr}(-\frac{1}{2}\Lambda H^2+V_0(H))}}, \\ K_{2\ell_1+1,\ldots,2\ell_n+1} &= \frac{\partial}{\partial t_{2\ell_1+1}}\cdots\frac{\partial}{\partial t_{2\ell_n+1}}\ln Z(\mathbf{t})\Big|_{\mathbf{t}=0} \\ &= \sum_{\mathbf{I}=\text{partitions of }[n]}(-1)^{|\mathbf{I}|-1}(|\mathbf{I}|-1)!\prod_{I\in\mathbf{I}}M_{(\ell_i)_{i\in I}}. \end{aligned}$$

Note that the hierarchy of equations (1.4) does not depend on N, Λ and V_0, though the particular solution $Z(\mathbf{t})$ does through the initial data $Z(\mathbf{0})$.

For instance, the first two BKP-equations are

$$\begin{aligned} 0 &= (D_1^6 - 5D_1^3D_3 - 5D_3^2 + 9D_1D_5)(Z,Z)(\mathbf{0}) \\ 0 &= (D_1^8 + 7D_1^5D_3 - 35D_1^2D_3^2 - 21D_1^3D_5 - 42D_3D_5 + 90D_1D_7)(Z,Z)(\mathbf{0}) \end{aligned} \tag{1.5}$$

in terms of the Hirota operators $D_k(\tau,\tau)(\mathbf{t}) = (\partial_{t_k} - \partial_{s_k})\tau(\mathbf{t})\tau(\mathbf{s})\big|_{\mathbf{s}=\mathbf{t}}$. For *even* V_0 we have $M_{2\ell_1+1,\ldots,2\ell_n+1} = 0$ for n odd, and (1.5) results in

$$\begin{aligned}
&0 = M_{1^6} + 15M_{1^4}M_{1,1} - 5M_{3,1^3} - 15M_{3,1}M_{1,1} - 5M_{3,3} + 9M_{5,1} \\
(1.6)\quad &0 = M_{1^8} + 28M_{1^6}M_{1,1} + 35(M_{1^4})^2 + 7M_{3,1^5} + 70M_{3,1^3}M_{1,1} + 35M_{3,1}M_{1^4} - 35M_{3,3}M_{1,1} \\
&\quad - 70(M_{3,1})^2 - 21M_{5,1^3} - 63M_{5,1}M_{1,1} - 42M_{5,3} + 90M_{7,1}.
\end{aligned}$$

Or, equivalently in terms of cumulants:

$$\begin{aligned}
0 &= K_{1^6} + 30K_{1^4}K_{1,1} + 60(K_{1,1})^3 - 5K_{3,1^3} - 5K_{3,3} - 30K_{3,1}K_{1,1} + 9K_{5,1}, \\
0 &= K_{1^8} + 56K_{1^6}K_{1,1} + 70(K_{1^4})^2 + 840K_{1^4}(K_{1,1})^2 + 840(K_{1,1})^4 + 7K_{3,1^5} \\
&\quad + 70K_{3,1}K_{1^4} + 420K_{3,1}(K_{1,1})^2 + 140K_{3,1^3}K_{1,1} - 35K_{3,3}K_{1,1} - 70(K_{3,1})^2 - 21K_{5,1^3} \\
&\quad - 126K_{5,1}K_{1,1} - 42K_{5,3} + 90K_{7,1}.
\end{aligned}$$

When V_0 is not even, many more terms contribute.

It is instructive to test these equation in the simplest case $V_0 = 0$. The moments can be found[1] e.g. in [16], with $p_k = \mathrm{Tr}\,\Lambda^{-k}$

$$\begin{array}{rclrcl}
M_{1,1} &=& p_1 & & & \\
M_{3,1} &=& 3p_1^2 & M_{1^4} &=& 3p_1^2 \\
M_{1^6} &=& 15p_1^3 & M_{3,1^3} &=& 6p_3 + 9p_1^3 \\
M_{3,3} &=& 3p_3 + 12p_1^3 & M_{5,1} &=& 5p_3 + 10p_1^3.
\end{array}$$

and it can be checked that they satisfy the first equation of (1.6), as expected.

2. Proof of Theorem 1.1

A hermitian matrix H decomposes as

$$H = \sum_{a=1}^{N} (\mathrm{Re}\,H_{a,a})\,E_{a,a} + \sum_{1\le a<b\le N} (\sqrt{2}\,\mathrm{Re}\,H_{a,b})\,\tfrac{1}{\sqrt{2}}(E_{a,b}+E_{b,a}) + (\sqrt{2}\,\mathrm{Im}\,H_{a,b})\,\tfrac{\mathrm{i}}{2}(E_{a,b}-E_{b,a}).$$

As $E_{a,a}$, $\frac{1}{\sqrt{2}}(E_{a,b}+E_{b,a})$ and $\frac{\mathrm{i}}{\sqrt{2}}(E_{a,b}-\mathrm{E}_{b,a})$ have unit norm for the standard Euclidean metric on $\mathrm{Mat}_N(\mathbb{C}) \cong \mathbb{R}^2$, the volume form on $\mathcal{H}_N$ induced by the Euclidean volume form on $\mathrm{Mat}_N(\mathbb{C}) \cong \mathbb{R}^{2N^2}$ is $2^{\frac{N(N-1)}{2}}\,\mathrm{d}H$. Denote $\mathcal{U}_N$ the unitary group and $\mathrm{d}\nu$ its volume form induced by the Euclidean volume form in $\mathrm{Mat}_N(\mathbb{C})$. The corresponding volume is

$$\mathrm{Vol}(\mathcal{U}_N) = \frac{(2\pi)^{\frac{N(N+1)}{2}}}{\prod_{n=1}^{N-1} n!}.$$

We also recall the Harish-Chandra–Itzykson–Zuber formula [13]

$$\frac{1}{\prod_{n=1}^{N-1} n!}\int_{\mathcal{U}_N} \frac{\mathrm{d}\nu(U)}{\mathrm{Vol}(\mathcal{U}_N)}\, e^{\mathrm{Tr}(AUBU^\dagger)} = \frac{\det(e^{a_ib_j})}{\Delta(\boldsymbol{a})\Delta(\boldsymbol{b})}.$$

where $A = \mathrm{diag}(a_1,\ldots,a_N)$ and $B = \mathrm{diag}(b_1,\ldots,b_N)$.

[1] Note that in [16], equation (45) follows from substituting $\Lambda \to \frac{1}{2}\Lambda$ in equation (44). Equation (45) is the one they use to compute moments and the Gaussian probability measure it induces agrees with our $\mathbb{P}$ defined in (1.1).

Diagonalising the matrix $H = UXU^\dagger$ with $X = \mathrm{diag}(x_1, \dots, x_N)$ and $U \in \mathcal{U}_N$ defined up to action of $\mathfrak{S}_N \times \mathcal{U}_1^N$ brings the partition function (1.2) in the form

$$
\begin{aligned}
Z(\mathbf{t}) &= \frac{\sqrt{\Delta(\boldsymbol{\lambda},\boldsymbol{\lambda})}}{2^{\frac{N}{2}}(2\pi)^{\frac{N^2}{2}}} \frac{1}{N!\,(2\pi)^N\, 2^{\frac{N(N-1)}{2}}} \int_{\mathbb{R}^N} \left(\int_{\mathcal{U}_N} \mathrm{d}\nu(U)\, e^{-\frac{1}{2}\mathrm{Tr}(\Lambda U X^2 U^\dagger)} \right) (\Delta(\boldsymbol{x}))^2 \prod_{i=1}^N e^{V_{\mathbf{t}}(x_i)} \mathrm{d}x_i \\
&= \frac{\sqrt{\Delta(\boldsymbol{\lambda},\boldsymbol{\lambda})}}{2^{\frac{N^2}{2}}\,(2\pi)^{\frac{N}{2}}\, N!\, \Delta(-\boldsymbol{\lambda}/2)} \int_{\mathbb{R}^N} \frac{\big(\Delta(\boldsymbol{x})\big)^2}{\Delta(\boldsymbol{x}^2)} \det_{1 \leq i,j \leq N} \left(e^{-\frac{1}{2}\lambda_i x_j^2}\right) \prod_{i=1}^N e^{V_{\mathbf{t}}(x_i)} \mathrm{d}x_i.
\end{aligned} \tag{2.1}
$$

Here we could use Fubini because the integrand in the first line of (2.1) is real positive, and in fact integrable due to the assumptions on V_0 and Λ. We observe that $\Delta(-\boldsymbol{\lambda}/2) = (-2)^{-\frac{N(N-1)}{2}} \Delta(\boldsymbol{\lambda})$ and recall Schur's Pfaffian identity [18], for N even

$$
\frac{\big(\Delta(\boldsymbol{x})\big)^2}{\Delta(\boldsymbol{x}^2)} = \prod_{1 \leq i < j \leq N} \frac{x_j - x_i}{x_j + x_i} = \operatorname*{Pf}_{1 \leq i,j \leq N} \left(\frac{x_j - x_i}{x_j + x_i} \right).
$$

So, up to a prefactor $Z(\mathbf{t})$ is an integral of the form

$$
\int_{\mathbb{R}^N} \operatorname*{Pf}_{1 \leq i,j \leq N} \big(S(x_i,x_j)\big) \det_{\substack{0 \leq m \leq N-1 \\ 1 \leq j \leq N}} \big(f_m(x_j^2)\big) \prod_{i=1}^N \rho(x_i) \mathrm{d}x_i \tag{2.2}
$$

De Bruijn's identity [8] would allow rewriting (2.2) as

$$
N! \operatorname*{Pf}_{0 \leq m,n \leq N-1} \left(\int_{\mathbb{R}^2} S(x,y)\, f_m(x^2) f_n(y^2)\, \rho(x)\rho(y)\, \mathrm{d}x\mathrm{d}y \right), \tag{2.3}
$$

but the proof in loc. cit. is solely based on algebraic manipulations, valid when $(f_n)_{n=0}^{N-1}$ is a sequence of measurable functions on $\mathbb{R}_{\geq 0}$ and $S(x,y) = -S(x,y)$ is a measurable function on $\mathbb{R}^2$ such that $\int_{\mathbb{R}^2} \big|S(x,y) f_m(x^2) f_n(y^2)\big| \rho(x)\rho(y) \mathrm{d}x\mathrm{d}y < +\infty$. The choice of $S(x,y) = \frac{x+y}{x-y}$ in general violates this integrability assumption due to the presence of the simple pole on the antidiagonal combined with the non-compactness of $\mathbb{R}^2$. Nevertheless, we show that the conclusion (2.3) remains valid provided the integral in the Pfaffian is understood as a Cauchy principal value, under a Schwarz-type condition.

Lemma 2.1. *Let $\rho > 0$ be a measurable function on $\mathbb{R}$ and $(f_n)_{n=0}^{N-1}$ be a sequence of $\mathcal{C}^{N-1}$-functions on $\mathbb{R}_{\geq 0}$ such that $f_m^{(\ell)}$ is bounded by a polynomial for any $m, \ell \in \{0, \dots, N-1\}$. Let $S(x,y) = \frac{\tilde{S}(x,y)}{x+y}$ where $\tilde{S}$ is a measurable function on $\mathbb{R}^2$ such that*

$$
\forall k,l \in \mathbb{N}, \qquad \int_{\mathbb{R}^2} |\tilde{S}(x,y) x^k y^l|\, \rho(x)\rho(y)\, \mathrm{d}x\mathrm{d}y < +\infty.
$$

Then, for N even

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \operatorname*{Pf}_{1 \leq i,j \leq N} \big(S(x_i,x_j)\big) \det_{\substack{0 \leq m \leq N-1 \\ 1 \leq j \leq N}} \big(f_m(x_j^2)\big) \prod_{i=1}^n \rho(x_i) \mathrm{d}x_i \\
&= N! \operatorname*{Pf}_{0 \leq m,n \leq N-1} \left(⨍_{\mathbb{R}^2} S(x,y) f_m(x^2) f_n(y^2)\, \rho(x)\rho(y)\, \mathrm{d}x\mathrm{d}y \right),
\end{aligned} \tag{2.4}
$$

where $⨍ = \lim_{\epsilon \to 0} \int_{|x+y| \geq \epsilon}$ and the integrand in the left-hand side is integrable.

Proof. Take $\epsilon > 0$ and set $S_\epsilon(x,y) = S(x,y)\cdot \mathbf{1}_{|x+y|\geq \epsilon}$. In this situation we can use de Bruijn's formula and write

$$\begin{aligned}&\int_{\mathbb{R}^N} \operatorname*{Pf}_{1\leq i,j\leq N}\big(S_\epsilon(x_i,x_j)\big) \operatorname*{det}_{\substack{0\leq m\leq N-1\\ 1\leq j\leq N}}\big(f_m(x_j^2)\big)\prod_{i=1}^N \rho(x_i)\mathrm{d}x_i \\ &= N!\operatorname*{Pf}_{0\leq m,n\leq N-1}\bigg(\int_{\mathbb{R}^2} S_\epsilon(x,y)\, f_m(x^2)f_n(y^2)\,\rho(x)\rho(y)\,\mathrm{d}x\mathrm{d}y\bigg).\end{aligned} \tag{2.5}$$

The right-hand side tends to $N!\,\mathrm{Pf}_{0\leq m,n\leq N-1}\big(⨍_{\mathbb{R}^2} S(x,y)f_m(x^2)f_n(y^2)\rho(x)\rho(y)\mathrm{d}x\mathrm{d}y\big)$ when $\epsilon\to 0$. Call $I_\epsilon(\mathbf{x})$ the integrand in the left-hand side of (2.5). We clearly have $\lim_{\epsilon\to 0} I_\epsilon(\mathbf{x}) = I_0(\mathbf{x})$ for $\mathbf{x}$ almost everywhere in $\mathbb{R}^N$. Provided we can find for $I_\epsilon(\mathbf{x})$ a uniform in ϵ and integrable on $\mathbb{R}^2$ upper bound, the lemma follows from dominated convergence.

To find such a bound, we introduce the matrix $W(\boldsymbol{\xi})$ with entries ξ_j^n at row index $n\in\{0,\ldots,N-1\}$ and column index $j\in[N]$, which satisfies $\Delta(\boldsymbol{\xi}) = \det W(\boldsymbol{\xi})$. Its inverse matrix is

$$(W(\boldsymbol{\xi})^{-1})_{i,n} = \frac{(-1)^{N-n-1}\, e_{N-n-1}(\boldsymbol{\xi}_{[i]})}{\prod_{j\neq i}(\xi_i-\xi_j)},$$

where e_k is the k-th elementary symmetric polynomial and $\boldsymbol{\xi}_{[i]} = (\xi_1,\ldots,\widehat{\xi_i},\ldots,\xi_N)$. Then

$$\begin{aligned}\operatorname*{det}_{\substack{0\leq m\leq N-1\\ 1\leq j\leq N}}\big(f_m(x_j^2)\big) &= \Delta(\boldsymbol{x}^2)\operatorname*{det}_{\substack{1\leq i\leq N\\ 0\leq n\leq N-1}}\big(f_n(x_i^2)\big)\cdot\det\big(W(\boldsymbol{x}^2)^{-1}\big) \\ &= \Delta(\boldsymbol{x}^2)\operatorname*{det}_{0\leq m,n\leq N-1}\bigg(\sum_{i=1}^N \frac{(-1)^{N-m-1}\, f_n(x_i^2)\, e_{N-m-1}(\boldsymbol{x}^2_{[i]})}{\prod_{j\neq i}(x_i^2-x_j^2)}\bigg).\end{aligned} \tag{2.6}$$

Up to a sign that we can take out of the determinant, the (m,n)-entry inside the determinant is

$$\begin{aligned}[u^{N-m-1}]\sum_{i=1}^N f_n(x_i^2)\prod_{j\neq i}\frac{1+ux_j^2}{x_i^2-x_j^2} &= [u^{N-m-1}]\prod_{i=1}^N(1+ux_i^2)\bigg(\sum_{i=1}^N\frac{f_n(x_i^2)}{1+ux_i^2}\,\frac{1}{\prod_{j\neq i}(x_i^2-x_j^2)}\bigg) \\ &= \sum_{k=0}^{N-m-1} e_{N-m-1-k}(\boldsymbol{x}^2)\bigg(\sum_{i=1}^N\frac{(-1)^k x_i^{2k} f_n(x_i^2)}{\prod_{j\neq i}(x_i^2-x_j^2)}\bigg).\end{aligned}$$

In the first two steps, $[u^m]$ acting on the formal power series of u to its right meant extracting the coefficient of u^m. Up to the use of squared variables, we recognize the divided difference

$$g[\xi_1,\ldots,\xi_N] := \sum_{i=1}^N \frac{g(\xi_i)}{\prod_{j\neq i}(\xi_i-\xi_j)}.$$

When g is $\mathcal{C}^{N-1}$, it can be written (see e.g. [12, Theorem 2, p250]) as an integral over the $(N-1)$-dimension simplex $\Delta_{N-1} = \{p\in[0,1]^N \mid p_1+\cdots+p_N = 1\}$, equipped with the volume form $\mathrm{d}\sigma(\boldsymbol{p}) = \mathrm{d}p_1\cdots\mathrm{d}p_{N-1}$:

$$g[\xi_1,\ldots,\xi_N] = \int_{\Delta_{N-1}} g^{(N-1)}(p_1\xi_1+\cdots+p_N\xi_N)\,\mathrm{d}\sigma(\boldsymbol{p}).$$

We use this for $g_{k,n}(\xi) = (-1)^k\xi^k f_n(\xi)$. Inserting the integral representation in (2.6) yields

$$\begin{aligned}&\big|I_\epsilon(\mathbf{x})\big| = \big|\Delta(\boldsymbol{x}^2)\big|\operatorname*{Pf}_{1\leq i,j\leq N}\big(S_\epsilon(x_i,x_j)\big) \\ &\times\bigg|\operatorname*{det}_{0\leq m,n\leq N-1}\bigg(\sum_{k=0}^{N-1-m} e_{N-1-m-k}(\boldsymbol{x}^2)\int_{\Delta_{N-1}} g_{k,n}^{(N-1)}(p_1x_1^2+\cdots+p_Nx_N^2)\mathrm{d}\sigma(\boldsymbol{p})\bigg)\bigg|\prod_{i=1}^N\rho(x_i).\end{aligned}$$

Since $|\Delta(\boldsymbol{x}^2)|$ cancels the denominators in S_ϵ, the first line of the right-hand side admits an upper bound by sum of terms, each of which is a polynomial in $\boldsymbol{x}$ multiplied by $\prod_{\{i,j\}\in\mathcal{P}} |\tilde{S}(x_i,x_j)|$, where $\mathcal{P}$ is a partition of $[N]$ into pairs. In the second line, we first expand the determinant inside the absolute value and use the triangular inequality to get an upper bound by a sum of finitely positive terms, each of which involves an N-fold product of simplex integrals of functions with at most polynomial growth, since the derivatives $f_n^{(\ell)}$ (and thus $g_{k,n}^{(N-1)}$) has at most polynomial growth. Therefore, they result in a polynomial upper bound in the variable $\boldsymbol{x}$. We are thus left with an upper bound by a sum of finitely many terms of the form:

$$\prod_{\{i,j\}\in\mathcal{P}} \big|\tilde{S}(x_i,x_j)\big| \prod_{i=1}^{N} x_i^{q_i}\rho(x_i) = \prod_{\{i,j\}\in\mathcal{P}} \big|\tilde{S}(x_i,x_j)\big|\, x_i^{q_i}x_j^{q_j}\,\rho(x_i)\rho(x_j)$$

for various N-tuples of integers $\boldsymbol{q}$ and pair partitions $\mathcal{P}$ of $[N]$. Integrating each term of this form over $\mathbb{R}^N$ factorizes into a product of $\frac{N}{2}$ two-dimensional integrals, each of them being finite by assumption. This provides the domination assumption to conclude $\lim_{\epsilon\to 0}\int_{\mathbb{R}^N} I_\epsilon(\boldsymbol{x})\prod_{i=1}^N \mathrm{d}x_i = \int_{\mathbb{R}^N} I_0(\boldsymbol{x})\prod_{i=1}^N \mathrm{d}x_i$ as desired. □

Remark 2.2. The proof can easily be adapted to obtain a analogous statement for kernels of the form $S(x,y) = \frac{\tilde{S}(x,y)}{x-y}$ with $\tilde{S}$, in which case one can use $f_n(x)$ instead of $f_n(x^2)$.

The assumptions of Lemma 2.1 are fulfilled for

$$S(x,y) = \frac{x-y}{x+y}, \qquad \rho(x) = e^{-\frac{1}{2}\lambda_{\min}x^2 + V_{\mathbf{t}}(x)}, \qquad f_m(\xi) = e^{-\frac{1}{2}(\lambda_m - \lambda_{\min})\xi}$$

where we stress that $\mathbf{t}$ are formal parameters. Therefore, coming back to (2.1) and tracking the N-dependent prefactors, we arrive to the identity of formal power series in the variables $\mathbf{t}$:

$$\begin{aligned} Z(\mathbf{t}) &= \frac{(-1)^{\frac{N(N-1)}{2}}\sqrt{\Delta(\boldsymbol{\lambda},\boldsymbol{\lambda})}}{2^N\,\pi^{\frac{N}{2}}\,\Delta(\boldsymbol{\lambda})} \operatorname*{Pf}_{0\le m,n\le N-1}\big(L_{m,n}\big) \\ L_{m,n} &= \bigg(\fint_{\mathbb{R}^2} \frac{x-y}{x+y}\, e^{-\frac{1}{2}\lambda_m x^2 - \frac{1}{2}\lambda_n y^2 + V_{\mathbf{t}}(x) + V_{\mathbf{t}}(y)}\,\mathrm{d}x\mathrm{d}y\bigg). \end{aligned}$$

We would like to rewrite this formula by absorbing the denominator $\Delta(\boldsymbol{\lambda})$ in the Pfaffian. Recall the transposed Vandermonde matrix $W(\boldsymbol{\lambda})^T$, whose entries are $W(\boldsymbol{\lambda})^T_{i,n} = \lambda_i^n$ indexed by $i\in[N]$ and $n\in\{0,\ldots,N-1\}$. We have

$$\frac{\mathrm{Pf}(L)}{\Delta(\boldsymbol{\lambda})} = \frac{\mathrm{Pf}(L)}{\det W(\boldsymbol{\lambda})^T} = \mathrm{Pf}\big((W(\boldsymbol{\lambda})^T)^{-1}LW(\boldsymbol{\lambda})^{-1}\big),$$

and by Cramer's formula for the inverse

$$\big((W(\boldsymbol{\lambda})^T)^{-1}LW(\boldsymbol{\lambda})^T\big)_{m,n} = v_m v_n \int_{\mathbb{R}^2} \frac{x-y}{x+y}\, F_m(x)F_n(y)\, e^{V_{\mathbf{t}}(x)+V_{\mathbf{t}}(y)}\,\mathrm{d}x\mathrm{d}y,$$

where v_n are non-zero constants to be chosen later, rows and columns are indexed by $m,n\in\{0,\ldots,N-1\}$, and we introduced:

$$F_m(x) = \frac{1}{v_m}\sum_{i=1}^{N}\big(W(\boldsymbol{\lambda})^T\big)^{-1}_{i,m}\, e^{-\frac{1}{2}\lambda_i x^2} = \frac{\det\big(\lambda_i^0\,\big|\,\lambda_i^1\,\big|\,\cdots\,\big|\,\lambda_i^{m-1}\,\big|\,e^{-\frac{1}{2}\lambda_i x^2}\,\big|\,\lambda_i^{m+1}\,\big|\,\cdots\,\big|\,\lambda_i^{N-1}\big)}{v_m\,\Delta(\boldsymbol{\lambda})}.$$

With Taylor formula in integral form at order N near 0, we can write

$$\begin{aligned} e^{-\frac{1}{2}\lambda_i x^2} &= P_{N-1}\big(-\tfrac{1}{2}\lambda_i x^2\big) + \frac{(-\frac{1}{2}\lambda_i x^2)^m}{m!} + R_N\big(-\tfrac{1}{2}\lambda_i x^2\big) \\ R_N(\xi) &= \frac{\xi^N}{(N-1)!}\int_0^1 (1-u)^{N-1} e^{\xi u}\mathrm{d}u. \end{aligned} \tag{2.7}$$

for some polynomial P_{N-1} of degree atmost $N-1$ and without its term of degree m (which we wrote separately). The contribution of P_{N-1} disappears as it is a linear combination of the other columns, while the contribution of the degree m term simply retrieves the Vandermonde determinant. Hence:

$$F_m(x) = \frac{(-1)^m x^{2m}}{2^m\, m!\, v_m} + \frac{\det\Big(\lambda_i^0 \,|\, \lambda_i^1 \,|\, \cdots \,|\, \lambda_i^{m-1} \,|\, R_N\big(-\tfrac{1}{2}\lambda_i x^2\big) \,|\, \lambda_i^{m+1} \,|\, \cdots \,|\, \lambda_i^{N-1}\Big)}{v_m\, \Delta(\boldsymbol{\lambda})}.$$

We now choose $v_m = \frac{(-1)^m}{2^m m!}$ to get $F_m(x) = x^{2m} + O(x^{2N})$ when $x \to 0$. Introducing the matrix

$$K_{m,n}(\mathbf{t}) = \int_{\mathbb{R}^2} \frac{x-y}{x+y}\, F_m(x) F_n(y)\, e^{V_{\mathbf{t}}(x)+V_{\mathbf{t}}(y)}\, \mathrm{d}x\mathrm{d}y,$$

we arrive to

$$\begin{aligned} Z_N(\mathbf{t}) &= \frac{(-1)^{\frac{N(N-1)}{2}}\, \sqrt{\Delta(\boldsymbol{\lambda},\boldsymbol{\lambda})}\, \prod_{n=0}^{N-1} v_n}{2^N\, \pi^{\frac{N}{2}}} \operatorname*{Pf}_{0\le m,n\le N-1} K_{m,n}(\mathbf{t}) \\ &= \frac{\sqrt{\Delta(\boldsymbol{\lambda},\boldsymbol{\lambda})}}{2^{\frac{N^2}{2}}\, (2\pi)^{\frac{N}{2}}\, \prod_{n=1}^{N-1} n!} \operatorname*{Pf}_{0\le m,n\le N-1} K_{m,n}(\mathbf{t}). \end{aligned}$$

3. Discussion

For the Kontsevich model with potential $V_0(x) = -\frac{\mathrm{i}}{6}x^3$, $Z(\mathbf{t})$ is a KdV τ-function with respect to the times $s_{2n+1} = -(2n-1)!!\,\mathrm{Tr}\,\Lambda^{-(2n+1)}$ [15, 14]. With the result of Alexandrov [1], it is also a BKP τ-function in the times $\frac{1}{2}\mathbf{s}$. In [16, Section 4] a relation between the Kontsevich model (presented with a shift $H \to H - 2\mathrm{i}\Lambda$) and Schur Q-functions was discovered. We note that Schur Q-functions are a key tool in the proof of the BKP relations.

The BKP hierarchy of Theorem 1.1 is independent of the KdV/BKP structure of the Kontsevich model $V(x) = -\frac{\mathrm{i}}{6}x^3$, since it rather governs the evolution under polynomial deformations of the potential (parameters $\mathbf{t}$), for fixed Λ. We are mainly interested in expansion at $\mathbf{t} = \mathbf{0}$ which amounts to quadratic BKP equations between moments. When V_0 is even these relations simplify considerably. It still remains a meaningful problem to understand whether, for arbitrary V_0, $Z(\mathbf{t})$ is governed by an integrable hierarchy with respect to the times $\mathbf{s}$ related to Λ.

Apart from $V_0 = 0$ and V_0 cubic, the simplest even case is $V_0(x) = -\frac{cN}{4}x^4$ for some parameter $c > 0$, and its formal large N topological expansion has been studied during the last years [10, 19], providing strong evidence [6] that the topological expansion of the cumulants obey the blobbed topological recursion [5], which is the general solution of abstract loop equations [4]. In [11] a recursive formula for meromorphic differentials which are generating series of the genus 1 cumulants was given, and a generalisation to higher genera was outlined.

On the other hand, BKP tau functions of hypergeometric type with mild analytic assumptions are known to satisfy abstract loop equations, and thus (perhaps blobbed) topological recursion [2]. In particular, this was applied to prove the conjecture of [9] that spin Hurwitz numbers (weighted by the parity of a spin structure) satisfy topological recursion. It is plausible that a suitably normalised version of $Z(\mathbf{t})$ is in fact a 2-BKP τ-function with respect to $\mathbf{t}$ and Λ, and therefore that (blobbed or not) topological recursion could be established for it, and we shall return to this question later.

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